On a Sturm–Liouville Framework for Continuous and Discrete Frequency Modulation

(Invited Paper)

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***Abstract*—It is well known that purely sinusoidal signals satisfy a linear second-order constant coefficient differential equation. It is also well known that a broad class of orthogonal special functions such as the Legendre and Hermite polynomials satisfy the second-order Sturm-Liouville differential equation. Both sinusoidal and AM–FM models have been used for analysis and synthesis of speech signals. In this paper, we present a Sturm-Liouville differential and difference equation approach to both continuous and discrete time frequency modulation. Orthogonal modes of frequency modulation that are not distorted by the Sturm-Liouville operator are described.**

Keywords: Frequency modulation, eigenvectors, Sturm-Liouville differential or difference equation, generalized Fourier series.

I. Introduction

Sinusoidal signals have a special connection with LTI systems in that they are *eigenfunctions* of a LTI system operator and form the basis for LTI system theory:

$$L(\exp{(j\omega_o t)}) = H(j\omega_o)\exp{(j\omega_o t)},$$

where $H(j\omega_o)$ represents the complex eigenvalue or gain.

A sinusoidal signal of the form:

$$x(t) = \cos(\omega_o t + \theta_o).$$

further satisfies the constant coefficient, homogenous, second-order differential equation of the classical harmonic oscillator:

$$\ddot{x} + \omega_o^2 x = 0.$$

Now consider a frequency modulated version of the sinusoidal signal of the form:

$$x(t) = \cos(\phi(t)) = \cos\left(\int_{-\infty}^{t} \omega_i(\tau)d\tau\right),$$

where $\omega_i(t)$ is the instantaneous frequency and $\phi_i(t)$ is the instantaneous phase. This signal satisfies a second-order differential equation with time-varying coefficients of the form:

$$\ddot{x} - \frac{\dot{\omega}_i(t)}{\omega_i(t)}\dot{x} + \omega_i^2(t)x = \left(\mathcal{D}^2 - \frac{\mathcal{D}\omega_i}{\omega_i}\mathcal{D} + \omega_i^2\right)x = 0, \quad (1)$$

where $\mathcal{D}$ denotes the derivative operator. It is known that even in the simple case, where the message waveform is sinusoidal, the bandwidth of the FM signal is infinite and requires truncation[1].

The *energy separation algorithm* (ESA) and its discrete version DESA were studied in [5] as a methodology for the demodulation of AM–FM signals. In [4], it was shown that AM–FM signals can only be approximate eigenfunctions of LTI systems and consequently they will undergo harmonic distortion when they are subjected to LTI filtering. Constraints on the frequency response of a filter for minimizing the error induced by the eigenfunction approximation and bounds on the demodulation error for AM–FM signals were developed. However, when these constraints are not met, the eigenfunction approximation incurs significant demodulation error. Orthogonal FM functions derived from simple permutations of the phase of the conventional DFT were investigated in [6] in the context of energy compaction.

In this paper, the goal is to develop and analyze a *Sturm–Liouville* (S-L) [9] framework for both continuous and discrete frequency modulation. This is accomplished by studying the generating differential or difference equation underlying the frequency modulated signal [11]. Orthogonal modes of frequency modulation that are not subject to distortion from the underlying S-L operator are described and are used to define a generalized Fourier series framework applicable to the processing of frequency modulated signals.

II. Continuous time FM

The FM differential equation described in Eq. (1) does not correspond to a self-adjoint operator. The self-adjoint form of the FM differential equation is [1]:

$$\mathcal{D}\left(\frac{1}{\omega_i(t)}\mathcal{D}x(t)\right) + \omega_i(t)x(t) = 0$$

The self-adjoint form of the FM differential equation for the FM signal $x(t) = \cos(n\phi(t))$ is given by:

$$\left(\frac{1}{\omega_i}\mathcal{D}^2 - \frac{\mathcal{D}\omega_i}{\omega_i^2}\mathcal{D}\right)x = -n^2\omega_i x, \ \mathcal{H}(\omega_i)x = -n^2\omega_i x. \quad (2)$$

Comparing this to the general differential form of the S-L differential equation:

$$\mathcal{D}\left(p(x)\mathcal{D}(y(x))\right) + q(x)y(x) = \lambda w(x)y(x),$$

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[1] Carson bandwidth of an FM signal retains just spectral components that have an amplitude of at least 10% of the maximum spectral amplitude

