Random Signals & Multirate Systems

Now that we have a basic understanding of the decimation and interpolation systems with deterministic inputs, we can now look at these systems when they are excited by random input signals. Consider a zero mean, discrete–time *wide sense stationary* (WSS) random signal $x[n]$ that is the input to a decimation system. The power spectrum of the input signal $x[n]$ is denoted $P_{xx}(e^{j\omega})$ and its average power is the area under the power spectrum function:

$$P^x_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{xx}(e^{j\omega})d\omega.$$

The output of the *lowpass filtering operation* (LPF) is denoted $z[n]$ and the average of the output of the LPF is given by:

$$P^z_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_{\text{lp}}(e^{j\omega})|^2 P_{xx}(e^{j\omega})d\omega.$$

In our case LPF operation is assumed to be an ideal LPF with cut-off frequency $\omega_c = \pi/M$ and this relation can be reduced to:

$$P^z_{\text{ave}} = \frac{1}{2\pi}\int_{-\pi/M}^{\pi/M} P_{xx}(e^{j\omega})d\omega.$$

The output of the downsampling system is denoted $y_d[n]$ and the *autocorrelation function* (ACF) of this sequence is given by:

$$R_{yy}[k] = E\{y_d[n]y_d^*[n-k]\} = E\{z[Mn]z^*[Mn-Mk]\} = R_{zz}[Mk].$$

This relation is significant in that it tells us that the ACF of the sequence $z[n]$ also gets downsampled in the process of downsampling $z[n]$. A direct consequence of this relation is that the average power of the output of the downsampler is the same as its input:

$$P^y_{\text{ave}} = R_{yy}[0] = R_{zz}[0] = P^z_{\text{ave}}$$

This results in a scaling of the power spectrum of $z[n]$ so that the average power remains invariant in the downsampling process:

$$P_{yy}(e^{j\omega}) = \frac{1}{M}\sum_{p=0}^{M-1} P_{zz}\left(e^{j(\frac{\omega-2p\pi}{M})}\right).$$

Substituting the expression for $P_{zz}(e^{j\omega})$ into this expression yields:

$$P_{yy}(e^{j\omega}) = \frac{1}{M}\sum_{p=0}^{M-1} \left|H\left(e^{j(\frac{\omega-2p\pi}{M})}\right)\right|^2 P_{xx}\left(e^{(\frac{\omega-2p\pi}{M})}\right), \quad \omega \in [-\pi,\pi]$$

This of course can be interpreted as a spectral zoom operation on the power spectrum of the random signal $x[n]$. The corresponding effect of interpolation on the random signal is more complex than the decimation operation. The output of the interpolator is not a wide sense stationary process but a cyclostationary process with cyclic parameter L. However, we can still use the time-averaged power spectral density representation:

$$\tilde{P}_o(e^{j\omega}) = \frac{1}{L} P_{xx}(e^{j\omega L})|H_i(e^{j\omega})|^2, \quad \omega \in [-\pi, \pi]$$

The average power associated with this time-averaged power spectral density is therefore given by:

$$\tilde{P}^o_{\text{ave}} = \frac{L}{2\pi} \int_{-\pi/L}^{\pi/L} P_{xx}(e^{j\omega L}) d\omega.$$

Using a substitution of variables $\rho = L\omega$ we obtain:

$$\tilde{P}^o_{\text{ave}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_{xx}(e^{j\rho}) d\rho = P^x_{\text{ave}}.$$

A special case of these relations occurs when the input signal is *white noise* obtained by sampling a bandlimited, zero mean continuous–time white noise process $x_c(t)$ with variance σ_x^2 and sampling period T_s. In this case the output of the decimation system is again white noise but with a reduced noise variance:

$$P^y_{\text{ave}} = \frac{1}{2\pi} \int_{-\pi/M}^{\pi/M} \frac{\sigma_x^2}{T_s} d\omega = \frac{\sigma_x^2}{T_s M}.$$

The corresponding interpolator output, however, is not white due to the correlation introduced during the interpolation process. These will be useful specifically when we look at noise shaping based quantization systems.