ECE-541
Fall 2005

Power Spectral Density

Define $X_T(t) = x(t)\Pi_T(t)$,

where $\Pi_T(t) = \begin{cases} 1, & |t| \le \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$

$$\mathcal{F}\{X_T(t)\} = X_T(\Omega)$$

$$= \int_{-T/2}^{T/2} X(t)\, e^{-j\Omega t}\, dt$$

$$|X_T(j\Omega)|^2 = X_T(j\Omega)\, X_T^*(j\Omega)$$

$$= \int_{-T/2}^{T/2} X(t)\, e^{-j\Omega t}\, dt \int_{-T/2}^{T/2} X^*(t)\, e^{j\Omega t}\, d\Omega$$

$$|X_T(j\Omega)|^2 = \int_{-T/2}^{T/2} X(t_1)\, e^{-j\Omega t_1}\, dt_1 \int_{-T/2}^{T/2} X^*(t_2)\, e^{j\Omega t_2}\, dt_2$$

$$= \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} X(t_1)\, X^*(t_2)\, e^{j\Omega(t_2 - t_1)}\, dt_1\, dt_2$$

$$E\{|X_T(j\Omega)|^2\} = \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} R_{xx}(t_1, t_2)\, e^{j\Omega(t_2 - t_1)}\, dt_1\, dt_2$$

If $X(t)$ is WSS then $R_{xx}(t_1, t_2) = R_{xx}(t_2 - t_1)$

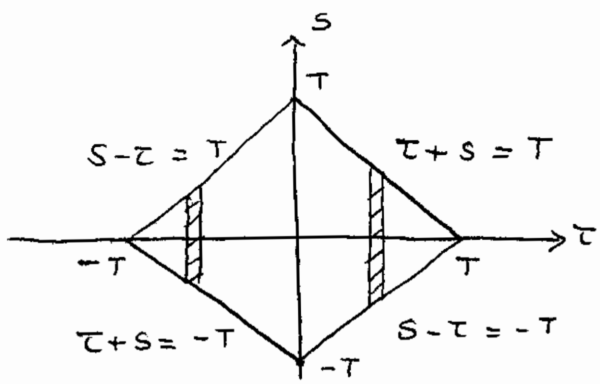
$$E\{|X_T(j\Omega)|^2\} = \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} R_{xx}(t_1-t_2)\, e^{+j\Omega(t_2-t_1)}\, dt_1\, dt_2$$

Using the coordinate transformation

$$\begin{pmatrix}\tau\\ s\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}t_1\\ t_2\end{pmatrix}$$

$$d\tau\, ds = 2\, dt_1\, dt_2$$

or $\quad dt_1\, dt_2 = \frac{1}{2}\, d\tau\, ds$



$$E\{|X_T(j\Omega)|^2\} = \int_{-T}^{0}\int_{-T-\tau}^{T+\tau} R_{xx}(\tau)\, e^{-j\Omega\tau}\, \frac{d\tau\, ds}{2} + \int_{0}^{T}\int_{-T+\tau}^{T-\tau} R_{xx}(\tau)\, e^{-j\Omega\tau}\, \frac{d\tau\, ds}{2}$$

$$E\{|X_T(j\Omega)|^2\} = \frac{1}{2}\int_{-T}^{0} \underbrace{(2T+2\tau)}_{R_{xx}(\tau)}\, e^{-j\Omega\tau}\, d\tau + \frac{1}{2}\int_{0}^{T} \underbrace{(2T-2\tau)}_{R_{xx}(\tau)}\, e^{-j\Omega\tau}\, d\tau$$

$$E\left\{|X_T(j\Omega)|^2\right\} = \int_{-T}^{0} (T+\tau)\, R_{xx}(\tau)\, e^{-j\Omega\tau}\, d\tau + \int_{0}^{T} (T-\tau)\, R_{xx}(\tau)\, e^{-j\Omega\tau}\, d\tau$$

$$\frac{1}{T}\, E\left\{|X_T(j\Omega)|^2\right\} = \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) R_{xx}(\tau)\, e^{-j\Omega\tau}\, d\tau$$

$$\lim_{T\to\infty} \frac{1}{T}\, E\left\{|X_T(j\Omega)|^2\right\} = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\Omega\tau}\, d\tau$$

$$= P_{xx}(j\Omega)$$

Notes

(a) $X_T(j\Omega)$ is a R.V. because it depends on the realization

(b) $P_{xx}(j\Omega)$ requires an infinite data record.