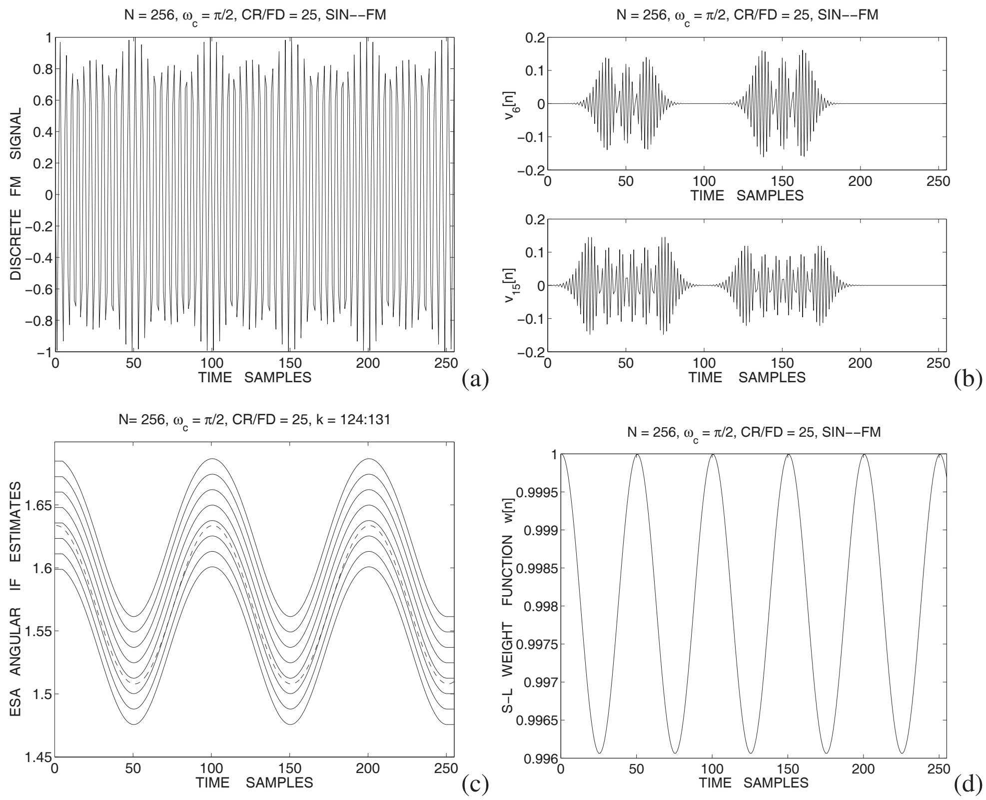

Fig. 1. Discrete S-L problem, sinusoidal-FM :(a) sinusoidal FM signal, (b) selected eigenvectors of the discrete S-L operator depicting different number of zero crossings, (c) IF of selected eigenvectors extracted using the ESA [5], and (d) weighting function of the discrete S-L problem,

where λ is the eigenvalue and $w(x)$ is the weight function we can see that Eq. (2) is a specific case of the S-L problem with

$$\lambda_n = -n^2,\ p(t) = \frac{1}{\omega_i(t)},\ q(t) = 0$$

and weight function[2] $w(t) = w_i(t)$. Eq. (2) can in turn be formulated as a S-L system with periodicity by periodic extension of the instantaneous frequency $\omega_i(t)$ or it can be treated as a S-L extrapolation problem, where this can be accomplished by repeating the values of the instantaneous frequency at the boundaries[3]. This S-L framework implies that the operator $\mathcal{H}$ has real and positive eigenvalues and a full set of orthogonal eigenfunctions $\psi_n(t)$ with respect to the weight function $\omega_i(t)$:

$$\begin{aligned} <\psi_m(t), \psi_n(t)> &= \int_{-\infty}^{\infty} w_i(t)\psi_m(t)\psi_n(t)dt \\ &= 0, \quad m \neq n. \end{aligned} \quad (3)$$

This result is consistent with earlier work on FAM-lets [3], where the sequence of functions:

$$\begin{aligned} \gamma_n(t) &= \sqrt{\omega_i(t)}\cos(n\phi(t)) \\ \zeta_n(t) &= \sqrt{\omega_i(t)}\sin(n\phi(t)), \end{aligned} \quad (4)$$

were shown to be an orthogonal sequence of functions with both amplitude and frequency modulation[4]. It is also well known that many of the special functions encountered in quantum mechanics such as Legendre or Hermite functions satisfy the S-L framework for specific discrete values of the eigenvalue λ and the weight function $w(x)$ [2], [9].

