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On discrete Gauss–Hermite functions and eigenvectors of the discrete Fourier transform

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ABSTRACT

The problem of furnishing an orthogonal basis of eigenvectors for the *discrete Fourier transform* (DFT) is fundamental to signal processing and also a key step in the recent development of discrete fractional Fourier transforms with projected applications in data multiplexing, compression, and hiding. Existing solutions toward furnishing this basis of DFT eigenvectors are based on the commuting matrix framework. However, none of the existing approaches are able to furnish a commuting matrix where both the eigenvalue spectrum and the eigenvectors are a close match to corresponding properties of the continuous differential Gauss–Hermite (G–H) operator. Furthermore, any linear combination of commuting matrices produced by existing approaches also commutes with the DFT, thereby bringing up issues of uniqueness.

In this paper, inspired by concepts from quantum mechanics in finite dimensions, we present an approach that furnishes a basis of orthogonal eigenvectors for both versions of the DFT. This approach furnishes a commuting matrix whose eigenvalue spectrum is a very close approximation to that of the G–H differential operator and in the process furnishes two generators of the group of matrices that commute with the DFT.

1. Introduction

Conventional Fourier analysis treats frequency and time as orthogonal variables and consequently is only suitable for the analysis of signals with stationary frequency content. The *fractional Fourier transform* (FRFT), an angular generalization of the Fourier transform, enables the analysis of waveforms, such as chirps, that possess time–frequency coupling. The continuous Fourier integral transform of a finite energy signal is defined via

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) \exp(-j\omega t)\, dt = \mathscr{F}(x(t)).$$

Gauss–Hermite (G–H) functions defined by

$$H_n(t) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} h_n(t) \exp\left(-\frac{t^2}{2}\right),$$

where $h_n(t)$ is the nth-order Hermite polynomial, are solutions to the second-order differential equation

$$\frac{d^2x}{dt^2} - (t^2 + \lambda)x(t) = 0.$$

They are eigenfunctions of the G–H differential operator

$$\mathscr{H}(x(t)) = (\mathscr{D}^2 - t^2\mathscr{I})x(t) = -(2n+1)x(t),$$

with a corresponding eigenvalue of $\lambda_n = -(2n+1)$, where $\mathscr{D}, \mathscr{I}$ denote the derivative and identity operators. They are also eigenfunctions of the Fourier integral operator

$$\mathscr{F}(H_n(t)) = \exp\left(-jn\frac{\pi}{2}\right) H_n(t),$$

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with a corresponding eigenvalue of $\lambda_n = \exp(-\mathrm{j}n\pi/2)$. These G–H functions are also eigenfunctions of the FRFT defined via

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t)K_\alpha(t,u)\,\mathrm{d}t,$$

$$K_\alpha(t,u) = \sum_{n=-\infty}^{\infty} \exp(-\mathrm{j}n\alpha)H_n(t)H_n(u).$$

Quantum mechanics as it pertains to the harmonic oscillator connects the canonical variables, position, and momentum through the Fourier integral operator $\mathscr{F}$ via [1,2]

$$\mathscr{F} = \exp\left(\mathrm{j}\frac{\pi}{4}(\hat{p}^2 + \hat{q}^2 - 1)\right),$$

where $\hat{q}$ and $\hat{p}$ are the position and momentum operators that are related through a similarity transformation [2]:

$$\hat{p} = \mathscr{F}\hat{q}\mathscr{F}^\dagger, \quad \hat{p} = -\mathrm{j}\frac{\mathrm{d}}{\mathrm{d}q},$$

where $\mathscr{F}^\dagger$ denotes the Hermitian adjoint of $\mathscr{F}$ and p, q denote the eigenvalues of their corresponding operators. In the continuous case the expression inside the exponential is exactly the G–H differential operator:

$$(\hat{q}^2 + \hat{p}^2)x(q) = -\frac{\mathrm{d}^2}{\mathrm{d}q^2}x(q) + q^2x(q) = -\mathscr{H}(x(q)).$$

Consequently, G–H functions are also the eigenfunctions of the quantum harmonic oscillator. The position and momentum[1] operators furthermore do not commute and their commutator corresponds to the identity [2]

$$[\hat{q},\hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = \mathrm{j}\mathbf{I}. \tag{1}$$

The connection between the G–H operator and the Fourier transform $\mathscr{F}$ can be further expressed as

$$(\hat{p}^2 + \hat{q}^2 - 1)x(q) = -(\mathscr{H} + 1)(x(q)) = \frac{-4\mathrm{j}}{\pi}\log\mathscr{F}(x(q)). \tag{2}$$

This relation implies that in the continuous case the G–H operator is related to the logarithm of the Fourier transform.

2. Prior work

The eigenvalues and eigenvectors of the discrete Fourier transform (DFT) matrix have been of interest from early work [3], where the DFT eigenvalue problem was discussed in detail. Work in [4] outlines an analytical expression for the eigenvectors of the DFT corresponding to distinct eigenvalues. However, this expression involves infinite sums and is not computable. Recent efforts to develop a discrete version of the FRFT have focussed on the DFT and its centralized version and on generating an orthogonal basis of eigenvectors for the DFT by furnishing a commuting matrix that has a non-degenerate eigenvalue spectrum and shares a common basis of eigenvectors with the DFT. These approaches, however, do not yield a unique discretization since the sum or the product of matrices that commute with the DFT also commutes with the DFT. Our goal in this paper is to define a discrete equivalent of the G–H differential operator $\mathscr{H}$ that will furnish the basis for both the centered and off-centered versions of the DFT matrix. This framework will enable the definition of a discrete version of the FRFT and also serve as the discrete equivalent of the G–H operator with eigenvalues and eigenvectors that closely resemble those of the continuous counterpart.

