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Space-Time Signals & Wave Propagation

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Spatial Array Processing

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Relevance of Array Processing

- Audio & speech: microphone arrays used for speech enhancement & source localization.
- Communications: spatial diversity, diversity combining for mitigating fading & co-channel interference.
- Biomedical problems: ultrasound imaging, fetal-maternal ECG problem.
- Geophysics: seismic event monitoring & empirical mode analysis.
- Radar & STAP: clutter suppression, side-lobe cancelation, DOA estimation.

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Review of Vector Calculus

- Gradient of a scalar field ϕ is defined via:

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i}_x + \frac{\partial\phi}{\partial y}\hat{i}_y + \frac{\partial\phi}{\partial z}\hat{i}_z.$$

- Divergence of a vector field defined via:

$$\text{div}(\vec{A}) = \vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

- Curl of a vector field:

$$\text{Curl}(\vec{A}) = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i}_x & \hat{i}_y & \hat{i}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_z & A_y & A_z \end{vmatrix}$$

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Review of Vector Calculus

- Laplacian of a scalar field:

$$\nabla^2\phi = \vec{\nabla}.\vec{\nabla}\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}.$$

- Laplacian of a vector field:

$$\nabla^2\vec{A} = \frac{\partial^2 A_x}{\partial x^2}\hat{i}_x + \frac{\partial^2 A_y}{\partial y^2}\hat{i}_y + \frac{\partial^2 A_z}{\partial z^2}\hat{i}_z.$$

- Null identities:

$$\begin{aligned} \text{div}(\text{curl}\vec{A}) &= \vec{\nabla}.(\vec{\nabla}\times\vec{A}) = 0 \\ \text{curl}(\text{grad}\phi) &= \vec{\nabla}\times\vec{\nabla}\phi = \mathbf{0} \end{aligned}$$

Coordinate Transformations

- Gradient vector in new (u_1,u_2,u_3) system:

$$\vec{\nabla} V = \frac{1}{h_1}\frac{\partial V}{\partial u_1}\hat{e}_{u_1} + \frac{1}{h_2}\frac{\partial V}{\partial u_2}\hat{e}_{u_2} + \frac{1}{h_3}\frac{\partial V}{\partial u_3}\hat{e}_{u_3}$$

- Divergence in new coordinate system:

$$\text{div}(\vec{A}) = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right)$$

- Curl in new system:

$$\text{curl}(\vec{A}) = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_{u_1} & h_2\hat{e}_{u_2} & h_3\hat{e}_{u_3} \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Maxwell's Equations

- Propagation governed by four fundamental equations:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial(\mu\vec{H})}{\partial t}$$
$$\vec{\nabla} \times \vec{H} = \frac{\partial(\epsilon\vec{E})}{\partial t}$$
$$\vec{\nabla} \,.\, (\epsilon\vec{E}) = 0$$
$$\vec{\nabla} \,.\, (\mu\vec{H}) = 0.$$

- ε & μ are the dielectric permitivity & magnetic permeability of free-space.

Wave Equation

- Solution to Maxwell's equations satisfy:

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}$$

- Component form of Wave-equation:

$$\frac{\partial^2 E_x}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

- Scalar form of wave equation:

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}.$$

Solution to Wave Equation

- Assuming solution in separable form:

$$s(x,y,z,t) = f(x)g(y)h(z)p(t)$$

- Monochromatic plane-wave: complex exponential

$$s(x,y,z,t) = A\exp\left(j(\omega t - k_x x - k_y y - k_z z)\right)$$

- Plane-wave a solution if wave-number vector satisfies:

$$|\vec{k}|^2 = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

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More on Plane Waves

- Superposition of plane waves a solution also:

$$s(\vec{x},t) = s(t-\vec{\alpha}.\vec{x}) = \sum_{n=-\infty}^{\infty} S_n \exp\left(jn\omega_o(t-\vec{\alpha}.\vec{x})\right)$$

- General aperiodic solution:

$$s(\vec{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\exp\left(j\omega(t-\vec{\alpha}.\vec{x})\right)d\omega$$

- The waveform s(u) should satisfy Dirichlet conditions

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Plane Waves

- In terms of temporal content plane waves are monochromatic, i.e., single frequency.
- For a fixed time-instant, locus of points defining the waveform is a plane defined by:

$$k_x x + k_y y + k_z z = C.$$

- The speed of propagation of the plane defined via:

$$\frac{\delta\vec{x}}{dt} = \frac{\omega}{|\vec{k}|} = c.$$

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Spherical Wavefronts

- Wave Equation in spherical polar coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial s}{\partial r}\right)+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial s}{\partial\phi}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2 s}{\partial\theta^2} = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}.$$

