DISCRETE GAUSS-HERMITE FUNCTIONS AND EIGENVECTORS OF THE CENTERED DISCRETE FOURIER TRANSFORM

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ABSTRACT

Existing approaches to furnishing a basis of eigenvectors for the discrete Fourier transform (DFT) are based upon defining tridiagonal operators that commute with the DFT. In this paper, motivated by ideas from quantum mechanics in finite dimensions, we define a symmetric matrix that commutes with the centered DFT, thereby furnishing a basis of eigenvectors for the DFT. We show that these eigenvectors in the limit converge to Gauss-Hermite (G-H) functions and that the eigenvalue spectrum of the commutor provides a very good discrete approximation to that of the continuous G-H differential operator.

Keywords: Gauss-Hermite functions, eigenvalues, eigenvectors, discrete Fourier transform, finite-space quantum mechanics.

1. INTRODUCTION

The continuous Fourier integral transform of a finite energy signal is defined via:

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) \exp(-j\omega t)\, dt = \mathcal{F}(x(t)).$$

Gaus–Hermite (G-H) functions defined by:

$$H_n(t) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} h_n(t) \exp\left(-\frac{t^2}{2}\right),$$

where $h_n(t)$ are the Hermite polynomials, are solutions to the second-order Hermite differential equation:

$$\frac{d^2x}{dt^2} - (t^2 + \lambda)x(t) = 0.$$

and eigenfunctions of the Hermite differential operator:

$$\mathcal{H}(x(t)) = (\mathcal{D}^2 - t^2\mathcal{I})x(t) = -(2n+1)x(t).$$

with a corresponding eigenvalue of $\lambda_n = -(2n+1)$. They are also eigenfunctions of the Fourier integral operator:

$$\mathcal{F}(H_n(t)) = \exp\left(-jn\frac{\pi}{2}\right) H_n(t),$$

with a corresponding eigenvalue of $\lambda_n = \exp(-jn\pi/2)$. These G-H functions are also eigenfunctions of the *fractional Fourier transform* (FRFT) defined via:

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t) \sum_{n=-\infty}^{\infty} \exp(-jn\alpha) H_n(t) H_n(u) dt.$$

2. EXISTING APPROACHES

The problem of obtaining a complete orthonormal basis of eigenvectors for the *discrete Fourier transform* (DFT) is a problem that is intrinsic to signal processing [1] and has become the focus of several efforts for defining a discrete version of the continuous-time FRFT. Existing approaches for furnishing a basis for the DFT that resemble discrete versions of G-H functions are based on defining a tridiagonal matrix that commutes with the DFT or its centralized version. The first approach called the Harper matrix approach is based on converting the G-H differential equation into a difference equation by replacing derivatives with finite differences [8, 7]. The resulting commuting matrix **S** referred to as the Harper matrix has one zero eigenvalue when N is a multiple of 4. This eigenvalue degeneracy was removed by resorting to the even and odd parts of the eigenvectors. The eigenvectors generated by this approach however do not converge to G-H functions and its eigenvalue spectrum does not possess the linear spacing needed for being considered a candidate for the discrete G-H operator.

The second approach outlined by Grunbaum [6] and later refined by Mugler and Clary [5] is an algebraic method that derives the general form of a tridiagonal matrix that commutes with the centered version of the DFT. As recognized by Grunbaum [6] it is this centered DFT formulation that leads to convergence to G-H functions in the limit. This approach also produces an orthogonal set of eigenvectors for any N. However, the eigenvalue spectrum of the Grunbaum commuting matrix does not have the linear spacing of the continuous G-H operator. Consequently the commuting matrix used in this approach cannot be a candidate for the discrete G-H operator.

Our goal in this paper is to define a discrete version of the Hermite-Gauss differential operator $\mathcal{H}$ that will furnish the basis for the centered version of the DFT matrix and simultaneously have eigenvalues and eigenvectors that very closely resemble those of the continuous G-H operator. Towards this purpose we utilize concepts borrowed from quantum mechanics in finite dimensions in the context of the discrete harmonic oscillator [3, 4]. The eigenvectors extracted from the proposed commuting matrix are then used to define a *discrete Fractional Fourier Transform* (DFRFT) that has the same abilities to represent multicomponent chirp signals as the Grunbaum approach.

