Wiener Process

$$Y(t) = \begin{cases} 0, & t = 0 \\ X[nT_s], & (n-1)T_s < t < nT_s \end{cases}$$

where $x[nT_s]$ is the random walk process

$$E\{Y(t)\}, \; t = nT_s \;=\; E\{x[nT_s]\} = 0$$

$$\sigma_Y^2(t) = E\{X^2[n]\} = \frac{td^2}{T_s} = nd^2$$

Definition

$$W(t) = \lim_{\substack{d \to 0 \\ T_s \to 0}} Y(t) \quad \text{with} \quad \frac{d^2}{T} = \alpha$$

Properties:

- $E\{W(t)\} = 0$
- $E\{W^2(t)\} = \alpha t < \infty$ for $t < \infty$
- $W(t)$ is a continuous-time, continuous-amplitude independent increments process
- $$f_{W(t)}(w;t) = \frac{1}{\sqrt{2\pi\alpha t}} \exp\left(\frac{-w^2}{2\alpha t}\right), \quad w \in \mathbb{R}^1$$

- For $t_1 < t_2 < t_3, \ldots\ t_k \in T$

$$Z(t_i) = x(t_i) - x(t_{i-1}) \sim N\left(0, \alpha(t_i - t_{i-1})\right)$$

- $$R_{ww}(t_1, t_2) = \alpha \min(t_1, t_2)$$

Alternative Definition

$$W(t) = \int_0^t v(\beta)\, d\beta \quad , \quad \text{where}$$

$v(t)$ is a WGN process with zero-mean and variance α, i.e,

$$R_{vv}(\tau) = \alpha\, \delta(\tau)$$

It can be easily seen that

$$f_{w(t)}(\omega; t) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\omega^2}{2\alpha t}\right)$$

$$f_{w(t)}\left(\omega / w(t_0) = \omega_0\right) \sim N\left(\omega_0, \alpha(t - t_0)\right)$$

Wiener Process

$$W(t) = \int_0^t v(\beta)\, d\beta$$

$$E\left\{W(t)\right\} = E\left\{\int_0^t v(\beta)\, d\beta\right\}$$

$$= \int_0^t E\big(v(\beta)\big)\, d\beta = 0$$

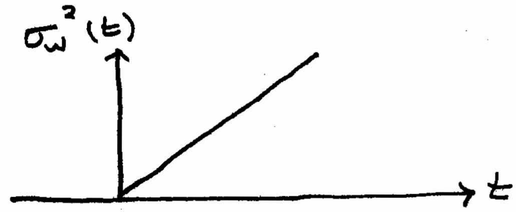
Variance :

$$\sigma_w^2(t) = E\big(W^2(t)\big) = E\left(\int_0^t W(\gamma)\,d\gamma \int_0^t W(\beta)\,d\beta\right)$$

$$\sigma_w^2(t) = \int_0^t \int_0^t E\left\{W(\gamma)W(\beta)\right\} d\gamma\, d\beta$$

$$\sigma_w^2(t) = \int_0^t \int_0^t \alpha\, \delta(\gamma - \beta)\, d\gamma\, d\beta$$

$$= \alpha \int_0^t d\beta = \alpha t$$



Correlation:

$t_1 < t_2$

$$R_{ww}(t_1, t_2) = E\{W(t_1)[W(t_2) - W(t_1) + W(t_1)]\}$$

$$= E\{[W(t_1) - W(0)][W(t_2) - W(t_1)]\}$$

$$+ E\{W^2(t_1)\}$$

$$R_{ww}(t_1, t_2) = \alpha t_1 + \overbrace{E\{[W(t_1) - W(0)][W(t_2) - W(t_1)]\}}^{0}$$

Independent increments

$$R_{ww}(t_1, t_2) = \begin{cases} \alpha t_1 , & t_1 < t_2 \\ \alpha t_2 , & t_2 > t_1 \end{cases}$$

$$R_{ww}(t_1, t_2) = \alpha \min(t_1, t_2)$$

$$S_{ww}(t_1, t_2) = \frac{\alpha \min(t_1, t_2)}{\alpha \sqrt{t_1 t_2}}$$

$$= \frac{\min(t_1, t_2)}{\sqrt{t_1 t_2}}$$