- Symmetry in ϕ & θ: solution independent of ϕ:

$$\frac{\partial^2(rs)}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2(rs)}{\partial t^2}.$$

- Solution of the form:

$$s(r,t) = \frac{A}{r}\exp\left(j(\omega t - kr)\right)$$

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Wavenumber-Frequency Space

- 4D Fourier transform pair:

$$S(\vec{k},\omega) = \int_{-\infty}^{\infty}\int_{x,y,z} s(\vec{x},t)\exp\left(-j(\omega t-\vec{k}.\vec{x})\right)d\vec{x}dt$$

$$s(\vec{x},t) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{k_x,k_y,k_z} S(\vec{k},\omega)\exp\left(j(\omega t-\vec{k}.\vec{x})\right)d\vec{k}d\omega.$$

- Wave equation can be interpreted as a bandpass filter in wave-number/frequency space
- Any S-T signal meeting the Dirichlet conditions, no matter what its shape, satisfies the wave equation !

Doppler Effect

- Consequence of relative motion between source of the wave-field & the sensor.
- Manifests as a frequency shift in the received signal.
- For moving sensor:

$$\omega_{\text{new}} = \omega_o \left(1 - \frac{v_{sen}}{c}\right)$$

- For moving source:

$$\omega_{\text{new}} = \omega_o \left(\frac{1}{1 - \frac{v_{sour}}{c}}\right)$$

Wave-number Frequency Spectrum

- For a monochromatic plane-wave:

$$\begin{aligned} s(\vec{x}, t) &= \exp\left(j(\omega_o t - \vec{k_o}.\vec{x})\right) \\ S(\vec{k}, \omega) &= \delta(\vec{k} - \vec{k_o})\delta(\omega - \omega_o). \end{aligned}$$

- For a propagating wave:

$$\begin{aligned} s(\vec{x}, t) &= s(t - \vec{\alpha}_o.\vec{x}) \\ S(\vec{k}, \omega) &= S(\omega)\delta(\vec{k} - \omega\vec{\alpha}_o) \end{aligned}$$

- Energy propagating along line: $\vec{k} = \omega\vec{\alpha}_o$

S-T Spectrum

- Narrowband & non-propagating signal: plane through the frequency $\omega = \omega_o$
- Wideband propagating space-time signal: plane directed along or parallel to direction of propagation.
- Same speed of propagation along different directions: cone-shaped spectrum.
- Propagation along a single direction : spectrum defined along a line.

Wavenumber-Frequency Space

- For a spherical wavefront:

$$s(r, t) = \frac{A}{r} \cos\left(\omega_o t - kr\right)$$

- Wavenumber-frequency spectrum:

$$\begin{aligned} S(k, \Phi, \Theta, \omega) &= \int_0^\infty \int_0^\pi \int_0^{2\pi} \int_{-\infty}^\infty s(r, \phi, \theta, t) \exp\left(-j\omega t + jkr\gamma(\phi, \theta, \Phi, \Theta)\right) r^2 sin\phi dr d\theta d\phi dt \\ \gamma(\phi, \theta, \Phi, \Theta) &= \sin\phi\cos\theta\sin\Phi\cos\Theta + \sin\phi\sin\theta\sin\Phi\sin\Theta + \cos\phi\cos\Phi \end{aligned}$$

$$S(k, \omega) = \left(\frac{2\pi^2}{jk}\delta(k - k^\circ) + \frac{4\pi}{k^2 - k_o^2}\right)\delta(\omega - \omega_o).$$

Phase & Group Velocity

- Phase velocity : speed of propagation of planes with constant phase:

$$\vec{v}_p = \frac{\omega}{k^2}\vec{k} \leftrightarrow \|\vec{v}_p\| = \frac{\lambda}{T} = \frac{\omega}{k}$$

- Group velocity: speed at which a group of closely spaced complex exponentials propagate:

$$\|\vec{v}_g\| = \frac{d\omega}{dk}$$

- Group velocity corresponds to the slope of the dispersion relation at a point.

Other forms of propagation

- Dispersion: arises when waves propagate thru medium with a string-like stiffness & is resistant to deformation by the propagating wave:

$$\nabla^2 s = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} + \frac{\omega_o^2}{c^2}s \longleftrightarrow \omega = c\sqrt{k^2 + \left(\frac{\omega_o}{c^2}\right)}$$

- Attenuation: arises when the propagating wave looses energy as it passes thru the medium:

$$\nabla^2 s = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} + \sigma\mu\frac{\partial s}{\partial t}$$

Other forms of propagation

- Some modes of propagation can be eliminated completely by the medium as in waveguides.
- Most general form of wave-propagation:

$$\nabla^2 s = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2} + c_1\frac{\partial s}{\partial t} + c_2 s$$

- Most general form of propagation combines attenuation & dispersion:
  - waveforms that decay in amplitude as they propagate.
  - waveforms that spread temporarily & deform.