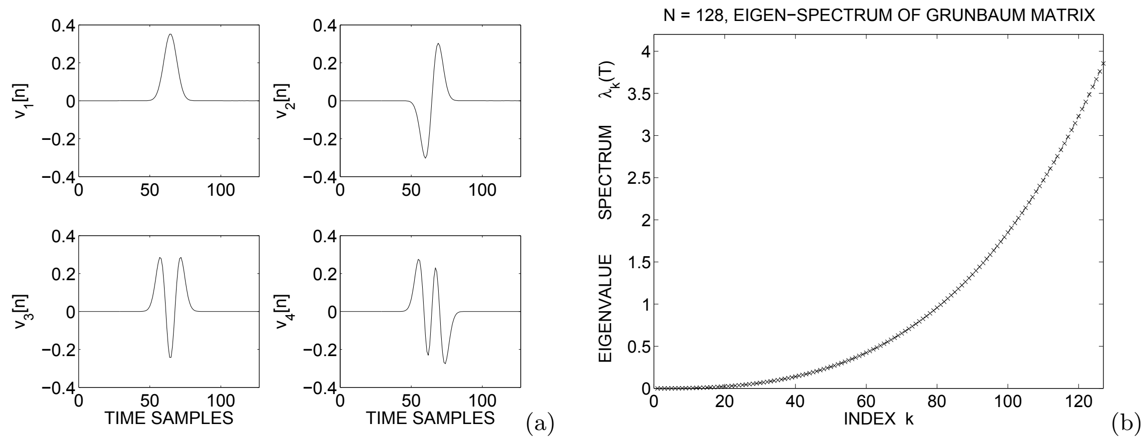

Figure 1: Eigenvectors and eigenvalues of the Grunbaum commuting matrix approach.

3. DEFINING THE COMMUTING MATRIX

Quantum mechanics in the infinite dimensional setting connects a pair of canonical variables such as position and momentum through the Fourier integral transform that is a fourth order involution operation:

$$\mathcal{F} = \exp\left(j\frac{\pi}{4}\left(\hat{p}^2 + \hat{q}^2 - 1\right)\right),$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators that are related through a basis change:

$$\hat{p} = \mathcal{F}\hat{q}\mathcal{F}^{\dagger},\ p = -j\frac{d}{dq}$$

and p, q denote the eigenvalues of their corresponding operators. In the continuous case the expression inside the exponential is exactly the G-H differential operator:

$$q^2 + p^2 = -\frac{d^2}{dq^2} + q^2 = -\mathcal{H}(q).$$

We seek a discrete analog to the operator $\mathcal{H}$. Towards this purpose we adopt Weyl's formulation of the discrete oscillator [3, 4] and define a diagonal matrix $\mathbf{Q} \in \mathbf{R}^{N \times N}$ whose entries are given by:

$$Q_{rr} = q[r] = \sqrt{\frac{2\pi}{N}}r,\quad r = -\frac{(N-1)}{2}, \ldots 0 \ldots \frac{(N-1)}{2}.$$

Then define the centered version of the DFT matrix $\mathbf{W}$ as:

$$\begin{aligned}\{W\}_{mn} &= \frac{1}{\sqrt{N}}\exp\left(-j\frac{2\pi}{N}(m-a)(n-a)\right),\\ a &= \frac{(N-1)}{2},\quad 0 \leq m, n \leq (N-1).\end{aligned}$$

The matrix $\mathbf{P}$ is defined by combining the matrices $\mathbf{Q}$ and $\mathbf{W}$ via a basis change as:

$$\mathbf{P} = \mathbf{W}\mathbf{Q}\mathbf{W}^H$$

The matrices $\mathbf{P}$ and $\mathbf{Q}$ can be interpreted as finite dimensional counterparts of the position and momentum operators and the relation between them and the centered DFT ensures correspondence with the continuous case where the position and momentum of the quantum mechanical oscillator are connected by the continuous Fourier integral operator. The matrix that commutes with the CDFT is then defined as:

$$\mathbf{T}_1 = \mathbf{P}^H\mathbf{P} + \mathbf{Q}^H\mathbf{Q}. \tag{1}$$

4. COMMUTATION PROPERTIES

Substituting the expression for $\mathbf{P}$ into the definition of the commutor we have:

$$\mathbf{T} = \mathbf{W}\mathbf{Q}^H\mathbf{Q}\mathbf{W}^H + \mathbf{Q}^H\mathbf{Q} = (\mathbf{Q}\mathbf{W}^H)^H\mathbf{Q}\mathbf{W}^H + \mathbf{Q}^H\mathbf{Q}.$$