Our goal is to develop a framework for discretisation of this differential operator $\mathcal{H}$ so that the eigenvectors of the resultant discrete system are discrete approximations to the FM differential equation. We accomplish this by expressing the operator in the form:

$$\mathcal{H}(\omega_i) = \mathcal{D}\left(\frac{1}{\omega_i}\mathcal{D}\right). \quad (5)$$

There are two important consequences of expressing the FM differential equation in the S-L form. The first implication is that if the FM signal $x(t)$ is input to the system $\mathcal{H}(\omega_i)$, then the output is just a scalar multiple of the input signal. In other words, the system does not introduce any frequency distortion and that instantaneous frequency of the input signal

[2]For the S-L framework to hold the weight function $\omega_i(t)$ should be strictly positive

[3]The instantaneous frequency, $\omega_i(t)$, is assumed to be slow time-varying.

[4]The distinguishing characteristic of FAM-lets is that the ratio of their center-frequency to the bandwidth is a constant

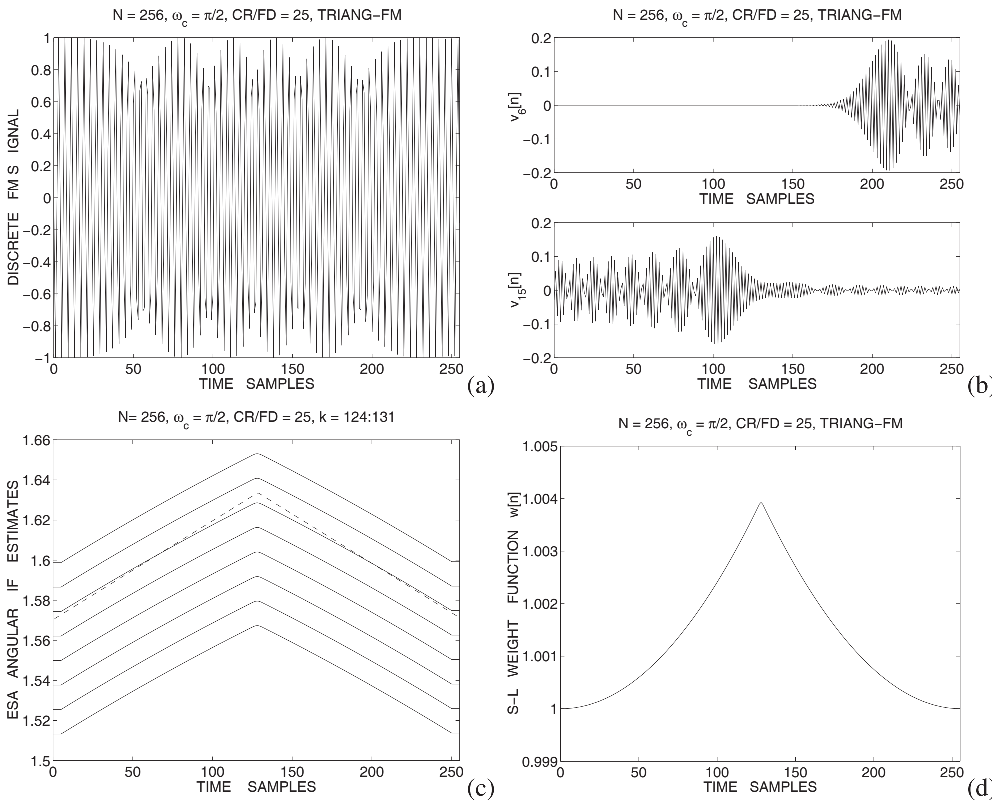

Fig. 2. Discrete S-L problem, triangular frequency modulation : (a) FM signal, (b) selected eigenvectors of the discrete S-L operator, (c,d) instantaneous frequency and envelope estimates of eigenvectors $k = 124 : 131$ of the discrete S-L operator using the DESA, (e) weight function associated with the discrete S-L operator.

