On a Pseudo-Subspace Framework for Discrete Fractional Fourier Transform Based Chirp Parameter Estimation

Balu Santhanam and Majeed Hayat
Dept. of E.C.E., University of New Mexico, Albuquerque, NM: 87131
Tel: 505 277 1611, Fax: 505 277 1439, Email: bsanthan@ece.unm.edu

***Abstract*—Sinusoids remain the prototypical waveform for signal modeling, analysis, detection, and estimation in stationary environments, but are unsuitable for the analysis of signals with non stationary frequency content. In [1], the MA-CDFRFT was introduced as a useful tool for the analysis of multicomponent chirp signals in the absence of noise. Subspace approaches derived from a eigenvalue decomposition of the correlation matrix of noisy observations of sinusoidal signals, such as the MUSIC or minimum-norm algorithms are popular approaches for estimating the parameters of multiple sinusoidal signals in white noise. In this paper, we extend the MA-CDFRFT methodology to develop a pseudo-subspace approach towards chirp parameter estimation.**

Keywords: Discrete Fractional Fourier Transform, subspace methods, multicomponent chirp parameter estimation.

I. Introduction

While Fourier analysis based techniques remain the norm for the representation of stationary signals, they are unsuitable for signals such as linear FM, i.e., chirps, where the frequency content is not stationary. Conventional techniques for the analysis of these signals such as the short time Fourier transform are based on assumptions of sinusoidal modeling over smaller windowed segments. Discrete Fractional Fourier analysis techniques have [2], [1] recently garnered attention for multicomponent chirp signal analysis because of their ability to concentrate chirps in a few transform coefficients. However, the underlying analysis is done in the absence of noise.

Subspace approaches such as MUSIC, eigenvector, or minimum variance techniques are based on eigenvalue decomposition of the sample covariance matrix of observations. These are popular, statistically efficient approaches towards sinusoidal parameter estimation, and are not plagued by the bias-variance problem prevalent in the periodogram based approaches [5]. Recent work in [6] extends the subspace framework to parameter estimation of chirp signals.

In this paper, we develop a framework for the analysis and comparison of subspace spectrum estimation approaches such as MUSIC, root-MUSIC, eigenvector, minimum-norm, and minimum variance algorithms for discrete Fractional Fourier transform based chirp parameter estimation. Simulation results with monocomponent and multicomponent chirp signals are used to demonstrate the efficacy of proposed approach.

This work was supported by the US Department of Energy (award no. DE-FG52-08NA28782) and by the National Science Foundation (award no. IIS-0813747).

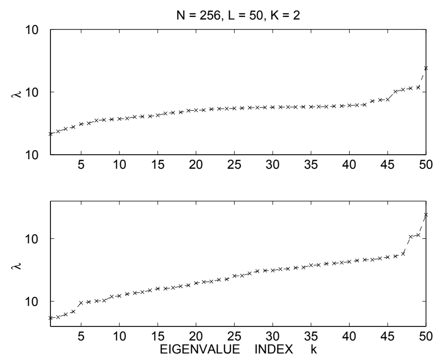

Fig. 1. Eigenvalue distribution of subspace covariance matrices: (a) eigenvalues for $\mathbf{R}_{cf}$ and (b) $\mathbf{R}_{cr}$ for $L = 50$ snapshots and $N = 256$ depicting a large spread with large values corresponding to the pseudo signal subspace and small values corresponding to the pseudo noise subspace.

II. Signal Model and MA-CDFRFT Primer

The class of signals that we consider here are multicomponent chirp signals of the form:

$$x[n] = \sum_{i=1}^{N_s} A_i \exp\left(j\omega_{ci} n + j c_{ri} m^2\right) + w[n] \tag{1}$$

where $m = n - (N-1)/2,\ 0 \leq n \leq N-1$, $w[n]$ is additive white Gaussian noise, and (ω_{ci}, c_{ri}) are the center frequency and chirp rate of the i-th chirp component.

