DKF: Assumptions

- Known state model for desired process:

$$\begin{aligned} \mathrm{x}[n+1] &= \mathrm{A}[n]\mathrm{x}[n] + \mathrm{f}[n] \\ y[n] &= \mathrm{C}[n]\mathrm{x}[n] + v[n]. \end{aligned}$$

- Underlying state-model observable:

$$\det(\mathrm{O}(\mathrm{C}, \mathrm{A})) \neq 0.$$

- f[n] & v[n] are zero mean, uncorrelated Gaussian noise sources.
- Associated covariance matrices: R_{vv}, R_{ff}

Discrete Kalman Filter (DKF)

- Linear state estimate:

$$\hat{\mathrm{x}}_+[n] = \mathrm{K}_1[n]\hat{\mathrm{x}}_-[n] + \mathrm{K}_2[n]y[n].$$

- Prior/posteriori state estimation errors:

$$\begin{aligned} \mathrm{e}_+[k] &= \hat{\mathrm{x}}_+[k] - \mathrm{x}[k] \\ \mathrm{e}_-[k] &= \hat{\mathrm{x}}_-[k] - \mathrm{x}[k] \end{aligned}$$

- Unbiased prior/posteriori state estimates provided:

$$\mathrm{K}_1[n] = \mathbf{I} - \mathrm{K}_2[n]\mathrm{C}[n]$$

DKF: State Estimation

- DKF state estimate:

$$\hat{\mathrm{x}}_+[n] = (\mathbf{I} - \mathrm{K}_2[n]\mathrm{C}[n])\hat{\mathrm{x}}_-[n] + \mathrm{K}_2[n]y[n]$$

- State estimate: alternative form

$$\hat{\mathrm{x}}_+[n] = \hat{\mathrm{x}}_-[n] + \mathrm{K}_2[n](y[n] - \mathrm{C}[n]\hat{\mathrm{x}}_-[n])$$

- Conversion factor:

$$\mathrm{e}_+[n] = (\mathbf{I} - \mathrm{K}_2[n]\mathrm{C}[n])\mathrm{e}_-[n] + \mathrm{K}_2[n]v[n]$$

- Error covariance update:

$$\mathrm{P}_+[n] = (\mathbf{I} - \mathrm{K}_2[n]\mathrm{C}[n])^T\mathrm{P}_-[n](\mathbf{I} - \mathrm{K}_2[n]\mathrm{C}[n]) + \mathrm{K}_2^T[n]\mathrm{R}_{vv}\mathrm{K}_2[n]$$

DKF: State Estimation

- Cost function: Trace of error covariance

$$J(\mathrm{K}_2[n], n) = E\{\mathrm{e}_+^T[n]\mathrm{e}_+[n]\} = \mathrm{Trace}(E\{\mathrm{e}_+[n]\mathrm{e}_+^T[n]\})$$

- Optimal solution:

$$\mathrm{K}_2^{\mathrm{opt}}[n] = \mathrm{P}_-[n]\mathrm{C}^T[n](\mathrm{R}_{vv} + \mathrm{C}[n]\mathrm{P}_-[n]\mathrm{C}^T[n])^{-1}$$

- Optimized error covariance:

$$\mathrm{P}_+[n] = (\mathbf{I} - \mathrm{K}_2[n]\mathrm{C}[n])\mathrm{P}_-[n]$$

DKF: State Extrapolation

- Alternative expression for optimal Kalman gain:

$$\mathbf{K}_2^{\text{opt}}[n] = \mathbf{P}_+[n]\mathbf{C}^T[n]\mathbf{R}_{vv}^{-1}$$

- Optimal state extrapolation/prediction:

$$\hat{\mathbf{x}}_-[n] = \mathbf{A}[n-1]\mathbf{x}_+[n-1]$$

- Error covariance after prediction:

$$\mathbf{P}_-[n] = \mathbf{A}[n-1]\mathbf{P}_+[n-1]\mathbf{A}^T[n-1] + \mathbf{R}_{ff}$$

DKF: Innovations

- Alternative form of state estimate:

$$\hat{\mathbf{x}}_+[n] = \hat{\mathbf{x}}_-[n] + \mathbf{K}_2[n]\alpha[n].$$

- Innovations process or measurement residual:

$$\alpha[n] = y[n] - \mathbf{C}[n]\hat{\mathbf{x}}_-[n]$$

- For smoothing problem: convex combination

$$\hat{\mathbf{x}}_+[n] = (\mathbf{I} - \mathbf{K}_2[n])\hat{\mathbf{x}}_+[n-1] + \mathbf{K}_2[n]y[n]$$

- Innovations : an orthonormal basis for regular or unpredictable part of y[n].

DKF: State Estimation

- Prior/Posteriori output errors:

$$e[n] = y[n] - \mathbf{C}[n]\hat{\mathbf{x}}_-[n] - \mathbf{C}[n]\mathbf{K}_2[n]\alpha[n]$$

- Conversion factor:

$$e[n] = (1 - \mathbf{C}[n]\mathbf{K}_2[n])\alpha[n].$$

- Conversion factor:

$$\gamma[n] = 1 - \frac{\mathbf{C}[n]\mathbf{P}_-[n]\mathbf{C}^T[n]}{\sigma_v^2 + \mathbf{C}[n]\mathbf{P}_-[n]\mathbf{C}^T[n]}$$

DKF: State Estimation

- Orthogonality principle:

$$E\{\hat{\mathbf{x}}_+[n]\mathbf{e}_+^T[n]\} = 0$$

- Choice of weight matrices guarantees no bias.
- Convergence in MS sense: $I - K_2[n]C[n]$ is a stable matrix.
- Optimal Kalman gain matrix $K_2[n]$ weights prior observations and current data automatically.