LMS Algorithm: Motivation

- Only a single realization of observations available.
- Statistics need to be estimated.
- Need to estimate the gradient vector
- Elaborate estimation : delay in tap-weight adjustment.
- Simplicity: real-time applications possible.

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- Use instantaneous estimates for statistics:

$$\hat{\mathbf{R}}_{uu}[n] = \mathbf{u}^*[n]\mathbf{u}^T[n], \quad \hat{\mathbf{r}}_{du}[n] = d[n]\mathbf{u}^*[n].$$

- Filter output:

$$y[n] = \mathbf{w}^T[n]\mathbf{u}[n].$$

- Estimation error:

$$e[n] = d[n] - \mathbf{w}^T[n]\mathbf{u}[n]$$

- Tap-weight update:

$$\mathbf{w}[n+1] = \mathbf{w}[n] + \mu e[n]\mathbf{u}^*[n]$$

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- Estimate of gradient used:

$$\hat{\mathbf{g}}[n] = e[n]\mathbf{u}^*[n].$$

- Estimate of gradient unbiased:

$$E\{\hat{\mathbf{g}}[n]\} = E\{e[n]\mathbf{u}^*[n]\} = \mathbf{g}[n].$$

- Gradient estimate contains gradient noise:

$$\hat{\mathbf{g}}[n] = \mathbf{g}[n] + \mathbf{v}_g[n].$$

- Tap-weight converges in the mean :

$$\lim_{n\to\infty} E\{\mathbf{w}[n]\} = \mathbf{w}_{\text{opt}}$$

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- Given w[0] = 0 for the LMS filter:

$$\mathbf{w}[n] = \mu \sum_{k=0}^{n-1} e[k]\mathbf{u}^*[k].$$

- Input-output relation nonlinear:

$$y[n] = \mathbf{w}^T[n]\mathbf{u}[n] = \mu \sum_{k=0}^{n-1} \mathbf{u}^T[n]e[k]\mathbf{u}^*[k].$$

- Feedback in recursive tap-weight update makes stability considerations important.

LMS : Independence Theory

- Tap inputs u[n] comprised of statistically independent random vectors.
- Tap inputs u[n] independent of previous samples of the desired process d[n].
- Tap inputs u[n] and desired process d[n] jointly Gaussian distributed.
- Assumptions violated in certain applications but sufficient for obtaining general design guidelines.

LMS: Independence Theory

- Tap-weight vector w[n] depends only on:

  (i) Prior tap-input samples

  (ii) Prior desired process samples

  (iii) Initial tap-weight: w[0].
- w[n+1] is independent of d[n+1], u[n+1]

Average Tap-weight Behavior

- LMS tap-weight update:

$$\mathbf{w}[n+1] = \mathbf{w}[n] + \mu e[n]\mathbf{u}^*[n].$$

- Tap-weight error update:

$$\epsilon[n+1] = (\mathbf{I} - \mathbf{u}^*[n]\mathbf{u}^T[n])\epsilon[n] + \mu e_o[n]\mathbf{u}^*[n].$$

- Average tap-weight trajectory:

$$E\{\epsilon[n+1]\} = (\mathbf{I} - \mu\mathbf{R}_{uu})E\{\epsilon[n]\}$$

Convergence in the Mean:

- LMS algorithm converges in the mean provided:

$$0 < \mu < \frac{2}{\lambda_{\max}}$$

- Alternative more useful bound:

$$0 < \mu < \frac{2}{LE\{|u[n]|^2\}}.$$

- Estimate average power via:

$$E\{|u[n]|^2\} \approx \frac{1}{N}\sum_{k=0}^{N-1} |u[k]|^2.$$

$$P_{\text{ave}}[n] = \rho P_{\text{ave}}[n-1] + (1-\rho)|u[n]|^2.$$

Convergence in the MS sense

- Unlike the SDA algorithm, the LMS algorithm does not converge to the Wiener solution in the MS sense:

$$\lim_{n\to\infty} J(\mathbf{w}[n]) = \epsilon^2_{\text{min}} + J_{\text{excess}}.$$

- The excess mean-squared error is the price that the designer pays for the simplicity of the LMS algorithm.

- Tap-weight error with respect to zero-order solution:

$$\epsilon[n+1] = (\mathbf{I} - \mu\mathbf{R}_{uu})\epsilon[n] - \mu e_o[n]\mathbf{u}^*[n]$$

Convergence in the MS sense.

- Tap-weight error in canonical coordinates:

$$\mathbf{q}[n+1] = (\mathbf{I} - \mu\mathbf{\Lambda})\mathbf{q}[n] + \mathbf{V}^H e[n]\mathbf{u}^*[n].$$

- Stochastic force vector has zero mean:

$$E\{\mathbf{\Phi}[n]\} = E\{\mathbf{V}^H e[n]\mathbf{u}^*[n]\} = 0$$

- Covariance of stochastic force vector is diagonal:

$$E\{\mathbf{\Phi}[n]\mathbf{\Phi}^H[n]\} = \mu^2\epsilon^2_{\text{min}}\mathbf{\Lambda}.$$

- ε[n] executes Brownian motion about optimal solution.

Misadjustment of LMS

- Misadjustment is normalized excess MSE:

$$M = \frac{J_{\text{excess}}}{J_{\text{min}}} = \frac{J(\infty)}{J_{\text{min}}}.$$

- Misadjustment proportional to μ for small μ:

$$M \approx \frac{1}{2}\mu\,\text{Trace}(\mathbf{R}_{uu}).$$

- Misadjustment inversely proportional to τ_k