The discrete *fractional Fourier transform* (DFRFT) of a sequence $x[n]$ is defined via [2]:

$$\mathbf{X}_{\alpha} = \mathbf{W}^{\frac{2\alpha}{\pi}} \mathbf{x} = \mathbf{V} \mathbf{\Lambda}^{\frac{2\alpha}{\pi}} \mathbf{V}^T \mathbf{x}, \tag{2}$$

where $\mathbf{V}$ are a basis of eigenvectors for the DFT matrix $\mathbf{W}$ and $\mathbf{\Lambda}$ is a diagonal matrix of eigenvectors of the DFT. Direct computation of the transform requires the computation of a full basis of non-degenerate eigenvectors for the DFT matrix [1], [4]. In [1], a fast algorithm for computing the DFRFT for a discrete set of angles, called the *multiangle centered* DFRFT (MA-CDFRFT), exploiting symmetries in the DFT eigenvectors, was developed. The multi-angle version of the CDFRFT is expressed as:

$$
\begin{aligned}
X_k[r] &= \sum_{p=0}^{N-1} z_k[p] e^{-j\frac{2\pi}{N}pr}, \\
z_k[p] &= v_{kp} \sum_{n=0}^{N-1} x[n] v_{np}
\end{aligned} \tag{3}
$$

and v_{kp} denotes the p-th component of the k-th DFT eigenvector. This multiangle transform has the capability to concentrate chirps in a few transform coefficients and its performance in noise can be improved to a certain extent by zooming in close to the peaks using the CZT [3]. However, being a FFT based transform, it is affected by noise, and its chirp parameter estimates will be either biased or statistically inconsistent. Subspace techniques [5] are expected to improve the performance of the underlying parameter estimators in noise.

III. Subspace Chirp Parameter Estimation

The DFRFT based subspace algorithm is obtained by calculating the inverse DFT of row and column projections of the magnitude of the MA-CDFRFT via:

$$
\begin{aligned}
x_{cr}[r] &= \mathbf{W}^{-1}\left(\sum_{k=0}^{N-1} |X_k[r]|\right), \\
x_{cf}[k] &= \mathbf{W}^{-1}\left(\sum_{r=0}^{N-1} |X_k[r]|\right),
\end{aligned} \tag{4}
$$

where $\mathbf{W}^{-1}$ denotes the unitary version of the inverse DFT operator defined in [2]. Effectively x_{cf} and x_{cr} are one dimensional time-series containing information regarding the center-frequency and chirp rate parameters. Now define the covariance matrices associated with the signals x_{cf} and x_{cr} as $\mathbf{R}_{cf}$ and $\mathbf{R}_{cr}$ respectively. In practice, these covariance matrices are estimated from noisy observations using sample covariance estimates [5]. As described in [5], biased covariance estimates are preferred over unbiased estimates for obtaining positive definite matrices. Fig. (1) describes the eigenvalue distribution of the covariance matrices $\mathbf{R}_{cf}$ and $\mathbf{R}_{cr}$ for a real chirp using the "correlation" estimate.

Eigenvalue decomposition of these covariance matrices yields the desired pseudo-subspace decomposition [5], [6]:

$$
\begin{aligned}
\mathbf{R}_{cf} &= \mathbf{V}_{cf}\mathbf{\Lambda}_{cf}\mathbf{V}_{cf}^T \\
&= \mathbf{V}_{cfs}\mathbf{\Lambda}_{cfs}\mathbf{V}_{cfs}^T + \mathbf{V}_{cfn}\mathbf{\Lambda}_{cfn}\mathbf{V}_{cfn}^T \\
\mathbf{R}_{cr} &= \mathbf{V}_{cr}\mathbf{\Lambda}_{cr}\mathbf{V}_{cr}^T \\
&= \mathbf{V}_{crs}\mathbf{\Lambda}_{crs}\mathbf{V}_{crs}^T + \mathbf{V}_{crn}\mathbf{\Lambda}_{crn}\mathbf{V}_{crn}^T,
\end{aligned} \tag{5}
$$

where the subscripts s, n denote the S+N and N subspaces. We use the pseudo-subspace terminology similar to the approach in [6] because unlike the case of sinusoids in noise, the noise and signal subspaces do not completely separate. Using these covariance matrices, one can now develop both pseudo-noise subspace approaches such as MUSIC, eigenvector, minimum norm and pseudo-signal subspace approaches such as principle components Blackman-Tukey or minimum variance for chirp parameter estimation. The pseudo-spectra for the signals x_{cf} and x_{cr} with respect to the MUSIC algorithm for example are given by:

$$
\begin{aligned}
P_{cf}^{MUSIC} &= \frac{1}{\sum_{k=N_s+1}^{L} |\mathbf{e}^H \mathbf{v}_k^{cf}|^2}, \\
P_{cr}^{MUSIC} &= \frac{1}{\sum_{k=N_s+1}^{L} |\mathbf{e}^H \mathbf{v}_k^{cr}|^2},
\end{aligned} \tag{6}
$$

