PSD Factorization

- PSD rational and analytic in the ring : $\rho \leq |z| \leq 1/\rho$.
- PSD is real and positive on the UC
- Then a spectral factorization is valid:

$$P_{xx}(z) = \sigma_v^2 H_{\mathsf{min}}(z) H_{\mathsf{max}}(z).$$

- For white-noise based processes:

$$P_{xx}(z) = \sigma_v^2 H(z) H^* \left(\frac{1}{z^*}\right).$$

Implications of PSD Factorization

- Random signal linearly equivalent to innovations:

$$x[n] = \sum_{k=0}^{\infty} h_{\mathsf{min}}[k] v[n-k].$$

- Orthonormal signal expansion:

$$E\{v[n]v^*[n-k]\} = 0, \quad k \neq 0.$$

- Poles and zeroes come in conjugate-reciprocal pairs:

$$P_{xx}(z) = \sigma_v^2 \frac{B(z)}{A(z)} \frac{B^*\left(\frac{1}{z^*}\right)}{A^*\left(\frac{1}{z^*}\right)}.$$

Innovations Process

- Wold's decomposition for a WSS process

$$x[n] = x_p[n] + x_r[n], \quad x_p[n] \perp x_r[n].$$

- Linear predictor:

$$x_s[n] = \sum_{k=1}^{\infty} a[k] x[n-k].$$

- Prediction error linearly related to signal:

$$e[n] = x[n] - \sum_{k=1}^{\infty} a[k] x[n-k] = \sum_{k=0}^{\infty} d[k] x[n-k].$$

Innovations Process

- Regular part has a rational PSD with factorization:

$$P_r(z) = \sigma_v^2 H_r(z) H_r^* \left(\frac{1}{z^*}\right)$$

- Regular part and innovations linearly equivalent:

$$x_r[n] = \sum_{k=0}^{\infty} h_r[k] v[n-k].$$

- Innovations form basis for regular part:

$$S_r = \{v[n-k], \quad k = 0, 1, \ldots, \infty\}$$

Innovations Process

- **Prediction error for optimal predictor is white:**

$$E\{e[n]e^*[n-k]\} = 0, k = 1, 2, \ldots, \infty$$

- **Scale prediction error to get innovations:**

$$v[n] = \frac{\sigma_v}{\sigma_e} e[n].$$

- **Generate regular component from innovations:**

$$x_r[n] = \sum_{k=0}^{\infty} h_r[k] v[n-k].$$

- **Predictable part is orthogonal complement:**

$$x_p[n] = x[n] - x_r[n].$$