$x(t)$ remains invariant:

$$\mathcal{H}\left(\sum_{k=0}^{\infty} a[k]\cos(k\phi(t))\right) = \sum_{k=0}^{\infty} a[k]\mathcal{H}(\cos(k\phi(t)))$$

$$= \quad \omega_i(t)\sum_{k=0}^{\infty} \underbrace{-k^2 a[k]}_{b[k]} \cos(k\phi(t)). \tag{6}$$

The second implication is that results analogous to LTI systems and sinusoids such as a Fourier series of FM modulated waveforms can be developed for modulated signals with $\psi_k(t) = \cos(k\phi(t))$:

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} c[k]\psi_k(t) \\ c[k] &= \frac{\int_{-\infty}^{\infty} x(t)\psi_k(t)\omega_i(t)dt}{\int_{-\infty}^{\infty} |\psi_k(t)|^2\omega_i(t)dt} \end{aligned} \tag{7}$$

III. Discrete time FM

One approach to generating a S-L framework for discrete time FM is to work directly with the difference equation satisfied by the signal. First consider the sinusoidal sequence $s[n] = \cos(\Omega_o n)$ which satisfies the second-order difference equation:

$$s[n] - 2\cos(\Omega_o)\, s[n-1] + s[n-2] = 0.$$

Now consider the discrete time FM sequence $x[n]$ given by:

$$x[n] = \cos(\Theta[n]) = \cos\left(\int_o^n \Omega_i[m]dm + \theta_o\right),$$

where the instantaneous phase $\Theta[n]$ is modeled as a first difference:

$$\Theta[n] = \Theta[n-1] + \Omega_i[n].$$

It is easily seen that this satisfies a second-order generating difference equation of the form [8]:

$$x[n] - c_1[n]x[n-1] + c_2[n]x[n-2] = 0,$$

where the time-varying coefficients are given by:

$$\begin{aligned} c_1[n] &= \frac{\sin(\Omega_i[n] + \Omega_i[n-1])}{\sin(\Omega_i[n-1])} \\ c_2[n] &= \frac{\sin(\Omega_i[n])}{\sin(\Omega_i[n-1])}. \end{aligned} \tag{8}$$

It can also be verified that this difference equation will reduce to that of the sinusoid in the stationary case, i.e., $\Omega_i[n] = \Omega_o$. The corresponding self-adjoint difference equation obtained

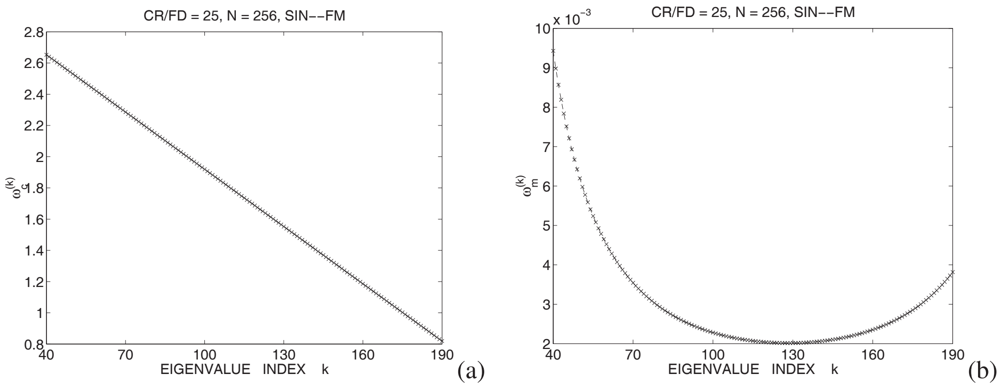

Fig. 3. Center frequency and frequency deviation of selected FM modes of the discrete S-L operator for the first sinusoidally modulated example.