where $\mathbf{V}_{cf}$ and $\mathbf{V}_{cr}$ are the eigenvector matrices for the covariance matrices $\mathbf{R}_{cf}$ and $\mathbf{R}_{cr}$, where N_s is the number of chirp components present, L is the number of snapshots, $\mathbf{e}$ is the conventional frequency vector, and $\mathbf{v}_k^{cr}$ denotes the k-th column of $\mathbf{V}_{cr}$. In a similar vein, the principle components Blackman-Tukey or minimum variance pseudo-spectra for example would be defined via:

$$
\begin{aligned}
P_{cf}^{PCMV} &= \frac{1}{\sum_{k=1}^{N_s} \frac{1}{\lambda_k} |\mathbf{e}^H \mathbf{v}_k^{cf}|^2}, \\
P_{cr}^{PCMV} &= \frac{1}{\sum_{k=1}^{N_s} \frac{1}{\lambda_k} |\mathbf{e}^H \mathbf{v}_k^{cr}|^2}.
\end{aligned} \tag{7}
$$

The center frequency estimate for each component is extracted directly from the difference between the corresponding peak locations in the center frequency pseudo spectrum, while the chirp rate parameters are estimated from the peaks in the chirp rate pseudo spectrum and the peak to parameter mapping approach employed in [6]. An example of the mapping between the pseudo-spectrum peaks for x_{cr} to the actual chirp rate for the MUSIC algorithm is illustrated in Fig. (2)(g). Note that this map for the MUSIC algorithm is bijective and maps a peak location uniquely to a chirp rate value. This parameter mapping in general depends on the specific algorithm, the covariance matrix size L, the number of subspace FFT points R, and the MA-CDFRFT size N.

Simulation results in Fig. (2,3) depict the application of the proposed subspace approach to both monocomponent and two component real chirps using the MUSIC, eigenvector, minimum norm, PC-BT, and minimum variance spectrum algorithms at a SNR of 30 dB. It can be seen from the monocomponent example, that while the performance of the center-frequency estimator for the different algorithms is more or less the same, the sharpness of the chirp rate spectral peaks are diminished in the case of PC-BT and minimum variance algorithms in comparison to the MUSIC and eigenvector approaches. Fig. (2,3)(b) describe the MA-CDFRFT spectra for the signals computed with a FFT size of $N = 256$. The two component example examines a cochannel parameter setting, where the carrier frequencies of the components are identical with $N_s = 4$. The pseudo-spectra for the different subspace

