Laplace transform

The bilateral Laplace transform of a function $f(t)$ is the function $F(s)$, defined by:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{-\infty}^{+\infty} e^{-st} f(t)\, dt.$$

The parameter s is in general complex:

$$s = \sigma + i\omega.$$

Table of common Laplace transform pairs

ID	Function	Time domain $x(t) = \mathcal{L}^{-1}\left\{X(s)\right\}$	Frequency domain $X(s) = \mathcal{L}\left\{x(t)\right\}$	Region of convergence *for causal systems*
1	ideal delay	$\delta(t-\tau)$	$e^{-\tau s}$	
1a	unit impulse	$\delta(t)$	1	$\text{all } s$
2	delayed *n*th power with frequency shift	$\frac{(t-\tau)^n}{n!} e^{-\alpha(t-\tau)} \cdot u(t-\tau)$	$\frac{e^{-\tau s}}{(s+\alpha)^{n+1}}$	$s > 0$
2a	*n*th power	$\frac{t^n}{n!} \cdot u(t)$	$\frac{1}{s^{n+1}}$	$s > 0$

2a.1	*q*th power	$\frac{t^q}{\Gamma(q+1)} \cdot u(t)$	$\frac{1}{s^{q+1}}$	$s > 0$
2a.2	unit step	$u(t)$	$\frac{1}{s}$	$s > 0$
2b	delayed unit step	$u(t-\tau)$	$\frac{e^{-\tau s}}{s}$	$s > 0$
2c	ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	$s > 0$
2d	*n*th power with frequency shift	$\frac{t^n}{n!} e^{-\alpha t} \cdot u(t)$	$\frac{1}{(s+\alpha)^{n+1}}$	$s > -\alpha$
2d.1	exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	$s > -\alpha$
3	exponential approach	$(1-e^{-\alpha t}) \cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$	$s > 0$
4	sine	$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0$
5	cosine	$\cos(\omega t) \cdot u(t)$	$\frac{s}{s^2+\omega^2}$	$s > 0$
6	hyperbolic sine	$\sinh(\alpha t) \cdot u(t)$	$\frac{\alpha}{s^2-\alpha^2}$	$s > \vert\alpha\vert$
7	hyperbolic cosine	$\cosh(\alpha t) \cdot u(t)$	$\frac{s}{s^2-\alpha^2}$	$s > \vert\alpha\vert$

8	Exponentially-decaying sine wave	$e^{-\alpha t}\sin(\omega t)\cdot u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$	$s > -\alpha$
9	Exponentially-decaying cosine wave	$e^{-\alpha t}\cos(\omega t)\cdot u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$	$s > -\alpha$
10	*n*th root	$\sqrt[n]{t}\cdot u(t)$	$s^{-(n+1)/n}\cdot\Gamma\left(1+\frac{1}{n}\right)$	$s > 0$
11	natural logarithm	$\ln\left(\frac{t}{t_0}\right)\cdot u(t)$	$-\frac{t_0}{s}\ [\ \ln(t_0 s)+\gamma\ ]$	$s > 0$
12	Bessel function of the first kind, of order *n*	$J_n(\omega t)\cdot u(t)$	$\frac{\omega^n\left(s+\sqrt{s^2+\omega^2}\right)^{-n}}{\sqrt{s^2+\omega^2}}$	$s > 0$ $(n > -1)$
13	Modified Bessel function of the first kind, of order *n*	$I_n(\omega t)\cdot u(t)$	$\frac{\omega^n\left(s+\sqrt{s^2-\omega^2}\right)^{-n}}{\sqrt{s^2-\omega^2}}$	$s > \lvert\omega\rvert$
14	Bessel function of the second kind, of order 0	$Y_0(\alpha t)\cdot u(t)$		
15	Modified Bessel function of the second kind, of order 0	$K_0(\alpha t)\cdot u(t)$		

16	Error function	$\mathrm{erf}(t) \cdot u(t)$	$\frac{e^{s^2/4}\mathrm{erfc}\,(s/2)}{s}$	$s > 0$

Explanatory notes:

- $u(t)$ represents the Heaviside step function.
- $\delta(t)$ represents the Dirac delta function.
- $\Gamma(z)$ represents the Gamma function.
- γ is the Euler-Mascheroni constant.
- t, a real number, typically represents *time*, although it can represent *any* independent dimension.
- s is the complex angular frequency.
- α, β, τ, and ω are real numbers.
- n is an integer.

- A causal system is a system where the impulse response *h*(*t*) is zero for all time *t* prior to *t* = 0. In general, the ROC for causal systems is not the same as the ROC for anticausal systems. See also causality.

Unilateral Z-Transform

Alternatively, in cases where *x*[*n*] is defined only for *n* ≥ 0, the *single-sided* or *unilateral* Z-transform is defined as

$$X(z) = Z\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

In signal processing, this definition is used when the signal is causal.

Table of common Z-transform pairs

	Signal, $x[n]$	Z-transform, $X(z)$	ROC
1	$\delta[n]$	1	all z

2	$\delta[n-n_0]$	$\frac{1}{z^{n_0}}$	all z
3	$u[n]$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
4	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
5	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
6	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
8	$\cos(\omega_0 n)u[n]$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
9	$\sin(\omega_0 n)u[n]$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
10	$a^n\cos(\omega_0 n)u[n]$	$\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
11	$a^n\sin(\omega_0 n)u[n]$	$\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$