Definition of a Stochastic Process

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Interpretation I

1. The stochastic process $X(t, \omega),\ t \in T, \omega \in \Omega$ can be viewed as a indexed collection of waveforms

$$x_i(t) = \{X(t, \omega) \ni \omega \in \Omega\}$$

2. When Ω is discrete then the number of sample functions is countably infinite & when Ω is a continuous set then the number is uncountably infinite.
3. The set of functions are also call as ensemble waveforms or member waveforms.
4. Member functions in themselves may or may not contain information. For example, the ensemble waveforms for noise are not informative.

Example: Interpretation I

1. The coin flip experiment has two possible outcomes $\omega \in \{T, H\}$ with an underlying Bernoulli probability law.
2. The σ-field of events for this experiment is $\mathcal{F} = \{\phi, T, F, S\}$, where S is the whole sample space.
3. Consider a stochastic process defined via the flip of a fair coin:

$$X(t,\omega) = \begin{cases} \cos(\omega_o t) & \omega \in H \\ -\cos(\omega_o t) & \omega \in T \end{cases}$$

4. This process has exactly two sample or member functions. Note that both sample functions are continuous.

Collection of Random Variables

1. Stochastic process $X(t, \omega)$ can be defined as a time-indexed collection of random variables:

$$X(t, \omega) = \{x_t(\omega), t \in T\}$$

2. For sampling times $t_1 < t_2 < t_3, \ldots t_n \in T$, the process is a n-component random vector:

$$X(t) = [X(t_1), X(t_2), \ldots X(t_n)]^T$$

3. Random vector view point convenient for characterizing statistics of the process. Gaussian stochastic process can be described via:

$$X(t) \sim N\left(\mathbf{m}_x, \mathbf{C}_{xx}\right),$$

where $\mathbf{m}_x$ is the mean-vector and $\mathbf{C}_{xx}$ is the covariance matrix associated with the process.

Interpretation II: Collection of Random Variables

1. Consider a Gaussian process as input to the linear transformation:

$$\mathbf{Y} = \mathbf{AX},$$

where $\mathbf{A}$ is a invertible linear transformation.

2. Output has Gaussian statistics $Y(t) \sim N(\mathbf{m}_y, \mathbf{C}_{yy})$ since linear combo of Gaussian random variables is Gaussian distributed:

$$\mathbf{m}_y = \mathbf{A}\mathbf{m}_x, \quad \mathbf{C}_{yy} = \mathbf{A}\mathbf{C}_{xx}\mathbf{A}^T$$

3. Strong white noise is a process whose components are independent and identically distributed:

$$F_X(x) = \prod_{i=1}^{n} F_{X_i}(x_i)$$

4. Weak white noise is a process whose components are uncorrelated:

$$\mathbf{C}_{xx} = \sigma^2 \mathbf{I}.$$

Formal definition of a stochastic process

- A stochastic process $X(t, \omega)$ can be formally defined as a measurable function from the product Cartesian space $T \times \Omega$ to the real line $\mathbf{R}$.
- t is the independent variable and ω is the stochastic parameter.
- Independent variable does not have to be "time".
- If the independent parameter is space then the process is a stochastic image. If the independent parameter is space-time then the process is called a stochastic field.

Statistical Characterization

1. The joint n-th order CDF of a stochastic process is the n-th order joint CDF of the random vector comprising the random variables in the collection:

$$F_{\mathbf{X}}(\mathbf{x};\mathbf{t}) = F_x\left(x_1, x_2, \ldots x_n; t_1, t_2, \ldots, t_n\right) \triangleq \Pr\left(X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots X(t_n) \leq x_n\right)$$

2. The joint n-th order PDF is the mixed partial derivative of the joint n-th order CDF:

$$f_{\mathbf{X}}(\mathbf{x};\mathbf{t}) = f_X(x_1, x_2, \ldots, x_n; t_1, t_2 \ldots, t_n) \triangleq \frac{\partial^n\left(F_{\mathbf{X}}(\mathbf{x};\mathbf{t})\right)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

First Order Statistics

1. Using interpretation II, the mean of a stochastic process is defined via:

$$\mu_x(t) = E\{X_t(\omega)\} = \int_{-\infty}^{\infty} x f_X(x;t)dx$$

2. Similarly the variance of a stochastic process is defined via:

$$\sigma_x^2 = E\{(X_t(\omega) - \mu_x(t))^2\} = \int_{-\infty}^{\infty} (x - \mu_x(t))^2 f_X(x;t)dx.$$

3. These two moments constitute the first-order statistics of the process and in general are functions of time.

Second-Order Statistics

1. The autocorrelation function of the process $X(t)$ is given by:

$$R_{xx}(t_2, t_2) = E\{X(t_1)X^*(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{XY}(x, y)dxdy$$