To demonstrate that this matrix indeed commutes with the matrix $\mathbf{W}$ we have:

$$\begin{aligned}\mathbf{TW} &= \mathbf{W}\mathbf{Q}^H\mathbf{Q} + \mathbf{Q}^H\mathbf{Q}\mathbf{W}\\ \mathbf{WT} &= \mathbf{W}\mathbf{Q}^H\mathbf{Q} + \mathbf{W}^2\mathbf{Q}^H\mathbf{Q}\mathbf{W}\end{aligned}$$

Since $q^2[r] = q^2[-r], 0 \leq r \leq (N-1)$, the matrix $\mathcal{J} = \mathbf{W}^2$ has no effect on the matrix $\mathbf{Q}^H\mathbf{Q}$ and we have:

$$\mathbf{W}^2\mathbf{Q}^H\mathbf{Q} = \mathbf{Q}^H\mathbf{Q} \implies [\mathbf{T}, \mathbf{W}] = (\mathbf{TW} - \mathbf{WT}) = \mathbf{0} \tag{2}$$

This implies that the commutor defined in Eq. (1) can be used to furnish the basis of eigenvectors for the centered DFT. If we now define $\mathbf{C}_1 = [\mathbf{Q}, \mathbf{P}]$ and look at the commutator:

$$\begin{aligned}[\mathbf{W}, \mathbf{C}_1] &= [\mathbf{W}, \mathbf{QP} - \mathbf{PQ}]\\ &= \mathbf{W}(\mathbf{QP} - \mathbf{PQ}) - (\mathbf{QP} - \mathbf{PQ})\mathbf{W}.\end{aligned}$$

Substituting the expression for $\mathbf{P}$ into this expression we have:

$$\begin{aligned}[\mathbf{W}, \mathbf{C}_1] &= \mathbf{WQWQW}^H - \mathbf{W}^2\mathbf{QW}^2\mathbf{WQ}\\ &- \mathbf{QWQ} + \mathbf{WQW}^H\mathbf{QW}\\ &= \mathbf{WQWQW}^H + \mathbf{WQW}^H\mathbf{QW}\\ &= \mathbf{WQWQW}^H + \mathbf{WQWW}^2\mathbf{QW}^2\mathbf{W}^H\\ &= \mathbf{0}\end{aligned} \tag{3}$$

