Random Walk

Model: $x[n] = 0 \qquad n = 0$

$$x[n+1] = x[n] + J[n], \qquad n > 0$$

where $J[n] = \begin{cases} d, & p = \frac{1}{2} \text{ (heads)} \\ -d, & p = +\frac{1}{2} \text{ (tails)} \end{cases}$

Equivalent Model: Toss a coin every T_s seconds and take a step up $(+d)$ if heads appears and step down $(-d)$ if tails appears.

$$x[1] = x[0] + J[0] = J[0]$$

$$x[2] = x[1] + J[1] = J[0] + J[1]$$

$$\vdots$$

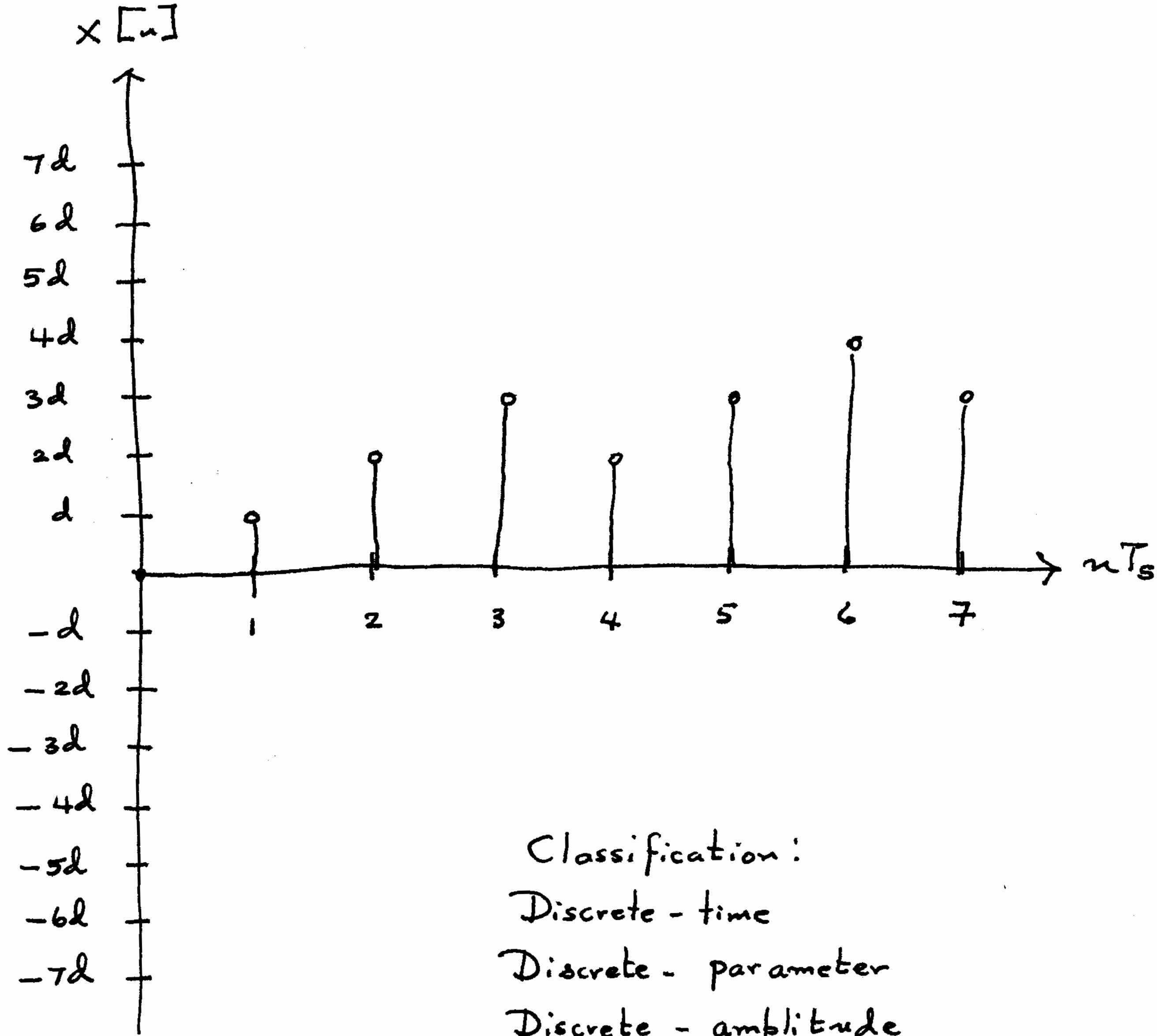
$$x[n] = \sum_{k=0}^{n-1} J[k], \qquad \text{where}$$

$$E\{J[k]\} = d\left(\tfrac{1}{2}\right) + d\left(-\tfrac{1}{2}\right) = 0$$

$$E\{J^2[k]\} = d^2\left(\tfrac{1}{2}\right) + (-d)^2\left(\tfrac{1}{2}\right) = d^2$$

$J[n]$ corresponds to a i.i.d random process with a Bernoulli distribution

Sample Functions



Classification:

