Hilbert Spaces and Random Vectors

Consder the random vectors of dimension n defined on $\mathbf{R}^n$ with an underlying sample space $\mathbf{S}$. We would like to characterize the Hilbert space of these random vectors with finite average power.

The inner product between two zero mean random vectors $\mathbf{X}$ and $\mathbf{Y}$ that belong to this space is given by:

$$< \mathbf{X}, \mathbf{Y} > = \; E\{X^T Y\},$$

where the notation $\mathbf{X}^T$ stands for the transpose of the vector $\mathbf{X}$. The corresponding average power of the random vector $\mathbf{X}$

$$< \mathbf{X}, \mathbf{X} > = \; \mathrm{P_{ave}} \; = \; E\{\mathbf{X}^T \mathbf{X}\} \; = \; E\{||\mathbf{X}||^2\} < \infty.$$

The angle between two random vectors $\mathbf{X}$ and $\mathbf{Y}$ is then given by:

$$\Theta_{XY} = \cos^{-1}\left(\frac{|E\{X^T Y\}|}{\sqrt{E\{||\mathbf{X}||^2\}E\{||\mathbf{Y}||^2\}}}\right).$$

The triangle inequality in this Hilbert space is given by:

$$E\{(\mathbf{X}+\mathbf{Y})^2\} \leq E\{||\mathbf{X}||^2\} + E\{||\mathbf{X}||^2\}.$$

The Cauchy Schwarz inequality on this Hilbert space of random vectors with finite average power is given by:

$$E^2\{X^T Y\} \leq E\{||\mathbf{X}||^2\}E\{||\mathbf{Y}||^2\},$$

where the equality holds when the vectors are linearly dependent.

Two random vectors $\mathbf{X}$ and $\mathbf{Y}$ are said to be statistically orthogonal if:

$$\Theta_{XY} = 90^\circ \Longleftrightarrow E\{X^T Y\} = 0.$$

In a similar fashion two random vectors $\mathbf{X}$ and $\mathbf{Y}$ are said to be statistically colinear if:

$$\Theta_{XY} = 0 \Longleftrightarrow E^2\{X^T Y\} = E\{||\mathbf{X}||^2\}E\{||\mathbf{Y}||^2\}.$$

When the random vectors are scalars. i.e., 1D random variables these results reduce back to the ones we saw with just two random variables.

Any random vector in the sample space has a *Karhunen Loeve* (KLT) expansion of the form:

$$\mathbf{X} = \sum_{i=1}^{n} \lambda_i \vec{v}_i,$$

where the λ_i are the expansion coefficients and $\vec{v}_i$ are the orthonormal eigenvectors of the covariance matrix $\mathbf{C}_X$ of the random vector $\mathbf{X}$