Least Squares : Motivation

- SDA/LMS assume a probabilistic model underlying the optimal filtering problem.
- SDA/LMS assume access to ensemble statistics and multiple realizations.
- SDA/LMS assume ergodicity in the absence of multiple realizations.
- SDA/LMS speed of convergence tied up with eigenvalue spread of R_{uu}.

Least-squares Algorithm

- Deterministic cost-function:

$$J(\mathbf{w}, M, N) = \sum_{i=M}^{N} |d[i] - \mathbf{w}^T[i]\mathbf{u}[i]|^2$$

- Deterministic orthogonality principle:

$$\sum_{i=M}^{N} e_o[i]\mathbf{u}^*[i] = 0$$

- Deterministic normal equations:

$$\begin{aligned}\tilde{\mathbf{R}}_{uu}\mathbf{w}_{\text{opt}} &= \tilde{\mathbf{r}}_{du}\\ \{\tilde{\mathbf{R}}_{uu}\}_{pq} &= \sum_{i=M}^{N} u[i-p]u^*[i-q],\ \{\tilde{\mathbf{r}}_{du}\}_q = \sum_{i=M}^{N} d[i]u^*[i-q].\end{aligned}$$

Least Squares Algorithm

- Optimal solution requires matrix inversion:

$$\mathbf{w}_{\text{opt}} = \tilde{\mathbf{R}}_{uu}^{-1}\tilde{\mathbf{r}}_{du}$$

- Data matrix and desired signal vector:

$$\begin{aligned}\mathbf{A}^T &= [\mathbf{u}[M], \mathbf{u}[M+1], \ldots \mathbf{u}[N]]\\ \mathbf{d}^T &= [d[0], d[1], \ldots, d[L-1]].\end{aligned}$$

- Optimal solution in terms of data matrix:

$$\mathbf{w}_{\text{opt}} = \mathbf{A}_l^{\dagger}\tilde{\mathbf{r}}_{du} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{d}$$

Least--squares Algorithm

- Least-squares recipe: SVD solution to $\mathbf{A}\mathbf{w} = \mathbf{d}$.
- Achievable MMSE:

$$\epsilon^2_{\text{min}} = ||\mathbf{d}||^2 - \mathbf{w}_{\text{opt}}^T\tilde{\mathbf{r}}_{du} = ||\mathbf{d}||^2 - \mathbf{w}_{\text{opt}}^T\tilde{\mathbf{R}}_{uu}\mathbf{w}_{\text{opt}}.$$

- Alternative form of MMSE:

$$\epsilon^2_{\text{min}} = \mathbf{d}^T\mathbf{d} - \mathbf{d}^T\mathbf{A}\mathbf{A}_l^{\dagger}\mathbf{d} = \mathbf{d}^T(\mathbf{I} - \mathbf{A}\mathbf{A}_l^{\dagger})\mathbf{d}.$$

- Regularized least-squares solution:

$$\mathbf{w}_r = (\mathbf{A}^T\mathbf{A} + \delta\mathbf{I})^{-1}\mathbf{A}^T\mathbf{d},\quad \delta > 0.$$

Properties: Least-squares

- Linear regression model:

  $$\mathrm{d} = \mathrm{A}\mathrm{w}_o + \epsilon_o.$$

- If measurement error is zero-mean, white w_{ls} is unbiased: $E\{\mathrm{w}_{ls}\} = \mathrm{w}_o$

- If measurement error is zero-mean, white covariance of w_{ls} is given by: $\mathrm{C}_{\mathrm{w}_{\mathrm{ls}}} = \sigma^2 \tilde{\mathrm{R}}_{uu}^{-1}$

- If error is further Gaussian, w_{ls} is MVUE.