Statistical Model for White Noise

A random signal $X(t)$ is said to be a *strictly white* random signal if the the constituent random variables of the random signal, i.e., $X(t), t \in \mathbf{R}^1$ are statistically independent, i.e.,

$$f_{\mathbf{X}}(\mathbf{x};\mathbf{t}) = \prod_{i=1}^{n} f_{X_{t_i}}(x_i),\ t_i \in \mathbf{T}.$$

A weaker, yet more practical condition is satisfied by *weakly white* random signals where the constituent random variables are statistically uncorrelated, i.e.,

$$\mathbf{C}_x = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

If the constituent random variables are further identically distributed then the covariance matrix becomes an identity matrix:

$$\mathbf{C}_x = \sigma_x^2 \mathbf{I}.$$

This classification of random signals is independent of the type of the underlying statistical distributions that characterize the random signal. The ensemble ACF of weakly white random signals takes the form:

$$R_{xx}(\tau) = E\{X(t)X^*(t-\tau)\} = \sigma_x^2 \delta(\tau). \tag{1}$$

The delta function associated with the ACF is used to model the fact that the random variables obtained by sampling the process at the time instants t and $t-\tau$ are uncorrelated. The PSD associated with the white noise random process is obtained by taking the Fourier transformation on both sides:

$$P_{xx}(\Omega) = \sigma_x^2, \quad \Omega \in \mathbf{R}^1. \tag{2}$$

This means that the PSD of white noise is flat across all frequencies, i.e., "white". The average power associated with the white-noise random process is therefore:

$$P_{\text{ave}}^x = R_{xx}(0) = \infty. \tag{3}$$

In other words, the continuous time white noise process does not have finite variance and therefore is not physically realizable. However, it serves as a convenient theoretical model for many applications. The discrete version of white noise process is the white noise sequence. As we saw in the case of Bernoulli white noise, the ensemble ACF of the process takes the form:

$$R_{xx}[i,j] = E\{X[i]X^*[j]\} = \sigma_x^2 \delta[i-j]. \tag{4}$$

As in the continuous case, the Kroneker delta models the fact the the random variables $X[i]$ and $X[j]$ are uncorrelated. The PSD of the white noise sequence is obtained by taking the DTFT on both sides:

$$P_{xx}(e^{j\omega}) = \sigma_x^2,\ \omega \in [-\pi, \pi]. \tag{5}$$

The average power of the white noise sequence is however, finite:

$$P_{\text{ave}}^x = R_{xx}[0] = \sigma_x^2 < \infty. \tag{6}$$

```
%*********************************************************************
%         This function pertains to the addition of zero mean
%         AWGN or AWUN to a deterministic input signal.
%                 AUTHOR: Santhanam, Balu
%                 SYNOPSIS:  [noisy, SNR] = awn(x,para,opt)
%                 opt: string with two options 'AWGN' and 'AWUN'
%                 AWGN ---> AWGN with parameters para = [mu,sigma]
%                 AWUN ---> AUN with parameters para = [a,b]
%*********************************************************************
function [noisy, SNR] = awn(x,para,opt) if nargin == 2
   opt = 'AWGN'; % Default option
end if nargin < 2
    error('insufficient Info')
elseif strcmp(opt,'AWGN') == 0 & strcmp(opt,'AWUN') == 0
    error('Invalid option')
elseif length(x) == 0
    error('Null Input')
elseif isnumeric(x) ~= 1
    error('Non-numeric Input')
elseif all(isfinite(x)) ~= 1
    error('Input contains Inf and NaN elements')
elseif strcmp(opt,'AWGN')
    % fprintf('Additive White Gaussian Noise (AWGN) Option\n')
    % fprintf('%s %f \t %s %f\n','Mean =' ,para(1),'std =',para(2))
    if para(2) <= 0
       error('STD has to be positive')
    end
    w = randn(1,length(x));
    mu = para(1); sigma = para(2);
elseif strcmp(opt,'AWUN')
     % fprintf('Additive White Uniform Noise (AWUN) Option \n')
     if para(1) > para(2)
        error('Invalid Interval')
     end
     w = rand(1,length(x)); mu = (para(1) + para(2))/2;
     sigma = (para(2) - para(1))/sqrt(12);
end
w = w - mean(w)*ones(size(w)); w = w / std(w);
w = mu*ones(size(w)) + sigma*w;
x = x(:); w = w(:); noisy =x+w;
SNR = 20*log10(std(x)/sigma);
%***************************************************************************
```