Other propagation phenomena

- Refraction occurs when there are spatial changes in the propagation speed due to change in medium.
- Although the ray of light bends at the interface, after refraction it still travels in a straight line.
- Diffraction occurs when rays of light are incident on a aperture whose size is comparable to its wavelength.
- Diffraction produces secondary wavefields that emanate from the aperture.

Filtering in Wavenumber-Frequency Space

- 4-D convolution in S-T space:

$$y(\vec{x},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\vec{x}-\vec{\zeta})x(\vec{\zeta},\tau)d\vec{\zeta}d\tau.$$

- Wavenumber-frequency Response:

$$Y(\vec{k},\omega) = H(\vec{k},\omega)X(\vec{k},\omega).$$

- Problems with 4-D framework:
  - Causality issues
  - Arbitrary filter unrealizable.

Random Fields

- 4D stochastic process is typically referred to as a random field.
- A random field is mathematically defined as a measurable transformation from the sample space to a set of 4D waveforms.
- For particular values of S-T, i.e., (x_o,y_o,z_o,t_o), we obtain a random variable.
- For a particular outcome $\omega_o \in \Omega$, we obtain a S-T waveform.

Statistics of S-T Random Fields

- S-T random process characterized by n-th order joint PDF of random variables obtained by S-T sampling.
- This information not available in practical situations & we must be content with first & second order statistics.
- First-order statistics:

$$\begin{aligned}
\mu_s(\vec{x}_o,t_o) &= \mathcal{E}(s(\vec{x}_o,t_o)) \\
\sigma_s^2(\vec{x}_o,t_o) &= \mathrm{Var}(s(\vec{x}_o,t_o)) \\
R_s(\vec{\chi},\tau) &= \mathcal{E}(s(\vec{x}_o,t_o)s(\vec{x}_o+\vec{\chi},t_o+\tau))
\end{aligned}$$

PSD of S-T Random Fields

- PSD & ACF form a Fourier transform pair:

$$\begin{aligned}
P_f(\vec{k},\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_f(\vec{\chi},\tau)\exp\left(-j(\omega\tau-\vec{k}.\vec{\chi})\right)dtd\vec{\chi} \\
R_f(\vec{\chi},\tau) &= \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{\infty}^{\infty} P_f(\vec{k},\omega)\exp\left(j(\omega\tau-\vec{k}.\vec{\chi})\right)d\vec{k}d\omega
\end{aligned}$$

- Separable S-T random field:

$$\begin{aligned}
R_f(\vec{\chi},\tau) &= R_s(\vec{\chi})R_t(\tau) \\
P_f(\vec{k},\omega) &= P_s(\vec{k})P_t(\omega)
\end{aligned}$$

S-T White noise

- Consider a S-T random field with PSD given by:

$$P_f(\vec{k},\omega) = \begin{cases} 1 & |k_x| \le B_x, |k_y| \le B_y, |k_z| \le B_z, |\omega| \le B \\ 0 & \text{otherwise} \end{cases}$$

- Corresponding ACF:

$$R_f(\vec{\chi},\tau) = \left(\frac{\sin B\tau}{\pi\tau}\right)\left(\frac{\sin B_x\chi_x}{\pi\chi_x}\right)\left(\frac{\sin B_y\chi_y}{\pi\chi_y}\right)\left(\frac{\sin B_z\chi_z}{\pi\chi_z}\right)$$

- S-T white noise is the limiting case:

$$w(\vec{x},t) = \lim_{B_x,B_y,B_z,B\to\infty} f(\vec{x},t) \leftrightarrow R_w(\vec{\chi},\tau) = \delta(\vec{\chi},\tau).$$

Isotrophic S-T Noise

- Consider a S-T random field with a PSD of the form:

$$P_f(\vec{k},\omega) = G(\omega)\delta\left(|\vec{k}| - \omega/c\right).$$

- Ensemble ACF of the form:

$$R_f(\chi,\tau) = \frac{A}{4\pi^3}\int_{-\infty}^{\infty} \frac{\omega}{c\chi}\sin\left(\frac{\omega\chi}{c}\right) G(\omega)\exp(j\omega\tau)\, d\omega$$

- For monochromatic temporal part:

$$R_f(\chi,\tau) = \frac{A\omega^2}{2\pi^2c^2}\cos\omega_o\tau \,\text{Sa}\left(\frac{\omega_o\chi}{c}\right)$$

- Spatial sampling at locations separated by $\lambda/2$ yield uncorrelated waveforms.

