Sampling of Random Signals

In the class we saw an argument from the frequency domain that specified that if we sample a zero-mean, WSS, continuous-time random signal $x_c(t)$, at twice the largest frequency component present in the PSD $P_{x_cx_c}(\Omega)$, i.e., $\Omega_s = 2\Omega_m$, then the discrete-time random sequence $x[n]$ obtained via impulse sampling of the random signal $x_c(t)$ contains all the information present in the continuous-time signal. This was a consequence of the alias-sum PSD formula given by:

$$P_{xx}(e^{j\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} P_{x_cx_c}\left(\frac{\omega - 2k\pi}{T_s}\right), \quad \omega \in [-\pi, \pi].$$

In this exercise, we shall look at a time-domain argument that describes the same. First consider the interpolation sum described in the class:

$$\hat{x}_c(t) = \sum_{k=-\infty}^{\infty} x_c(kT_s)\text{sinc}\left(\frac{t}{T_s} - k\right). \tag{1}$$

The equality above however, is in the mean-square sense and not in a pointwise sense. From the discussion in the class we saw that, the ACF of the continuous-time random signal was also sampled during the process

$$R_{xx}[k] = E\{x[n]x^*[n-k]\} = R_{x_cx_c}(kT_s).$$

Consequently, if we sampled the continuous-time random signal at an appropriate rate we could recover the ACF of the continuous-time random signal, i.e.,

$$\hat{R}_{x_cx_c}(\tau) = \sum_{k=-\infty}^{\infty} R_{x_cx_c}(kT_s)\text{sinc}\left(\frac{\tau}{T_s} - k\right). \tag{2}$$

Note that his expansion holds good for any lowpass waveform that is bandlimited and satisfies that Nyquist sampling criteria. The signal $R_{x_cx_c}(\tau - t)$ being a lowpass waveform and can therefore be expanded via:

$$\hat{R}_{x_cx_c}(\tau - t) = \sum_{k=-\infty}^{\infty} R_{x_cx_c}(kT_s - t)\text{sinc}\left(\frac{\tau}{T_s} - k\right)$$

Replacing the variable τ with the variable $\tau + t$ we obtain:

$$\hat{R}_{x_cx_c}(\tau) = \sum_{k=-\infty}^{\infty} R_{x_cx_c}(kT_s - t)\text{sinc}\left(\frac{\tau + t}{T_s} - k\right)$$

If the above expression is evaluated at $\tau = 0$ we obtain:

$$\hat{R}_{x_cx_c}(0) = \sum_{k=-\infty}^{\infty} R_{x_cx_c}(kT_s - t)\text{sinc}\left(\frac{t}{T_s} - k\right) \tag{3}$$

If the sum in Eq. (1) were to converge to the continuous-time random signal in the MS sense we require that:

$$\lim_{N\to\infty} E\left\{\left(x_c(t) - \sum_{p=-N}^{N} x_c(pT_s)\text{sinc}\left(\frac{t}{T_s} - p\right)\right)^2\right\} = 0.$$

Expanding the square in the expectation and rewriting the requirement:

$$\lim_{N\to\infty} E\left\{[x_c(t) - \hat{x}_c(t)]x_c(t)\right\} - \lim_{N\to\infty} E\left\{\hat{x}_c(t)[x_c(t) - \hat{x}_c(t)]\right\} = 0. \tag{4}$$

If we look at just the first sum in Eq. (4) and incorporating Eq. (3):

$$\begin{aligned}\lim_{N\to\infty} E\left\{[x_c(t) - \hat{x}_c(t)]x_c(t)\right\} &= R_{x_cx_c}(0) - \lim_{N\to\infty}\sum_{p=-N}^{N} R_{x_cx_c}(t - pT_s)\text{sinc}\left(\frac{t}{T_s} - p\right) \\ &= 0.\end{aligned}$$

In a similar fashion the second sum in Eq. (4) can be rewritten in the following form:

$$\lim_{N\to\infty} E\left\{\hat{x}_c(t)[x_c(t)-\hat{x}_c(t)]\right\} = \sum_{m=-\infty}^{\infty} E\{x_c(mT_s)(x_c(t)-\hat{x}_c(t))\}\mathrm{sinc}\left(\frac{t}{T_s}-m\right) \tag{5}$$

Expanding $R_{xx}(t-mT_s)$ in terms of the sinc function basis we obtain:

$$R_{x_cx_c}(t-mT_s) = \sum_{n=-\infty}^{\infty} R_{x_cx_c}((n-m)T_s)\mathrm{sinc}\left(\frac{t}{T_s}-n\right). \tag{6}$$

Expanding the expectation in Eq. (5) and incorporating Eq. (6) we have:

$$\begin{aligned}\lim_{N\to\infty} E\{x_c(mT_s)(x_c(t)-\hat{x}_c(t))\} &= R_{x_cx_c}(t-mT_s) - \lim_{N\to\infty}\sum_{n=-N}^{N} R_{x_cx_c}((m-n)T_s)\mathrm{sinc}\left(\frac{t}{T_s}-n\right)\\ &= 0.\end{aligned}$$

The frequency domain interpretation of the sampling theorem, that dealt with the alias sum formula and the minimum sampling rate required for signal recovery, in the time domain implies that $\hat{x}_c(t)$ converges to $x_c(t)$ or is equivalent to $x_c(t)$ in the mean-square sense, if we sample at an appropriate rate dictated by the sampling theorem.