by the S-L difference equation framework described in [7] is given by:

$$\nabla_- \left(p[n]\Delta_+(x[n])\right) + w[n]C[n]x[n] = 0, \tag{9}$$

where the weight function $w[n]$, $p[n]$, and $C[n]$ are given by:

$$\begin{aligned} w[n] &= \prod_{r=0}^{n-1} \frac{\sin(\Omega_i[r])}{\sin(\Omega_i[r+2])} = \frac{\sin(\Omega_i[0])\sin(\Omega_i[1])}{\sin(\Omega_i[n])\sin(\Omega_i[n+1])} \\ p[n] &= \sin\left(\Omega_i[n]\right) w[n] = \frac{\sin(\Omega_i[0])\sin(\Omega_i[1])}{\sin(\Omega_i[n+1])} \\ C[n] &= \sin(\Omega_i[n]) + \sin(\Omega_i[n+1]) \\ &- \sin(\Omega_i[n+1] + \Omega_i[n]) \end{aligned} \tag{10}$$

and the symbols ∇_- and Δ_+ denote the one-sample backward and forward difference operators. It should be noted here that the form of the FM difference equation and as a result the self-adjoint S-L difference equation are sensitive to the form of discretization of the instantaneous phase $\Theta[n]$. As in the continuous case, the difference equation in Eq. (9) can be formulated as a periodic S-L system by either extrapolation ot periodic extension of the instantaneous frequency $\Omega_i[n]$ at the boundaries [2], [10]. The solution to the discrete S-L system is then formulated as the solution to a weighted, tridiagonal eigenvalue problem of the form:

$$\mathcal{L}(\mathbf{x}) = \lambda \mathbf{W}\mathbf{x}, \tag{11}$$

where $\mathbf{W} = \mathrm{diag}(w[0], \ldots, w[N-1])$ is a diagonal matrix of the positive weights and λ is the eigenvalue[5]. Furthermore, as in the continuous case, the eigenvectors of the S-L operator:

$$\mathcal{L}(p[n]) = \nabla_- p[n]\Delta_+ + w[n]C[n]$$

corresponding to distinct eigenvalues are orthogonal with respect to the positive weight function $w[n]$:

$$< v_p[n], v_q[n] > = \sum_{n=0}^{N-1} w[n]v_p[n]v_q[n] = 0, \quad p \neq q. \tag{12}$$

[5]For situations where the signal of interest and the estimate of the IF, $\Omega_i[n]$, are noisy, a generalized SVD version of Eq. (11) is employed

The corresponding expansion of the discrete FM signal in terms of the eigenvectors $v_k[n]$ of the S-L operator is:

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} c[k]v_k[n], \\ c[k] &= \frac{\sum_{n=0}^{N-1} w[n]x[n]v_k[n]}{\sum_{n=0}^{N-1} w[n]|v_k[n]|^2} \end{aligned} \tag{13}$$

These eigenvectors contain both amplitude and frequency modulation and the IF of the eigenvectors of the matrix $\mathcal{L}$ furthermore have a form specified by the original IF, $\omega_i[n]$:

$$v_k[n] = a_k[n] \cos\left(\frac{\pi}{N}nk + \phi_k[n]\right)$$

Fig. (1), fig. (2), and fig. (4) describe the application of the discrete S-L approach to a monocomponent: (a) sinusoidally modulated FM signal, (b) FM signal with a triangular IF, and (c) FM signal with a triangular IF in noise. Note that the eigenvectors corresponding to smaller eigenvalues have instantaneous frequencies in the high frequency range, while the ones corresponding to the larger eigenvalues have IF's in the low-frequency range as depicted in Fig. (3)(a). Also note that the frequency deviation of the IF's of the eigenvectors is symmetric about a central mode as depicted in Fig. (3)(b).

Orthogonality and the self-adjoint form of the operator $\mathcal{L}$ have specific implications in terms of signal processing of the frequency modulated eigenvector: (a) the eigenvalues of $\mathcal{L}$ are both real and positive and can be put into an ascending order, where the lower eigenvalues correspond to IF's of modes with more zero-crossings and the higher eigenvalues correspond to IF's at low frequency or fewer zero-crossings, (b) the eigenvectors of $\mathcal{L}$ will not be subject to distortion of the IF by the system $\mathcal{L}$, which is in direct contrast to the AM–FM demodulation algorithms using the quasi-eigenfunction approximation and incur significant error, (c) polynomial compositions of the FM system operator $\mathcal{L}$ can be used to process the eigenvectors in a manner analogous to digital filter design.