2. The autocovariance function of the process $X(t)$ is given by:

$$C_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) - \mu_x(t_1)\mu_x^*(t_2).$$

3. The autocoherence function of the process $X(t)$ is given by:

$$\rho_{xx}(t_1, t_2) = \frac{C_{xx}(t_1, t_2)}{\sigma_x(t_1)\sigma_x(t_2)}.$$

Stochastic process example

1. Consider the two-part stochastic process defined in viewpoint I, defined on the toss of a fair coin. Its first-order PDF is given by:

$$f_X(x;t) = \frac{1}{2}\delta\left(x - \cos(\omega_o t)\right) + \frac{1}{2}\delta\left(x + \cos(\omega_o t)\right)$$

2. The corresponding first-order CDF is given by:

$$F_X(x;t) = \frac{1}{2}u\left(x - \cos(\omega_o t)\right) + \frac{1}{2}u\left(x + \cos(\omega_o t)\right).$$

3. The mean and variance of the process are given by:

$$\begin{aligned}\mu_x(t) &= \frac{1}{2}\cos(\omega_o t) + \frac{1}{2}(-\cos(\omega_o t)) = 0 \\ \sigma_x^2(t) &= \frac{1}{2}\cos^2(\omega_o t) + \frac{1}{2}\cos^2(\omega_o t) = \cos^2(\omega_o t).\end{aligned}$$

Stochastic Process Example I

1. Joint probability density function for continuous sample functions for $t_1, t_2 \in \mathbf{R}$ is given by:

$$f_{X_1,X_2}(x_1, x_2; t_1, t_2) = \frac{1}{2}\delta(x_1 - \cos(\omega_o t_1), x_2 - \cos(\omega_o t_2)) + \frac{1}{2}\delta(x_1 + \cos(\omega_o t_1), x_2 + \cos(\omega_o t_2))$$

2. Other transitions have zero probability due to the continuity assumption of the sample functions.
3. Joint density is not separable and does not factor into marginal densities:

$$f_{X_1,X_2}(x_1, x_2; t_1, t_2) \neq f_{X_1}(x_1; t_1) f_{X_2}(x_2; t_2)$$

1. The autocorrelation function for this example is computed as:

$$\begin{aligned} R_{xx}(t_1, t_2) &= E\{X(t_1)X(t_2)\} \\ &= \frac{1}{2}\cos(\omega_o t_1)\cos(\omega_o t_2) + \frac{1}{2}\cos(\omega_o t_1)\cos(\omega_o t_2) \\ &= \cos(\omega_o t_1)\cos(\omega_o t_2). \end{aligned}$$

2. The autocovariance function is given by:

$$C_{xx}(t_1, t_2) = R_{xx}(t_1, t_2) - \mu_x(t_1)\mu_x(t_2) = \cos(\omega_o t_1)\cos(\omega_o t_2).$$

3. The temporal coherence function for this process is given by:

$$\rho_{xx}(t_1, t_2) = \frac{\cos(\omega_o t_1)\cos(\omega_o t_2)}{|\cos(\omega_o t_1)\cos(\omega_o t_2)|}.$$

1. Consider the oscillator process defined via:

$$X(t,\omega) = \cos\left(\Omega(\omega)t\right),$$

where the frequency is a normal random variable, $\Omega \sim N(\Omega_c, \sigma^2)$.

2. Unlike the previous example, this process has a uncountably infinite number of realizations due to fact that the normal random variable is defined on the entire real line.

3. The ensemble mean of the process us given by:

$$\mu_x(t) = E\{\cos(\Omega(\omega)t)\} = \int_{-\infty}^{\infty} \cos\left(xt\right) N(\Omega_c, \sigma^2)dx.$$

4. Employing Euler identities this integral can be expressed as:

$$\mu_x(t) = \frac{1}{2}\int_{-\infty}^{\infty} N(\Omega_c, \sigma^2)\exp\left(jxt\right)dx + \frac{1}{2}\int_{-\infty}^{\infty} N(\Omega_c, \sigma^2)\exp-\left(jxt\right)dx$$

1. Each of the individual integrals corresponds to the characteristic function of a normal random variable with mean $\mu_x = \Omega_c$ and variance $\sigma_x^2 = \sigma^2$:

$$\Psi_N(jt) = \exp\left(j\Omega_c t\right)\exp\left(-\frac{1}{2}\sigma^2 t^2\right).$$

2. The ensemble mean of the process $X(t)$ can then evaluated via:

$$\mu_x(t) = \frac{1}{2}\left(\Psi_N(jt) + \Psi_N(-jt)\right) = \cos\left(\Omega_c t\right)\exp\left(-\frac{1}{2}\sigma^2 t^2\right)$$

3. The ensemble variance σ_x^2 of this process is given by:

$$\sigma_x^2(t) = E\{X_t^2(\omega)\} - \mu_x^2(t) = E\{\cos^2\left(\Omega(\omega)t\right)\} - \mu_x^2(t)$$

