Problem Set # 2.0
ECE-595, Fall 2006
Spatial Array Processing
University of New Mexico, Albuquerque
Date Assigned : 09/06/2006
Date Due : 09/13/2006

Background

Maxwell's equations for the propagation of a electromagnetic wave in free space can be condensed into the wave equation for the electric field given by:

$$\nabla^2(\vec{E}) = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2},$$

where $c = \frac{1}{\sqrt{\epsilon_o \mu_o}}$ is the speed of light in free space, ϵ_o is the dielectric permittivity of free space and μ_o is the magnetic permeability of free space. However, this is a vector-valued equation and we are more interested in the scalar counterpart of the transverse electromagnetic wave. The solution for the scalar field takes the form of a plane-wave:

$$E(x, y, z, t) = K \exp\left(jw_c t - j\vec{k}_o.\vec{r}\right).$$

This corresponds to the monochromatic plane wave with temporal frequency ω_c and associated wave-number vector $\vec{k}$. Let us now assume that such a plane wave is incident on a linear array of L sensors located on the x-axis spaced apart by a distance d at an angle of incidence $\phi_o = 25°$ measured with respect to the normal at the point of incidence. Each sensor then samples the signal at the Nyquist rate to yield N snapshots in time given by the baseband model:

$$y_m[n] = A[n]\exp(-j\vec{k}_o.\vec{x}_m) + w_m[n], \quad m = 1, 2, \ldots, L, \quad n = 1, 2, \ldots, N \quad ,$$

where $A[n]$ is assumed to be a complex white Gaussian process, i.e.,

$$A[n] = A_r[n] + jA_i[n], \quad A_r[n], A_i[n] \sim N\left(0, \frac{\sigma^2}{2}\right)$$

and $w_m[n]$ is a zero-mean complex Gaussian measurement white noise source associated with the m-th sensor assumed to be uncorrelated with the actual measurement. Furthermore it is assumed that the different branch noise sources are uncorrelated, i.e., spatially white noise. These branch outputs are stacked to form the array output vector:

$$\mathbf{y}[n] = [y_1[n], y_2[n], \ldots, y_L[n]]^T.$$

The spatial DFT of the array output at temporal snapshot n is defined via:

$$Y(\vec{k}, n) = \sum_{m=1}^{L} y_m[n] \exp(j\vec{k}.\vec{x}_m) = [e^{j\vec{k}.\vec{x}_1} e^{j\vec{k}.\vec{x}_2} \dots e^{j\vec{k}.\vec{x}_L}]\mathbf{y} = \mathbf{a}^H \mathbf{y}.$$

One measure of the average power of the array output in the $\vec{k}$ or wave-number domain is obtained by averaging the magnitude of the spatial DFT over N snap-shots and given by:

$$P_{\text{ave}}(\vec{k}) = \frac{1}{N} \sum_{n=1}^{N} |Y(\vec{k}, n)|^2 = \mathbf{a}^H \frac{1}{N} \left(\sum_{n=1}^{N} \mathbf{y}\mathbf{y}^H \right) \mathbf{a} = \mathbf{a}^H \mathbf{R}_{yy} \mathbf{a}.$$

The locations of the peaks of this average power measure for the array will yield estimates of the angle of arrival ϕ_o that is related to the wave-number vector $k_{ox} = \frac{2\pi}{\lambda} \sin \phi_o$. For this reason, the vector $\mathbf{a}$ defined above is sometimes referred to as the *steering vector* since it can be used to steer the array to the direction of the incoming signal. This algorithm for determining the direction of arrival (DOA) is analogous to the process of estimating the frequency of a sinusoidal signal from the peaks of it spectrum, the difference being that we are now looking at a spatial DFT rather than a temporal DFT.

Program

1. Write a MATLAB script `doa.m` that asks the user to enter d, the sensor spacing in units of half wavelength, N the number of temporal snapshots, L, the number of sensors, σ_a the amplitude of the signal and σ_w the strength of the noise. For different steering angles ϕ, compute the average power measure P_ϕ and plot the power measure as function of angle both in polar and rectangular form. Determine the location of the peak and consequently the DOA estimate.

2. Repeat the experiment with two source signals at angles of $\phi_1 = 40°$ and $\phi_2 = -10°$ for $L = 4$, $d = 0.5$, $N = 20$, $\sigma_a = 1$ and $\sigma_w = 0.2$.

3. Repeat the experiment with two source signals at angles of $\phi_1 = 15°$ and $\phi_2 = 20°$. Are the peaks in the average power measure resolved?

4. Investigate the effect of the number of snap-shots on the DOA estimation algorithm.