Example of a Stochastic Process

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White Gaussian Noise Sequence

In class, we discussed two special classes of processes: (a) Gaussian random process (GRP) and (b) white noise process. We saw in class that a white noise process is not physically realizable because its autocorrelation function is of the form:

$$R_{xx}(\tau) = \sigma^2 \delta(\tau).$$

The white noise sequence on the other hand, is physically realizable. Here we investigate the special combination of these two types of processes, the *white Gaussian noise* (WGN) process.

MATLAB Script

```
% MATLAB script that generates sample functions
% of a zero-mean Gaussian White Noise
% sequence and computes the sample autocorrelation
% and power spectral density of the process:
% Author: Balu Santhanam
% Course: ECE-541
% Date  : 08/29/18
%-------------------------------------------------------------------------
format long
N = 2500; % # of samples in each realization:
M = 5; % # of sample realiztions to be generated
mu = zeros(1,M); nvar = 0.01; % Mean and Variance
C = nvar*eye(5,5); % Covariance matrix of multivariate Gaussian
x = mvnrnd(mu,C,N);
figure(1)
subplot(3,2,1), plot([0:1:299],x(1:300,1))
xlabel('TIME  SAMPLES'), ylabel('x_{1}[n]')
title('\mu = zeros(1,5), C = 0.01*{\bf I}_{5 \times 5}, N = 300, M = 5')
changefont(gca,16,'Helvetica')
subplot(3,2,2), plot([0:1:299],x(1:300,2))
xlabel('TIME  SAMPLES'), ylabel('x_{2}[n]')
changefont(gca,16,'Helvetica')
subplot(3,2,3), plot([0:1:299],x(1:300,3))
xlabel('TIME  SAMPLES'), ylabel('x_{3}[n]')
changefont(gca,16,'Helvetica')
subplot(3,2,4), plot([0:1:299],x(1:300,4))
xlabel('TIME  SAMPLES'), ylabel('x_{4}[n]')
changefont(gca,16,'Helvetica')
subplot(3,2,5), plot([0:1:299],x(1:300,5))
xlabel('TIME  SAMPLES'), ylabel('x_{5}[n]')
changefont(gca,16,'Helvetica'), print -deps stoc_wgn_real.eps
figure(2)
subplot(2,1,1), plot([-400:400],xcorr(x(:,1),400))
xlabel('LAG   SAMPLES  k'), ylabel('SAMPLE ACF, r_{xx}[k]')
title('ACF of WGN, \mu = zeros(1,5), C = 0.01*eye(5,5)')
```

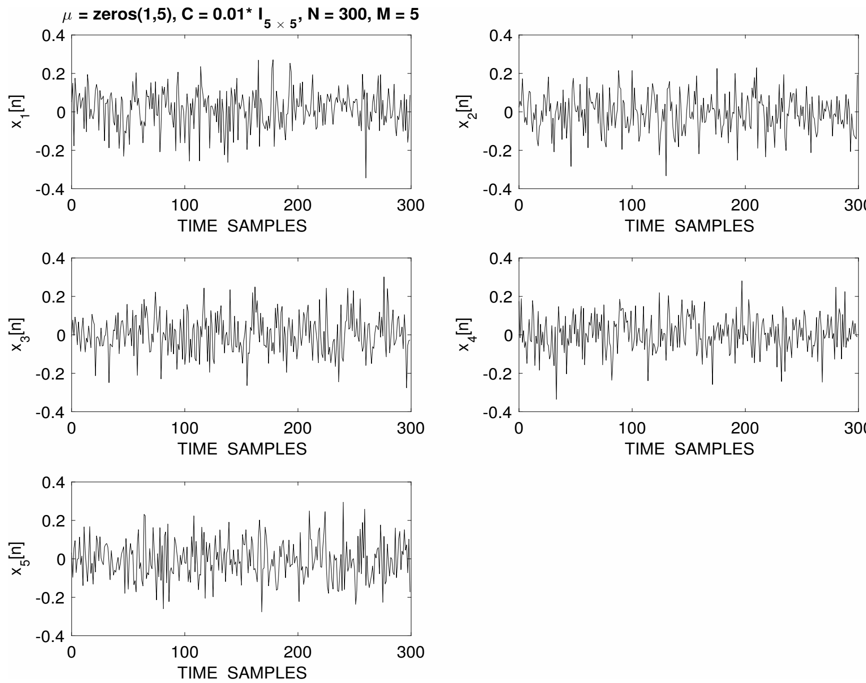

Figure 1: Five sample realizations of a white Gaussian stochastic sequence with mean $\mathbf{m}_x =$ `zeros(1,5)` and a covariance matrix of $\mathbf{C}_x = 0.01\mathbf{I}$ using the multivariate normal MATLAB function `mvnrnd.m`.

```
changefont(gca,16,'Helvetica')
subplot(2,1,2), pwelch(x(:,1)), axis([0 1 -40 0])
changefont(gca,16,'Helvetica'), print -deps stoc_wgn_acf.eps
figure(3)
hist(x(:,1)) xlabel('DATA  BINS')
title('HISTOGRAM OF x_{1}[n], \mu = zeros(1,5), C = 0.01*eye(5,5)')
changefont(gca,16,'Helvetica'), print -deps stoc_wgn_hist.eps
```

ACF and PSD

Since WGN is composed of uncorrelated random variables, the *autocorrelation function* (ACF) of the WGN process is given by:

$$r_{xx}[k] = E\{x[n]x^*[n-k]\} = \sigma_x^2 \delta[k], \quad -\infty \leq k \leq \infty.$$

The corresponding *power spectral density* (PSD) of the process is the DTFT of this ACF sequence:

$$P_{xx}(e^{j\omega}) = \sigma_x^2, \quad \omega \in [-\pi, \pi].$$

The corresponding average power of the WGN sequence is finite and given by:

$$P_{\text{ave}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_{xx}(e^{j\omega}) d\omega = \sigma_x^2 < \infty$$

The sample autocorrelation function $\hat{r}_{xx}[k]$ of the stochastic sequence using the sample function $x_i[n]$ is given by:

$$\hat{r}_{xx}^{(i)}[k] = \frac{1}{N} \sum_{n=0}^{N-1} x_i[n] x_i[n-k].$$

The sample ACF and PSD of $x_1[n]$ is depicted in Fig. (2). As can be observed, the ACF and PSD estimates are very close to the theoretical values.

