RLS Algorithm: Motivation

- Least-squares cost-function & solution non-iterative and not amenable to adaptation.
- Desire cost-function & solution to respond to change in signal environment.
- Want to avoid the least-squares prescription of matrix inversion.

Recursive Least Squares (RLS)

- Cost-function:

$$J(\mathbf{w}[n], n) = \sum_{i=0}^{n} \lambda^{n-i}|e[i]|^2 = \sum_{i=0}^{n} \lambda^{n-i}(d[i] - \mathbf{w}^T[i]\mathbf{u}[i])^2$$

- Time-varying ACF and cross-correlation:

$$\{\tilde{\mathbf{R}}_{uu}\}_{pq} = \sum_{i=0}^{n} \lambda^{n-i}u[i-p]u[i-q], \ \{\tilde{\mathbf{r}}_{du}\}_q = \sum_{i=0}^{n} \lambda^{n-i}d[i]u[i-q].$$

- Deterministic Normal Equations:

$$\tilde{\mathbf{R}}_{uu}[n]\mathbf{w}[n] = \tilde{\mathbf{r}}_{du}[n]$$

RLS Algorithm

- Rank-one update:

$$\begin{aligned}\tilde{\mathbf{R}}_{uu}[n] &= \lambda\tilde{R}_{uu}[n-1] + \mathbf{u}[n]\mathbf{u}^T[n] \\ \tilde{\mathbf{r}}_{du}[n] &= \lambda\tilde{r}_{du}[n-1] + d[n]\mathbf{u}[n]\end{aligned}$$

- RLS gain vector:

$$\mathbf{g}[n] = \frac{\mathbf{P}[n-1]\mathbf{u}[n]}{\lambda + \mathbf{u}^T[n]\mathbf{P}[n-1]\mathbf{u}[n]}$$

- Inverse update using the matrix inversion lemma:

$$\mathbf{P}[n] = \frac{1}{\lambda}\mathbf{P}[n-1] - \frac{1}{\lambda}\mathbf{g}[n]\mathbf{u}^T[n]\mathbf{P}[n-1]$$

RLS Algorithm

- RLS gain vector: solution to linear system:

$$\mathbf{g}[n] = \mathbf{P}[n]\mathbf{u}[n]$$

- Innovations process:

$$\alpha[n] = d[n] - \mathbf{w}^T[n-1]\mathbf{u}[n]$$

- Tap-weight update:

$$\mathbf{w}[n] = \mathbf{w}[n-1] + \alpha[n]\mathbf{g}[n]$$

- Whitening form of update:

$$\mathbf{w}[n] = \mathbf{w}[n-1] + \mathbf{P}[n]\mathbf{u}[n]\alpha[n].$$

RLS Algorithm

- Recursion for MMSE:

$$J_{\mathsf{min}}[n] = \lambda J_{\mathsf{min}}[n-1] + \alpha[n]e[n]$$

- Conversion factor:

$$e[n] = (1 - \mathrm{g}^T[n]\mathbf{u}[n])\alpha[n]$$

- Tap-weight update weights and smoothes innovations:

$$\mathrm{w}[n] = \mathrm{w}[0] + \sum_{i=0}^{n} \mathrm{g}[i]\alpha[i]$$

RLS Algorithm

- Forget factor λ weights prior information relative to current information.
- Choice of λ determines speed of adaptation.
- Initialization of the RLS algorithm:

$$\mathrm{w}[0] = \mathbf{0}, \quad \mathbf{P}[n] = \delta\mathbf{I}, \quad \delta > 0.$$

- Whitening approach accounts for an order of magnitude improvement in convergence.