1 Stationarity

A random signal $X(t)$ is said to be *strict sense stationary* (SSS) if the n^{th}-order PDF of the process is invariant to a translation of the time origin, i.e.,

$$f_{\mathbf{X}}(\mathbf{x};\mathbf{t}) = f_{\mathbf{X}}(\mathbf{x};\mathbf{t}+\alpha\mathbf{1})\ ,\alpha \in \mathbf{R}, \tag{1}$$

where $\mathbf{1}$ is a vector of ones. In other words all the n^{th}-order statistics of the SSS process are invariant to a shift in the time origin. This condition is a very stringent requirement and many real-world signals do not satisfy this requirement. Furthermore there is no practical way to verify whether all the n^{th}-order PDF's are invariant to a time-shift. A weaker yet more realistic requirement is satisfied by *wide sense stationary* (WSS) processes for just $n = 2$. It can be easily shown that first-order stationarity requires that

$$\mu_X(t) \ = \ \mu_X(t+\alpha) \ \ , \ \ \sigma_X^2(t) = \sigma_X^2(t+\alpha). \tag{2}$$

It can also be shown that second-order stationarity requires that

$$R_{XX}(t_1,t_2) \ = \ R_{XX}(t_1+\alpha,t_2+\alpha). \tag{3}$$

This inturn implies that for WSS and SSS processes $R_{XX}(t_1,t_2)$, $C_{XX}(t_1,t_2)$ and $\rho_{XX}(t_1,t_2)$ are dependent only on the variable $\tau = |t_1 - t_2|$. A random process that is not stationary is referred to as a *non-stationary* process. Note that SSS implies WSS but the converse is not true except in the case where the process is a Gaussian.

Although on the surface, this may not appear to be a significant requirement from the perspective of measurement it indeed is. Stationarity allows the engineer to make measurements of the random process and its parameters without having to depend on the specific definition of the origin of measurement. The first-order statistics become constants and the second-order statistics are only dependent on the difference between the sampling instants. We are not required to have specific knowledge of the origin of measurement.

The *power spectral density* (PSD) of a WSS random signal is defined via the Fourier transform pair:

$$\begin{aligned} P_{xx}(\Omega) &= \int_{-\infty}^{\infty} R_{xx}(\tau)\exp\left(-j\Omega\tau\right)d\tau \\ R_{xx}(\tau) &= \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} P_{xx}(\Omega)\exp\left(j\Omega\tau\right)d\Omega. \end{aligned} \tag{4}$$

Of course the PSD of the random signal has the same interpretation of being a power-spectrum describing the average of power of the random process that resides in different frequency bands. The average power of a zero mean WSS random signal, P_{ave} can then be evaluated via either of the expressions:

$$\begin{aligned} P_{\text{ave}} &= E(X^2(t)) = R_{xx}(0) \\ P_{\text{ave}} &= \left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} P_{xx}(\Omega)d\Omega \end{aligned}$$