Power Spectral Factorization

Consider a zero-mean, WSS, discrete-time, random signal with a power spectrum $P_{xx}(z)$ that is real and positive on the unit circle, which has a finite average power P^x_{ave}, where both $P_{xx}(z)$ and $\log(P_{xx}(z))$ are analytic in the region $\rho < |z| < \frac{1}{\rho}$. It can be shown that a power spectrum that satisfies these requirements can be factorized into the form:

$$P_{xx}(z) = \sigma_o^2 H_{\min}(z) H_{\max}(z), \quad \rho < |z| < \frac{1}{\rho},\ 0 < \rho < 1$$

where $H_{\min}(z)$ is the monic minimum phase part of $P_x(z)$, $H_{\max}(z)$ is the monic maximum phase part of $P_{xx}(z)$ and σ_o^2 is the variance of the innovations process.

The *cepstrum* representation of a random signal is defined via the DTFT pair:

$$C(e^{j\omega}) = \log(P_{xx}(e^{j\omega})) = \sum_{n=-\infty}^{\infty} c[n] \exp(-j\omega n) \quad (1)$$

$$c[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} C(e^{j\omega}) \exp(j\omega n) d\omega, \quad (2)$$

where $c[n]$ are referred to as the cepstral coefficients. The parameters of the power spectral factorization of the random signal can be related to the cepstrum representation via:

$$\sigma_o^2 = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left(P_{xx}(e^{j\omega})\right) d\omega\right).$$

This parameter is sometimes referred to as the *geometric mean* (GM) of the power spectrum. The minimum phase part is defined via:

$$H_{\min}(z) = \exp\left(\sum_{k=1}^{\infty} c[k] z^{-k}\right), \quad |z| > \rho.$$

Notice from the ROC that this system function corresponds to a causal system, furthermore it can be shown that is a monic system function. The fact that this is a minimum-phase system function follows from the fact that both $H_{\min}(z)$ is minimum-phase as well as its logarithm. In a similar fashion, the maximum phase part of the power spectrum is given by:

$$H_{\max}(z) = \exp\left(\sum_{k=-\infty}^{-1} c[k] z^{-k}\right), \quad |z| < \frac{1}{\rho}.$$

In a manner similar to the minimum-phase part it can be shown that this system function corresponds to a monic maximum phase system function.

Ramifications of PSD Factorization

In addition to allowing us to factor the PSD of a random signal into three parts, the PSD factorization theorem also implies the following:

1. The random signal $x[n]$ and its innovations process $v[n]$ are linearly equivalent, i.e.,

$$x[n] = \sum_{k=-\infty}^{\infty} h_{\min}[k]v[n-k] \quad (3)$$

$$v[n] = \sum_{k=-\infty}^{\infty} \gamma[k]x[n-k] \quad (4)$$

2. The innovations equivalent of the random signal $v[n], n \in \mathbf{I}$ constitutes an orthonormal basis for the Hilbert space of finite average power random signals that satisfy the criteria for PSD factorization. This innovations process is sometimes referred to as the *Kalman innovation* process and will play a major role in optimal estimation.

3. The power spectrum $P_{xx}(z)$ of the WSS random signal is composed of poles and zeroes that come in complex conjugate reciprocal pairs:

$$P_{xx}(z) = \sigma_o^2 \frac{B(z)B^*(\frac{1}{z^*})}{A(z)A^*(\frac{1}{z^*})}.$$

4. Since we require the PSD, which is a continuous function of ω, to be positive and real on the unit-circle, the zeros of $P_{xx}(z)$ on the unit circle come in pairs, i.e., the multiplicity of the zeroes of $P_{xx}(z)$ on the UC is even.