Spatio-Temporal Correlation Matrix

- Define cross-covariance matrix of signals at sensor pair (m_1,m_2) with elements:

$$\{\mathbf{R}_{m_1,m_2}\}(n_1,n_2) = R_f(\vec{x}_{m_1},\vec{x}_{m_2},t_{n_1},t_{n_2}), \;\; 0 \le n_1,n_2 \le N-1$$

- Define S-T correlation matrix as:

$$\mathrm{R}(t_o) = \begin{bmatrix} \mathbf{R}_{0,0} & \mathbf{R}_{0,1} & \dots & \mathbf{R}_{0,M-1} \\ \mathbf{R}_{1,0} & \mathbf{R}_{1,1} & \dots & \mathbf{R}_{1,M-1} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{R}_{M-1,0} & \mathbf{R}_{M-1,1} & \dots & \mathbf{R}_{M-1,M-1} \end{bmatrix}$$

- For a stationary & separable S-T field: $\mathrm{R} = \rho \bigotimes \mathrm{R}_{0,0}$

S-T Correlation Matrix

- Suppose the received waveform is a superposition of several plane-waves contaminated by AWGN:

$$y_m(t_n) = \sum_{i=0}^{N_d-1} s_i(m,t_n) + \zeta_m(t_n), \;\; 0 \le m \le M-1.$$

- Define a signal matrix:

$$\mathrm{S}(t_o) = \begin{bmatrix} \mathrm{s}_0(0) & \mathrm{s}_1(0) & \dots & \mathrm{s}_{N_d-1}(0) \\ \mathrm{s}_0(1) & \mathrm{s}_1(1) & \dots & \mathrm{s}_{N_d-1}(1) \\ \vdots & \vdots & \dots & \vdots \\ \mathrm{s}_0(M-1) & \mathrm{s}_1(M-1) & \dots & \mathrm{s}_{N_d-1}(M-1) \end{bmatrix}$$

S-T Correlation Matrix

- Observations rewritten in signal matrix:

$$\mathrm{y}(n_oT) = \mathrm{S}(n_oT)\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$$

- S-T correlation matrix of observations:

$$\mathbf{R}_y = \mathrm{S}(t_o)\mathbf{1}\mathbf{1}^T\mathrm{S}^T(t_o) + \mathbf{R}_n.$$

Linear Array Signal Model

- Consider an array of sensors located spatially along the X-axis and spaced apart by d.
- Assume a plane-wave with temporal frequency ω_o & wave-number k is incident on the linear-array.
- Assume that it is incident with an angle ϕ with respect to the normal at the surface of incidence.
- Incident wave-fields wave-number vector:

$$\vec{k} = k_x\hat{i}_x + k_y\hat{i}_y = |\vec{k}|\sin\phi\ \hat{i}_x - |\vec{k}|\cos\phi\ \hat{i}_y$$

Linear Array Signal Model

- Wave-field measured at the m-th sensor location:

$$\begin{aligned}
f(\vec{x}_m,t) &= \exp\left(j\omega_o t - j\vec{k}.\vec{x}_m\right),\ 0 \le m \le (M-1)\\
&= \exp\left(j\omega_o t - j|\vec{k}|\sin\phi(m-(M-1)/2)d\right)\\
&= \exp(j\omega_o t)\exp\left(-j\frac{\omega_o d}{c}\sin\phi\ (m-(M-1)/2)\right)\\
&= \exp(j\omega_o(t-\tau_m)),\ \tau_m = \frac{d\sin\phi}{c}(m-(M-1)/2)
\end{aligned}$$

- Wave-field at sensor locations differs only by a phase factor dependent on the angle of incidence ϕ

Linear Array Signal Model

- Sensor output matrix:

$$\begin{aligned}
\mathrm{S}(t) &= [s_0(t), s_1(t), s_3(t), \ldots, s_{M-1}(t)]\\
&= [s(t-\tau_o), s(t-\tau_1), \ldots s(t-\tau_{M-1})]\\
&= [1\ \ \exp(-j\omega_o\tau)\ \ \exp(-j2\omega_o\tau)\ldots]s(t-\tau_o), \tau = \frac{d\sin\phi}{c}\\
&= a(\phi)s(t-\tau_o) = a(\phi)s_o(t)
\end{aligned}$$

- For n_d plane-waves impinging on the linear-array at angles ϕ_i, i=1,2, ... ,M:

$$\tilde{\mathrm{S}}(t) = \sum_{r=1}^{n_d}\mathrm{a}(\phi_r)s_r(t) = \mathbf{A}(\phi)\mathbf{S}(t) + \mathbf{N}(t)$$