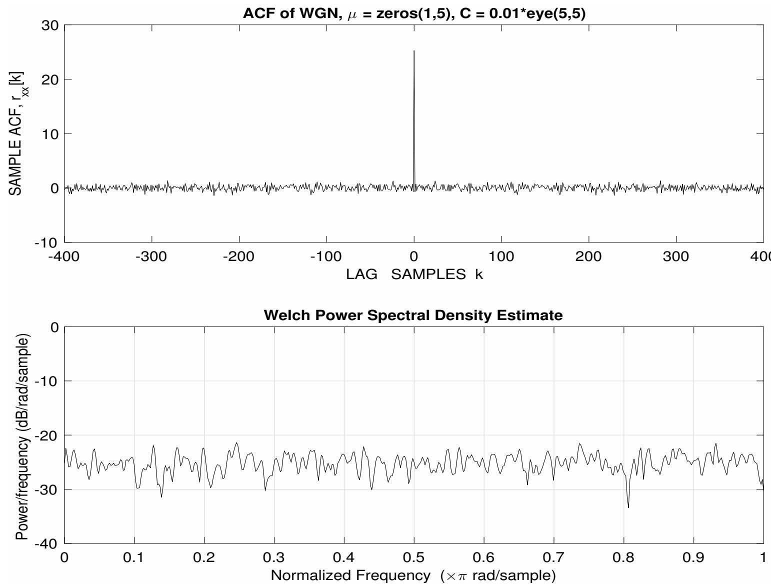

Figure 2: (a) Sample autocorrelation sequence $\hat{r}_{xx}[k]$ and (b) Welch's power spectral density estimate $\hat{P}_x(e^{j\omega})$ of white Gaussian noise sequence using the MATLAB functions `xcorr.m` and `welch.m`.

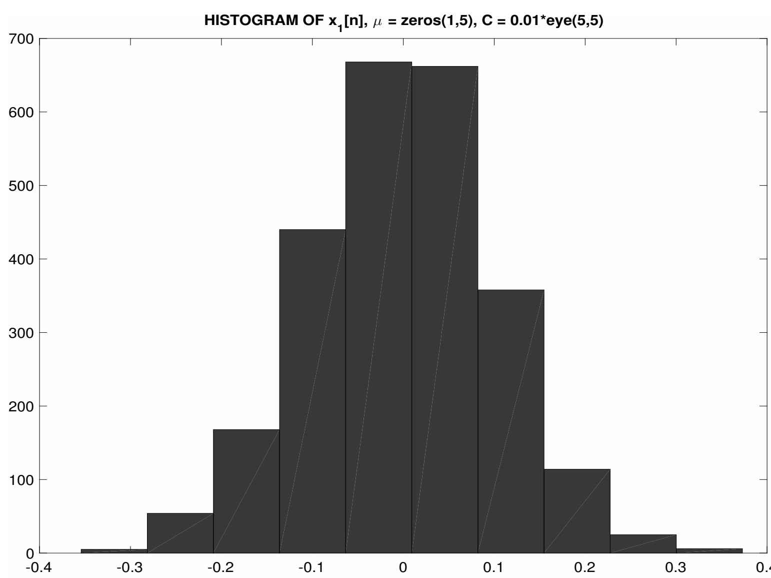

Figure 3: Histogram of a single sample realization of WGN $x_1[n]$, depicting Gaussian statistics with zero mean. In most applications we will have access to may be a few realizations.

Gaussian Statistics

The first-order PDF of the WGN stochastic process is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{1}{2\sigma_x^2}(x-\mu_x)^2\right).$$

The n-th order PDF of the WGN process is given by:

$$f_X(\mathbf{x}) = \frac{1}{2\pi|\det(\mathbf{C}_x)|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m}_x)^T\mathbf{C}_x^{-1}(\mathbf{x}-\mathbf{m}_x)\right).$$

In our particular situation, the theoretical mean vector $\mathbf{m}_x$ is a vector of zeros and the covariance matrix is of the form $\sigma_x^2\mathbf{I}$. The joint n-th order PDF of WGN therefore factors:

$$f_X(\mathbf{x}) = \prod_{i=1}^{n} f_{X_i}(x_i).$$

This is not a surprising result because for the Gaussian random vector the conditions of *uncorrelatedness* and *independence* are equivalent. The sample estimate of the autocorrelation matrix using the "covariance method" is evaluated in MATLAB using the function `corrmtx.m` as:

```
[X,R_hat] = corrmtx(x(:,1),4,'covariance');
>> R_hat
R_hat =

    0.0101   -0.0003    0.0003   -0.0002    0.0001
   -0.0003    0.0101   -0.0003    0.0003   -0.0002
    0.0003   -0.0003    0.0101   -0.0003    0.0003
   -0.0002    0.0003   -0.0003    0.0101   -0.0003
    0.0001   -0.0002    0.0003   -0.0003    0.0101
```

The histogram associated with a single sample realization $x_1[n]$ is depicted in Fig. (3) clearly indicating Gaussian statistics.