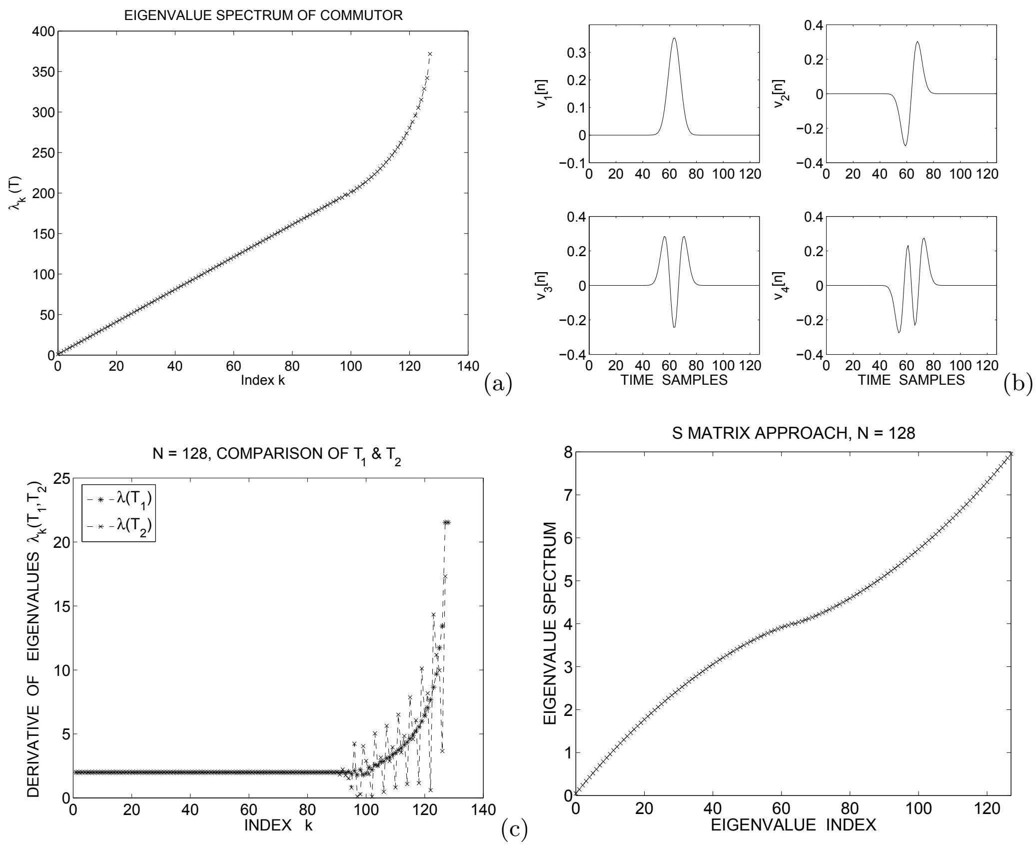

Figure 2: Decomposition of the commutor: (a) eigenvalues of the commutor $\mathbf{T}_1$, (b) eigenvectors of the commutor, (c) comparison of the symmetric difference of the eigenvalues of the matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ for $N = 128$, (d) eigenvalue spectrum of the Harper matrix approach.

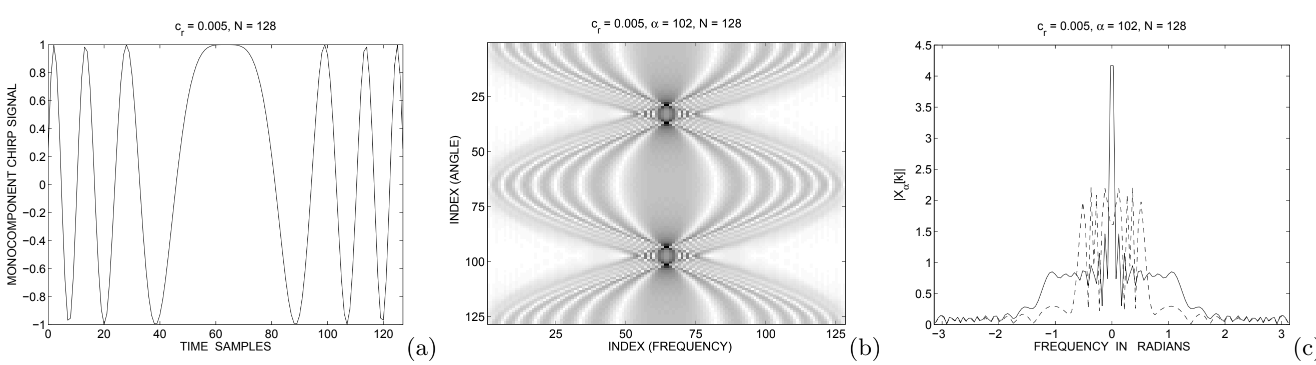

Figure 3: Concentrating a chirp: (a) chirp signal with a chirp rate of $c_r = 0.005$ and zero average frequency, (b) MA-CDFRFT of the chirp signal in (a), (c) cross-section of the MA-CDFRFT through the row containing the peak and the conventional CDFT.

where we have used the property that $q[-r] = -q[r]$, i.e., $\mathbf{W}^2\mathbf{Q}\mathbf{W}^2 = -\mathbf{Q}$. This in turn implies that $\mathbf{C}_1$ and $\mathbf{W}$ share a common basis of eigenvectors. The commutor matrix in its most general form can therefore be written as:

$$\mathbf{T}_2 = c_1(\mathbf{P}^2 + \mathbf{Q}^2) + c_2\mathbf{C}_1^H\mathbf{C}_1 + c_3\mathbf{I}, \tag{4}$$

where the constants c_1, c_2, c_3 are chosen appropriately to obtain a matrix with a non-degenerate eigenvalue spectrum that is as close as possible to that of the G-H operator $\mathcal{H}$. Figure (2) depicts the eigenvalues of the two commuting matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ for $N = 128$ and for $c_1 = 1$, $c_2 = -c_3 = -\frac{\pi^2}{N^2}$. Note that the largest eigenvalue of the commutor $\mathbf{T}_2$ is significantly smaller than that of the $\mathbf{T}_1$. This implies that the effect of the term $[\mathbf{Q}, \mathbf{P}]$ is to truncate the eigenvalue spectrum of the commutor at the end. The eigenvectors of $\mathbf{T}_2$ are the same as that of $\mathbf{T}_1$ except for a reordering of the eigenvectors.