Discrete - time

Discrete - parameter

Discrete - amplitude

Mean of Walk process

$$E\{x[n]\} = E\left\{\sum_{k=0}^{n-1} J[k]\right\}$$

$$E\{x[n]\} = \sum_{k=0}^{n-1} E\{J[k]\} = 0$$

Variance of Walk process

$$E\{x^2[n]\} = E\left\{\sum_{i=0}^{n-1} J[i] \sum_{j=0}^{n-1} J[j]\right\}$$

$$= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} R_{JJ}(i,j)$$

$$= \sum_{i=0}^{n-1} d^2 = nd^2$$

Since the variance of $x[n]$ varies with time, $x[n]$ corresponds to a non-stationary process

First - Order Distribution

$$\text{Prob}\{x[n] = md\} = \Pr\{x[n] = (2k-n)d\}$$

$$= \Pr\{x[n] = kd - (n-k)d\}$$

$$= \Pr\{k \text{ heads in } n \text{ tosses of coin}\}$$

$$= \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = \binom{n}{k}\left(\frac{1}{2}\right)^n$$

$$= \left(\frac{n!}{k!\,(n-k)!}\right)\left(\frac{1}{2}\right)^n$$

$$f_{x[n]}(x;n) = \sum_{i=0}^{n} \binom{n}{i}\left(\frac{1}{2}\right)^n \delta(x-i)$$

OBSERVATION:

$J[k]$ has a Bernoulli distribution

$x[n]$ has a Binomial distribution

(for $n=1$ order statistics)

For $n_2 = n_1 + 1$

$$x[n_2] = x[n_1+1] = x[n_1] + J[n_1]$$

If $x[n_1] = x_1$

$$x[n_2] = \begin{cases} x_1 + d, & p = \frac{1}{2} \\ x_1 - d, & p = \frac{1}{2} \end{cases}$$

$$f_{x[n_1],\, x[n_2]}(x_1, x_2; n_1, n_2) = f_{x[n_2]/x[n_1]}(x_2/x_1)\, f_{x[n_1]}(x_1; n_1)$$

$$= \left\{ \frac{1}{2}\,\delta(x_2 - x_1 - d) + \frac{1}{2}\,\delta(x_2 - x_1 + d) \right\} \left(\sum_{i=0}^{n_1} \binom{n_1}{i} \left(\frac{1}{2}\right)^{n_1} \delta(x_1 - i) \right)$$

For n_2 & n_1 general, $n_2 > n_1$

$$x[n_2] = x[n_1] + \sum_{i=n_1}^{n_2-1} J[i]$$

$$P_r\left(x[n_2] = x_2 \,\middle/\, x[n_1] = x_1 \right) = P_r\left\{ \sum_{i=n_1}^{n_2-1} J[i] = x_2 - x_1 \right\}.$$

2^{nd} - order PDF

$$\Pr\left\{ x[n_2] = x_2 \,\middle|\, _{x[n_1] = x_1} \right\}$$

$$= \Pr\left\{ \sum_{i=n_1}^{n_2 - 1} J[i] = x_2 - x_1 \right\}$$

$$= \binom{n_2 - n_1}{x_2 - x_1} \left(\frac{1}{2}\right)^{n_2 - n_1}$$

$$\Pr\left\{ x[n_1] = x_1, \; x[n_2] = x_2 \right\}$$

$$= \binom{n_2 - n_1}{x_2 - x_1} \left(\frac{1}{2}\right)^{n_2 - n_1} \binom{n_1}{x_1} \left(\frac{1}{2}\right)^{n_1}$$

$$= \binom{n_1}{x_1} \binom{n_2 - n_1}{x_2 - x_1} \left(\frac{1}{2}\right)^{n_2}$$

$$f_{x[n_1],\, x[n_2]}(x_1, x_2 ; n_1, n_2) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \binom{n_1}{i} \binom{n_2 - n_1}{j - i} \delta[x_1 - i, x_2 - j]$$

Correlation Funtion

$$R_{xx}[n_1, n_2] = E\{x[n_1]\, x[n_2]\} \qquad n_1 < n_2$$

$$R_{xx}[n_1, n_2] = E\{x[n_1]\,(x[n_2] - x[n_1] + x[n_1])\}$$

$$= E\{x^2[n_1]\} + E\{x[n_1]\,(x[n_2] - x[n_1])\}$$

$x[n_1]$ corresponds to # of heads from 1st toss to n_1^{th} toss

$x[n_2] - x[n_1]$: Corresponds to # of heads $n_1 + 1 < n < n_2$

$x[n_1]$ is statistically independant of $x[n_2] - x[n_1]$

$$\Rightarrow \quad E\{x[n_1](x[n_2] - x[n_1])\} = 0 \qquad \left(\begin{array}{l}\text{Independent} \\ \text{Increments Process}\end{array}\right)$$

$$R_{xx}[n_1, n_2] = \begin{cases} E\{x^2[n_1]\} = n_1 d^2, & n_1 < n_2 \\ E\{x^2[n_2]\} = n_2 d^2, & n_1 > n_2 \end{cases}$$

$$R_{xx}[n_1, n_2] = \min(n_1, n_2)\, d^2$$