1. The mean-squared value of the process is first computed as:

$$E\{\cos^2(\Omega(\omega)t)\} = \int_{-\infty}^{\infty} \cos^2(\Omega(x)t)\, N\left(\Omega_c, \sigma^2\right) dx.$$

2. Substituting the double-angle formula for the trigonometric function we have:

$$E\{\cos^2(\Omega(\omega)t)\} = \frac{1}{2} + \frac{1}{2}\int_{-\infty}^{\infty} \cos(2\Omega(x)t)\, N\left(\Omega_c, \sigma^2\right) dx$$

3. Using the expression for the characteristic function of a Gaussian random variable:

$$E\{\cos^2(\Omega(\omega)t)\} = \frac{1}{2} + \frac{1}{4}\left(\Psi_N(j2t) + \Psi_N(-j2t)\right)$$

4. We can finally evaluate the mean-squared value:

$$E\{X_t^2(\omega)\} = \frac{1}{2} + \frac{1}{2}\cos(2\Omega_c t)\exp\left(-2\sigma^2 t^2\right)$$

1. The ensemble variance can now be evaluated as:

$$\sigma_x^2(t) = \frac{1}{2} + \frac{1}{2}\cos\left(2\Omega_c t\right)\exp\left(-2\sigma^2 t^2\right) - \cos^2\left(\Omega_c t\right)\exp\left(-\sigma^2 t^2\right)$$

2. The ensemble ACF of this process is given by:

$$R_{xx}(t_1, t_2) = E_\Omega\{\cos\left(\Omega t_1\right)\cos\left(\Omega t_2\right)\}$$

3. Using trigonometric identities we can evaluate this expression as:

$$R_{xx}(t_1, t_2) = \frac{1}{2}E_\Omega\{\cos\left(\Omega(t_1 + t_2)\right)\} + \frac{1}{2}E_\Omega\{\cos\left(\Omega(t_1 - t_2)\right)\}$$

4. Rewriting each of these expressions in terms of the mean we have:

$$\begin{aligned} R_{xx}(t_1, t_2) &= \frac{1}{2}\cos\left(\Omega_c(t_1 + t_2)\right)\exp\left(-\frac{\sigma^2}{2}(t_1 + t_2)^2\right) \\ &+ \frac{1}{2}\cos\left(\Omega_c(t_1 - t_2)\right)\exp\left(-\frac{\sigma^2}{2}(t_1 - t_2)^2\right). \end{aligned}$$

1. If constituent random variables of a process are Gaussian random variables then resultant process called a *Gaussian random process* (GRP).
2. n-th order joint statistics specified by n-variate Gaussian distribution:

$$f_{\mathbf{X}}(\mathbf{x};\mathbf{t}) = (2\pi)^{-n/2}|\det(\mathbf{C}_{xx})|^{-1/2}\exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m}_x)^T\mathbf{C}_{xx}^{-1}(\mathbf{x}-\mathbf{m}_x)\right)$$

3. Distribution completely specified by knowledge of $\mathbf{m}_x$ and $\mathbf{C}_{xx}$. No additional information in higher-order moments.
4. Finds application in numerous problems such as modeling Brownian motion, in modeling superposition of i.i.d. random variables.

1. When random variables comprising the stochastic process are i.i.d this process is called strong-sense white-noise.
2. In this case, the n-th order PDF of the process factors are product of marginals as:

$$f_X(x;t) = \prod_{i=1}^{n} f_{X_i}(x_i;t_i)$$

3. A weaker form of this process occurs when the constituent random variables are uncorrelated instead:

$$\sigma(X_t, X_{t-\tau}) = \sigma_x^2 \delta(\tau).$$

   Note that the average power of the process, i.e., $R_{xx}(0)$ is not finite. These processes are therefore not physically realizable.

4. A weak-sense white noise stochastic sequence is one whose pair-wise covariance matrix is diagonal:

$$\mathbf{C}_x = \mathrm{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2).$$

1. The pair-wise auto-correlation sequence corresponding to a weak-sense white-noise sequence is:

$$r_{xx}[n,k] = E\{x[n]x^*[n-k]\} = \sigma_x^2\delta[n-k].$$

2. Unlike the continuous-time white noise process, the discrete sequence has finite average power:

$$P_{\mathrm{ave}} = r_{xx}[0] = \sigma_x^2 < \infty$$

3. In both cases, the corresponding *power spectral density* (PSD) is flat:

$$P_{xx}(j\Omega) = \sigma_x^2,\ \Omega \in \mathbf{R},\ P_{xx}(e^{j\omega}) = \sigma_x^2,\ \omega \in [-\pi,\pi].$$

   Hence the name "white noise".

4. Observation of this process at two time-instants provides no additional information over observation at a single instant. No information can be gleaned from additional samples.