Ensemble Averages and Statistics

Using the first and second-order distributions we can then define the following ensemble averages for the random process. The ensemble mean of the random process is defined as the expected value of the random variable $X_t(\lambda)$, i.e.,

$$\mu_X(t) \;=\; E_\lambda(X_t(\lambda)) \;=\; \int_{-\infty}^{\infty} x f_{X(t)}(x;t)dx. \tag{15}$$

The ensemble variance of the random process $X(t)$ is defined as the variance of the random variable $X_t(\lambda)$ via:

$$\sigma_X^2(t) \;=\; E_\lambda\left[|X_t(\lambda) - \mu_X|^2\right] \;=\; \int_{-\infty}^{\infty} |x-\mu_X|^2 f_{X(t)}(x;t)dx. \tag{16}$$

The ensemble autocorrelation function of the random process $X(t)$ denoted $R_{XX}(t_1,t_2)$ is defined via:

$$R_{XX}(t_1,t_2) \;=\; E_\lambda\left[X(t_1)X^*(t_2)\right] \;=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2^* f_{X(t_1),X(t_2)}(x_1,x_2;t_1,t_2)dx_1dx_2 \tag{17}$$

The ensemble autocovariance function of the random process $X(t)$ denoted $C_{XX}(t_1,t_2)$ is defined via:

$$\begin{aligned} C_{XX}(t_1,t_2) &= E_\lambda\left[(X(t_1)-\mu_X(t_1))\,(X(t_2)-\mu_X(t_2))^*\right] \\ C_{XX}(t_1,t_2) &= R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2). \end{aligned} \tag{18}$$

The ensemble temporal *autocoherence* function of the random process $X(t)$ denoted $\rho_{XX}(t_1,t_2)$ is defined via:

$$\rho_{XX}(t_1,t_2) \;=\; \left|\frac{C_{XX}(t_1,t_2)}{\sigma_X(t_1)\sigma_X(t_2)}\right|. \tag{19}$$

The temporal coherence function is essentially the correlation coefficient between the random variables $x(t_1)$ and $x(t_2)$ and is an indicator of the statistical dependence of $x(t_1)$ on $x(t_2)$. A non-zero coherence function indicates the presence of temporal redundance in the process.

The cross-correlation function between two random processes $X(t)$ and $Y(t)$ is defined using joint distributions of the two processes as:

$$R_{XY}(t_1,t_2) = E\left(X(t_1),Y(t_2)\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy^* f_{X(t_1),Y(t_2)}(x,y;t_1,t_2)dxdy. \tag{20}$$

The cross-covariance function is defined similarly as:

$$\begin{aligned} C_{XY}(t_1,t_2) &= E_\lambda\left[(X(t_1)-\mu_X(t_1))\,(Y(t_2)-\mu_Y(t_2))^*\right] \\ C_{XY}(t_1,t_2) &= R_{XY}(t_1,t_2) - \mu_X(t_1)\mu_Y^*(t_2). \end{aligned} \tag{21}$$

The temporal cross-coherence function for the random process pair is then defined via:

$$\rho_{XY}(t_1,t_2) \;=\; \left|\frac{C_{XY}(t_1,t_2)}{\sigma_X(t_1)\sigma_Y(t_2)}\right|. \tag{22}$$