Laplace transform

The bilateral Laplace transform of a function f(t) is the function F(s), defined by:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{-\infty}^{+\infty} e^{-st} f(t) \, dt.$$

The parameter *s* is in general <u>complex</u>:

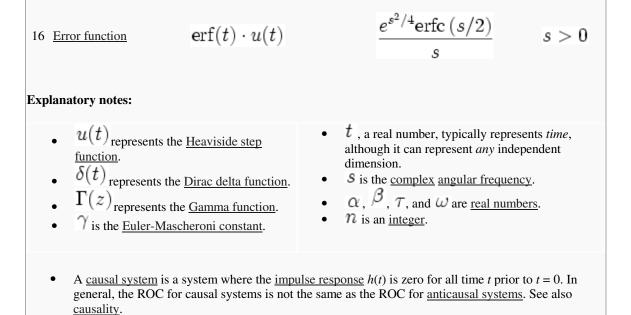
$$s = \sigma + i\omega$$
.

ID	Function	Time domain $x(t) = \mathcal{L}^{-1} \left\{ X(s) \right\}$	Frequency domain $X(s) = \mathcal{L} \left\{ x(t) \right\}$	Region of convergence for <u>causal</u> systems
1	ideal delay	$\delta(t-\tau)$	$e^{-\tau s}$	
1a	unit impulse	$\delta(t)$	1	all s
2	delayed <i>n</i> th power with frequency shift	$\frac{(t-\tau)^n}{n!}e^{-\alpha(t-\tau)}\cdot u(t-\tau)$	$\frac{e^{-\tau s}}{(s+\alpha)^{n+1}}$	s > 0
2a	<i>n</i> th power	$\frac{t^n}{n!} \cdot u(t)$	$\frac{1}{s^{n+1}}$	s > 0

Table of common Laplace transform pairs

2a.1	<i>q</i> th power	$\frac{t^q}{\Gamma(q+1)}\cdot u(t)$	$\frac{1}{s^{q+1}}$	s > 0
2a.2	<u>unit step</u>	u(t)	$\frac{1}{s}$	s > 0
2b	delayed unit step	u(t- au)	$\frac{e^{-\tau s}}{s}$	s > 0
2c	ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	s > 0
2d	nth power with frequency shift	$\frac{t^n}{n!}e^{-\alpha t}\cdot u(t)$	$\frac{1}{(s+\alpha)^{n+1}}$	$s > -\alpha$
2d.1	<u>exponential</u> <u>decay</u>	$e^{-lpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	$s > -\alpha$
3	exponential approach	$(1-e^{-\alpha t})\cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$	s > 0
4	sine	$\sin(\omega t)\cdot u(t)$	$\frac{\omega}{s^2 + \omega^2}$	s > 0
5	<u>cosine</u>	$\cos(\omega t)\cdot u(t)$	$\frac{s}{s^2 + \omega^2}$	s > 0
6	hyperbolic sine	$\sinh(\alpha t)\cdot u(t)$	$rac{lpha}{s^2-lpha^2}$	$s > \alpha $
7	hyperbolic cosine	$\cosh(\alpha t)\cdot u(t)$	$rac{s}{s^2-lpha^2}$	$s > \alpha $

8	Exponentially- decaying sine wave	$e^{-\alpha t}\sin(\omega t)\cdot u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$	$s > -\alpha$
9	Exponentially- decaying cosine wave	$e^{-\alpha t}\cos(\omega t)\cdot u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$	$s > -\alpha$
10	<i>n</i> th root	$\sqrt[n]{t} \cdot u(t)$	$s^{-(n+1)/n} \cdot \Gamma\left(1+\frac{1}{n}\right)$	s > 0
11	<u>natural</u> logarithm	$\ln\left(\frac{t}{t_0}\right)\cdot u(t)$	$-\frac{t_0}{s} \ [\ \ln(t_0 s) + \gamma \]$	s > 0
12	Bessel function of the first kind, of order <i>n</i>	$J_n(\omega t) \cdot u(t)$	$\frac{\omega^n \left(s + \sqrt{s^2 + \omega^2}\right)^{-n}}{\sqrt{s^2 + \omega^2}}$	s > 0 $(n > -1)$
13	<u>Modified</u> <u>Bessel</u> <u>function</u> of the first kind, of order <i>n</i>	$I_n(\omega t)\cdot u(t)$	$\frac{\omega^n \left(s + \sqrt{s^2 - \omega^2}\right)^{-n}}{\sqrt{s^2 - \omega^2}}$	$s > \omega $
14	Bessel function of the second kind, of order 0	$Y_{0}(\alpha t)\cdot u(t)$		
15	Modified <u>Bessel</u> <u>function</u> of the second kind, of order 0	$K_{\rm D}(\alpha t)\cdot u(t)$		



Unilateral Z-Transform

Alternatively, in cases where x[n] is defined only for $n \ge 0$, the *single-sided* or *unilateral* Z-transform is defined as

$$X(z) = Z\{x[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n}$$

In signal processing, this definition is used when the signal is causal.

Table of common Z-transform pairs