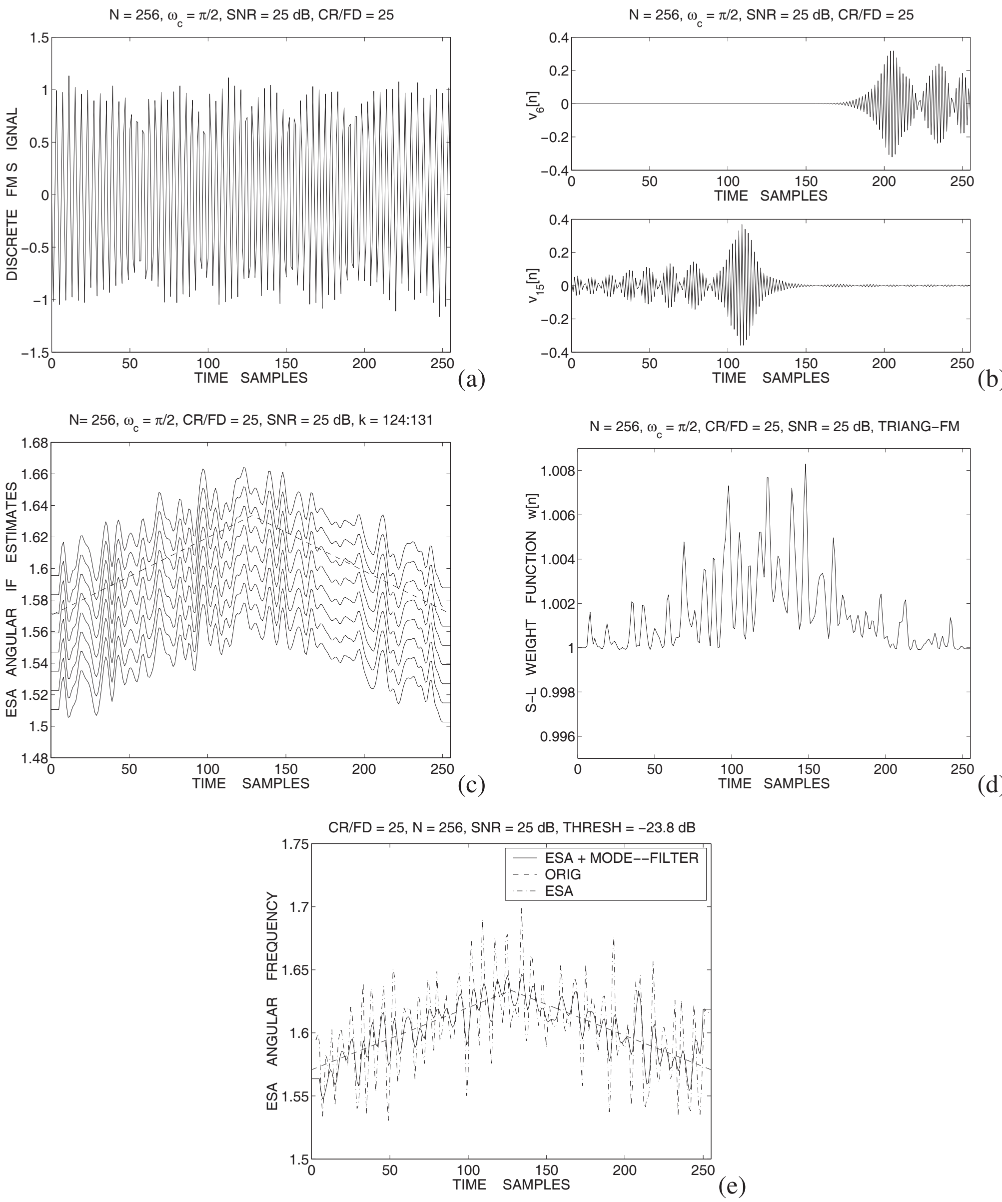

Fig. 4. FM orthogonal mode decomposition in noise using the generalized SVD version of Eq. (11): (a) noisy FM signal with SNR = 25 dB, (b,c) ESA IF and envelope estimates of selected eigenvectors, where the dashed line represents the ESA-IF estimate of the FM signal in part (a), (d) corresponding discrete S-L weight function, and (e) ESA estimate after FM mode rejection below a threshold of -23.8 dB.

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