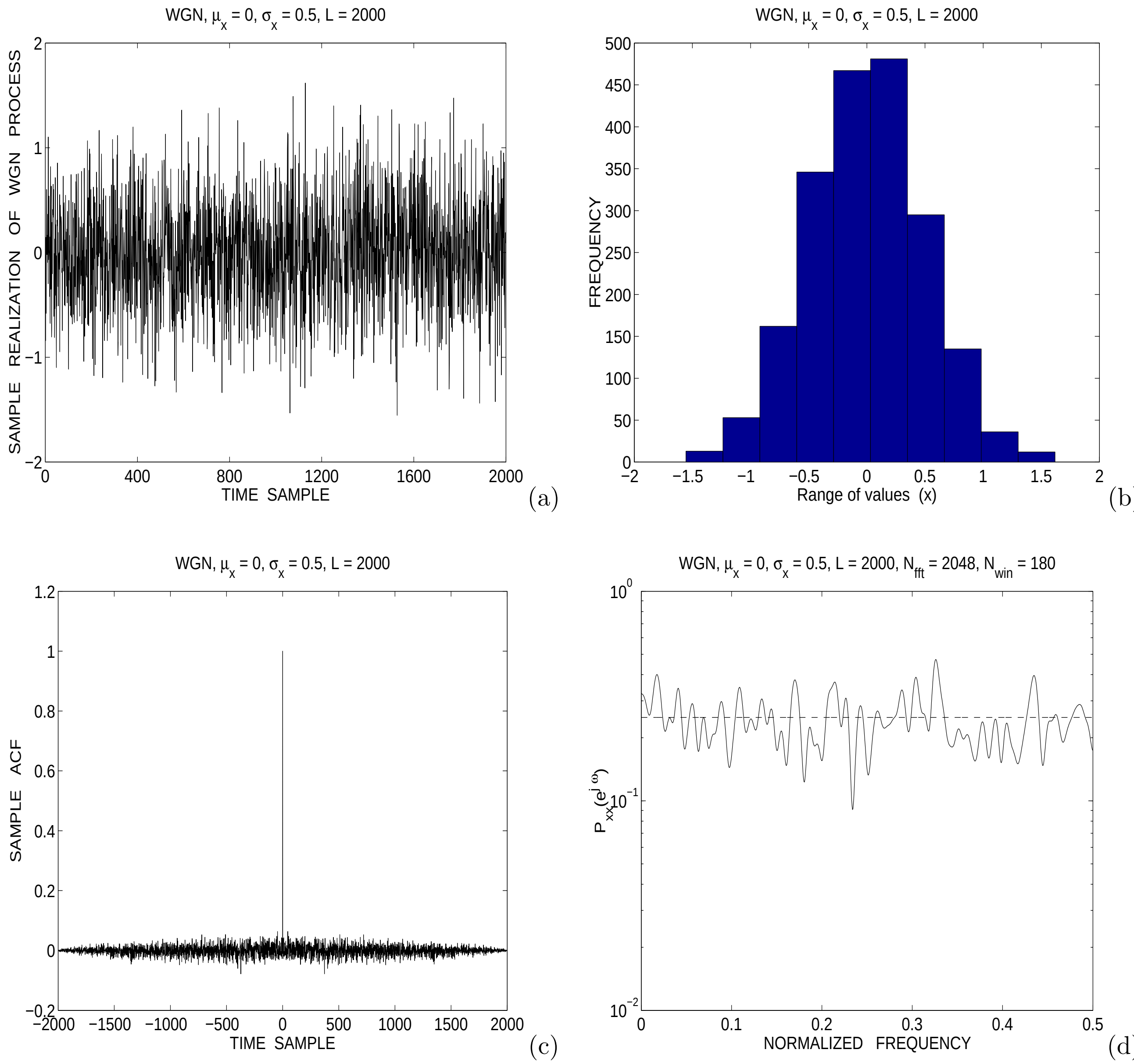

Figure 1: White noise example: (a) sample realization of white Gaussian noise, (b) histogram of WGN, (c) sample ACF of WGN, and (d) sample PSD of WGN. These figures were generated using `awn.m` using a input of `zeros(1,L)`.