Gradient Adaptive Lattice (GAL)

- FIR lattice propagation equations:

$$f_i[n] = f_{i-1}[n] + \kappa_i g_{i-1}[n-1]$$
$$g_i[n] = \kappa_i f_{i-1}[n] + g_{i-1}[n-1]$$

- Modularity of the lattice structure:

$$\begin{pmatrix} A_i(z) \\ B_i(z) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \kappa_i z^{-1} \\ \kappa_i & z^{-1} \end{pmatrix}}_{\mathrm{M}_i(z)} \begin{pmatrix} A_{i-1}(z) \\ B_{i-1}(z) \end{pmatrix}$$

- Lossless property of lattice filters:

$$|\det(\mathrm{M}_i(e^{j\omega}))| = (1-\kappa_i^2)$$

GAL Filters

- Cost function:

$$J(f_m[n], g_m[n], m) = \frac{1}{2}\left(E\{f_m^2[n]\} + E\{g_m^2[n]\}\right)$$

- Optimal reflection coefficient:

$$\kappa_m^{\text{opt}} = \frac{-2E\{f_{m-1}[n]g_{m-1}[n-1]\}}{E\{f_{m-1}^2[n] + g_{m-1}^2[n-1]\}}$$

- Time-averaged estimate of coefficient:

$$\hat{\kappa}_m[n] = -2\sum_{i=1}^{n} g_{m-1}[i-1]f_{m-1}[i] / \sum_{i=1}^{n}(f_{m-1}^2[i] + g_{m-1}^2[i-1])$$

GAL Filters

- Total energy at (m-1)-th stage:

$$E_{m-1}[n] = \sum_{i=1}^{n}(f_{m-1}^2[n] + g_{m-1}^2[n-1])$$

- Recursive computation of total energy:

$$E_{m-1}[n] = E_{m-1}[n-1] + f_{m-1}^2[n] + g_{m-1}^2[n-1]$$

- Reflection coefficient revisited:

$$\hat{\kappa}_m[n] = -\frac{2\sum_{i=1}^{n-1} f_{m-1}[i]g_{m-1}[i-1] + 2f_{m-1}[n]g_{m-1}[n-1]}{E_{m-1}[n]}$$

GAL Filters

- Recursion for the reflection coefficient:

$$\hat{\kappa}_m[n] = \frac{E_{m-1}[n-1]}{E_{m-1}[n]}\hat{\kappa}_m[n-1] - 2\frac{f_{m-1}[n]g_{m-1}[n-1]}{E_{m-1}[n]}$$

- Incorporating step-size:

$$\hat{\kappa}_m[n] = \frac{E_{m-1}[n-1]}{E_{m-1}[n]}\hat{\kappa}_m[n-1] - \frac{\mu}{E_{m-1}[n]} f_{m-1}[n]g_{m-1}[n-1]$$

- Time-varying environments:

$$E_{m-1}[n] = \beta E_{m-1}[n-1] + (1-\beta)(f_{m-1}[n] + g_{m-1}^2[n-1])$$

GAL Filters

- Under stationary conditions:

$$\hat{\kappa}_m[n] = \hat{\kappa}_m[n-1] - \tilde{\mu}_m[n] f_m[n] g_{m-1}[n-1]$$

- Regression using reverse outputs:

$$y_m[n] = \sum_{k=0}^{m} h_k[n] g_k[n] = y_{m-1}[n] + h_m[n] g_m[n]$$

- Estimation error at m-th stage:

$$e_m[n] = d[n] - y_m[n] = d[n] - \sum_{k=0}^{m} h_k[n] g_k[n]$$

GAL Filters

- NLMS update for regression coefficients:

$$\mathrm{h}[n+1] = \mathrm{h}[n] + \left(\frac{\beta}{||\mathrm{g}_m[n]||^2 + \delta}\right) \mathrm{g}_m[n] e_m[n]$$

- Recursion for norm:

$$||\mathrm{g}_m[n]||^2 = ||\mathrm{g}_{m-1}[n]||^2 + g_m^2[n]$$

- Initialization:

$$\beta \in [0,1],\ f_0[n] = g_0[n] = u[n], \mu < 0.1,\ f_m[0] = g_m[0] = 0,$$
$$E_{m-1}[0] = a > 0,\ \kappa_m[0] = 0, h_m[0] = 0.$$

GAL Filters

- Normalized step-size tracks variations in environment:

$$\mu_m[n] = \frac{\mu}{E_{m-1}[n]}$$

- Prediction errors are the cue for adaptation.
- Prediction errors small : $E_{m-1}[n]$ small and $\mu_m[n]$ large in magnitude, i.e, fast adaptation mode.
- Noisy environments: $E_{m-1}[n]$ large and $\mu_m[n]$ smaller in magnitude, i.e, noise rejection mode.

GAL Filters

- Superior to the LMS : lower noise sensitivity of $\kappa_m[n]$ and better tracking capabilities via $\mu_m[n]$.
- Computationally simple and attractive for practical implementation.
- Convergence of GAL inferior to RLS-based lattice structures.