5. CONVERGENCE TO THE G-H OPERATOR

Here we show that the commutor matrix $\mathbf{T}_1$ defined here in the continuous limit converges to the Gauss-Hermite operator. This can be demonstrated by looking at:

$$P_r(\mathbf{x}) = \sum_{s=0}^{N-1} \sum_{\tilde{m}=-(N-1)/2}^{(N-1)/2} \sqrt{\frac{2\pi}{N}} \frac{\tilde{m}}{N} \exp\left(j\frac{2\pi}{N}\tilde{m}(r-s)\right) x[s],$$

To enable the passage to the continuous limit, we define the following quantities:

$$q = \sqrt{\frac{2\pi}{N}}\tilde{r}, \quad \tilde{q} = \sqrt{\frac{2\pi}{N}}\tilde{s}, \; p = \sqrt{\frac{2\pi}{N}}\tilde{m},$$

where the tilde expressions correspond to centralized variables. Consequently we can write the sum as:

$$\begin{aligned}\lim_{N\to\infty} P_r\mathbf{x} &= \int_{-\infty}^{\infty} d\tilde{q} \frac{1}{2\pi} \int_{-\infty}^{\infty} p \exp\left(jp(q-\tilde{q})\right) dp x(\tilde{q}) \\ &= -j\frac{d}{dq}x(q).\end{aligned}$$

Therefore the commuting matrix in the limit can be written as:

$$\begin{aligned}\lim_{N\to\infty} \mathbf{T}\mathbf{x} &= \lim_{N\to\infty} (\mathbf{P}^2 + \mathbf{Q}^2)x(q) \\ &= -\frac{d^2}{dq^2}x(q) + q^2x(q) = -\mathcal{H}(x(q))\end{aligned}$$

The commuting matrix $\mathbf{T}_1$ in the limit therefore converges to the G-H differential operator. Grunbaum in his algebraic approach has shown that a linear combination of his commutor and identity converges to the G-H differential operator. The difference in the proposed approach is that the eigenvalue spectrum of the proposed commuting matrix is closer to the linear eigenvalue spectrum of the continuous G-H operator than that of Grunbaum approach or the Harper matrix approach as can be observed in Fig. (1) and Fig. (2).

Now that we have defined the symmetric commuting matrix that shares a basis of eigenvectors with the CDFT that also serves as a discrete approximation to the G-H operator, we can define a DFRFT based on the set of derived eigenvectors via:

$$\mathbf{A}_\alpha = \mathbf{W}^{\frac{2\alpha}{\pi}} = \mathbf{V}\mathbf{\Lambda}^{\frac{2\alpha}{\pi}}\mathbf{V}^H,$$

where $\mathbf{V}$ is the set of eigenvectors extracted from either $\mathbf{T}_1$ or $\mathbf{T}_2$. An efficient algorithm for computing the multiangle version of the DFRFT (MA-CDFRFT) for a general eigenvector set utilizing the DFT was developed in [2]. The associated chirp-rate/frequency representation was shown to be a very useful tool for analysis of multicomponent chirp signals. The eigenvectors derived from the proposed approach have the same symmetries as those of the Grunbaum eigenvectors and also have same ability to concentrate and represent a chirp as illustrated in Fig. (3).

6. CONCLUSION

In this paper, using ideas inspired from quantum mechanics in finite dimensions, we have presented a symmetric commuting matrix framework that furnishes a full orthogonal basis of eigenvectors for the centered DFT. We have shown that this converges to the Gauss-Hermite differential operator in the limit and have also shown that the eigenvalue spectrum of commutor very closely resembles that of the differential operator, a feature not shared by either of the existing DFRFT approaches. We have also demonstrated that the associated DFRFT has the same capability of representing chirp signals seen in the Grunbaum approach.

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