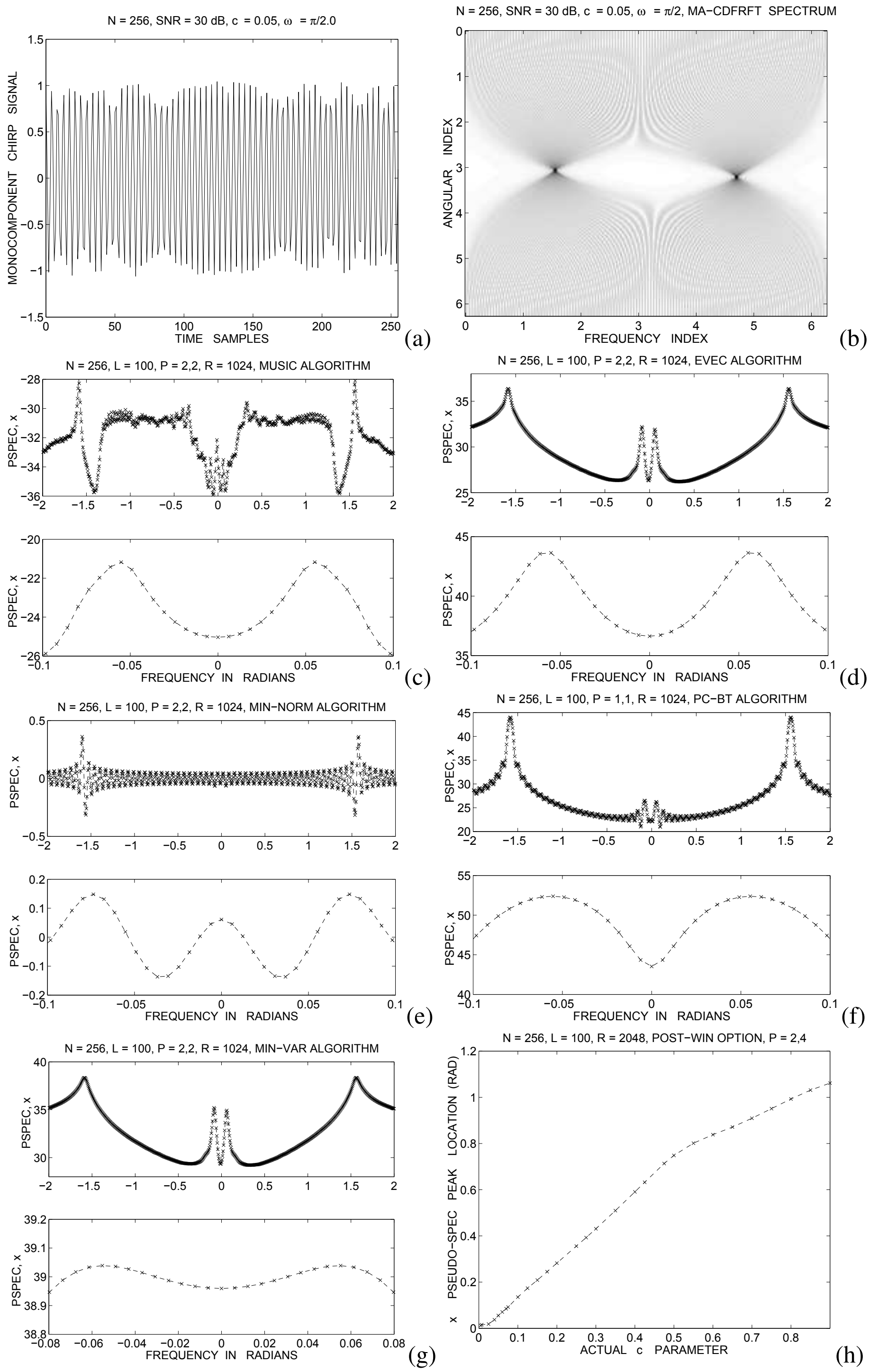

Fig. 2. Subspace chirp parameter estimation: (a) real chirp signal with $\omega_c = \frac{\pi}{2.0}$, chirp rate $c_r = 0.05$, $SNR = 30\ dB$, (b) MA-CDFRFT spectrum using FFT's of size $N = 256$, (c) MUSIC pseudo-spectra for the signals x_{cf} and x_{cr} using a model order of $N_s = 2$ depicting peaks at locations related to the chirp parameters, (d,e,f,g) corresponding x_{cf} and x_{cr} pseudo-spectra for the minimum norm, eigenvector, principal components Blackman-Tukey, and minimum variance spectrum estimation methods. The sample covariance matrices were computed using the "post-windowed" option in MATLAB with $L = 100$ snapshots, and (g) peak to chirp parameter mapping for the MUSIC algorithm using spline interpolation. The estimated chirp rate for the MUSIC algorithm using the mapping is $\acute{c_r} = 0.054$ while the actual value is $\acute{c_r} = 0.05$.

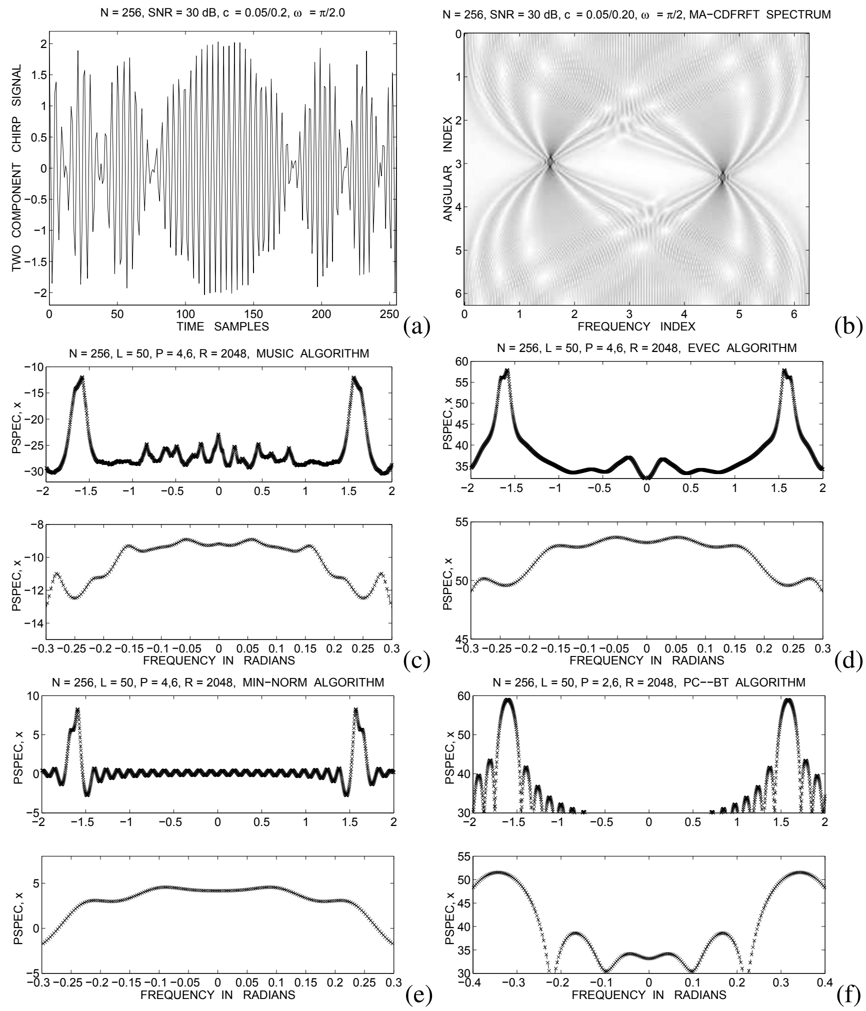

Fig. 3. Two component chirp example: (a) composite two component signal at a SNR of 30 dB, (b) MA-CDFRFT spectrum depicting two distinct peaks corresponding to each chirp component, (c,d,e) MUSIC, eigenvector, minimum-norm, and principal components B-T pseudo-spectra for the signals x_{cf}, and x_{cr} using a model order of $N_s = 4$ and a sample covariance matrix with $L = 40$ snapshots and the "post-windowed" option in MATLAB.

algorithms using the post-windowed option and $L = 50$ are depicted in Fig. (3). The MUSIC, eigenvector approaches clearly distinguish between the chirp components present, while the minimum norm and PC-BT approaches are only able to detect one. The identical carrier frequencies of the components manifest as a single peak in the carrier frequency pseudo-spectra. The chirp-rate pseudo-spectra specifically depict a certain amount of peak suppression due in part to the magnitude projection operation used in the approach.

IV. Conclusions

In this paper, we have presented a pseudo-subspace approach for discrete Fractional Fourier transform based chirp parameter estimation in noise, a topic more or less restricted to sinusoids in current literature. Simulation results with both monocomponent and multicomponent chirp signals demonstrate the ability of the approach to estimate the center frequencies and chirp rates associated with each chirp component. Statistical analysis of the performance of the approach, comparison to the CRLB, and other issues such as the effect of: the sample covariance matrix, the number of snapshots, the DFT eigenvectors, and model mismatch on the parameter estimates are currently being investigated.

References

[1] J. G .Vargas and B. Santhanam, "On the Multiangle Centered Discrete Fractional Fourier Transform," *IEEE Sig. Process. Lett.*, Vol. 12, No. 4, pp. 273-276, 2005.

[2] B. Santhanam and J. H. McClellan, "Discrete Rotational Fourier Transform, " *IEEE Trans. on Sig. Process.*, Vol. 44, No. 4, pp. 994-997, 1996.

[3] S. L. Reddy, B. Santhanam, and H. Hayat, "CoChannel FM Demodulation Via the Multiangle Discrete Fractional Fourier Transform," *Proc. of IEEE DSP & SP Educ. Wksp.*, pp. 535-539, Jan. 2009.

[4] B. Santhanam and T. S. Santhanam, "On Discrete Gauss–Hermite Functions and Eigenvectors of the Discrete Fourier Transform," *Sig. Process.*, Vol. 88, pp. 2738-2746, Nov. 2008.

[5] M. H. Hayes, "Statistical Digital Signal Processing and Modeling," John Wiley & Sons Inc., New York, 1996.

[6] B. Volcker and B. Ottersten, "Chirp Parameter Estimation from a Sample Covariance Matrix," *IEEE Trans. on Sig. Process.*, Vol. 49, No. 3, pp. 603-612, 2001.