Existing approaches toward obtaining an orthogonal basis of eigenvectors for the DFT can be grouped into two basic categories. The first approach called the **S** matrix approach or the Harper matrix approach [5–7] is based on replacing the derivatives in the G–H differential equation with finite differences thereby converting the differential equation into a difference equation. The approach furnishes an almost tridiagonal (Harper) matrix that commutes with the DFT matrix and consequently furnishes a basis of orthogonal DFT eigenvectors when N is not a multiple of four. Other numerical approaches that use orthogonal projections to furnish the eigenvectors of the Harper matrix **S** have been recently investigated in [8]. As shown in [5], the Harper matrix does not converge to the G–H operator in the limit, but rather to the Mathieu differential operator. Furthermore, the eigenvalue spectrum is not the linear spectrum with uniform spacing needed for consideration as the discrete G–H operator as described in Fig. 1.

The second approach pioneered by Grünbaum [9] and later refined in [10] is an algebraic approach that furnishes tridiagonal matrices that commute with both the centered and the off-centered versions of the DFT. It was shown in [9] that the commuting matrix in the limit converges to the G–H differential operator. However, the eigenvalues of the matrix do not exhibit the uniform integer spacing needed to be considered a viable candidate for the discrete G–H operator. Since the sum and the product of the different commuting matrices also commute with the DFT, numerous other commuting matrices can be furnished and the question of uniqueness of the commuting matrix approach arises. Recently, a combination of the commuting matrices from the Harper and Grünbaum matrix approaches have been used to furnish a basis of eigenvectors for the DFT [11,12], where the squared norm of error between the eigenvectors and the corresponding discrete G–H function was used to quantify the accuracy of the eigenvectors.

In this paper, we adopt a physical approach to develop a unique commuting matrix framework for both the CDFT and the DFT that: (a) furnishes a full orthogonal basis of eigenvectors resembling G–H functions via the eigenvalue problem for generalized K-symmetric matrices [13], (b) has an eigenvalue spectrum very close to that of $\mathscr{H}$, (c) converges to $\mathscr{H}$ in the limit, and (d) is quadratic in position and momentum analogous to the Hamiltonian of the quantum-mechanical harmonic oscillator.

3. Discrete G–H operator

3.1. Centered case

Toward formulating a physically meaningful, discrete, and computable version of the G–H operator, we borrow

[1] Although the analysis done here is done in terms of the quantum-mechanical variables $\hat{p}, \hat{q}$, they can be any pair of canonical variables such as time and frequency.

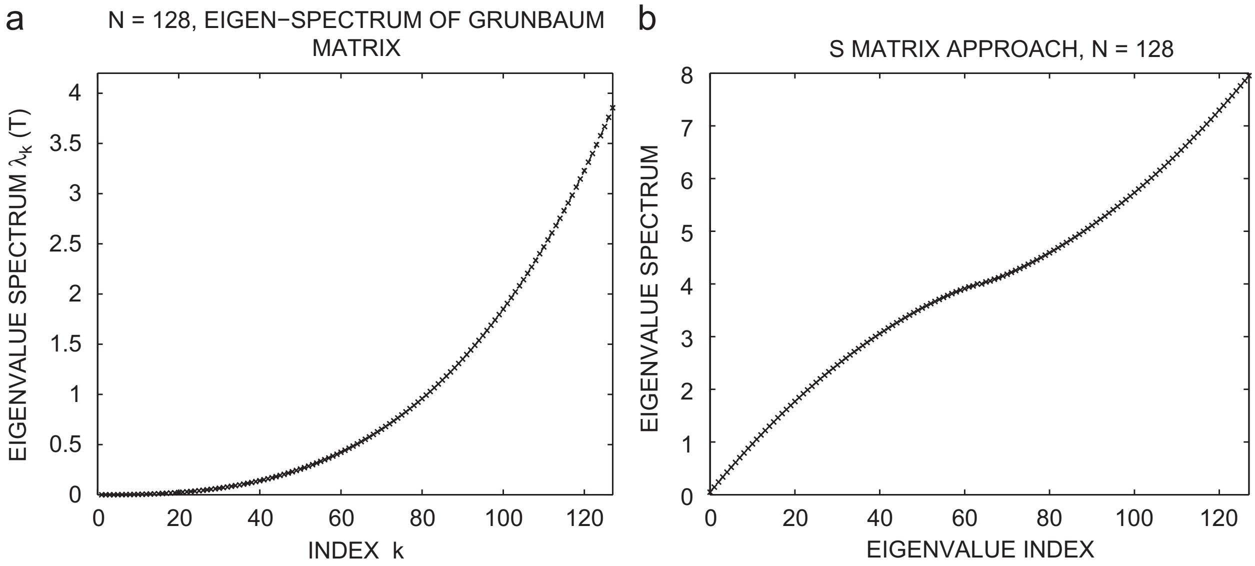

Fig. 1. Comparison of the eigenvalue spectra: (a) eigenvalues of the Grünbaum commuting matrix approach in [10] for $N=128$, (b) eigenvalues of the Harper matrix approach in [5] for $N=128$.

some ideas from quantum mechanics in finite dimensions [1]. First we define a diagonal matrix $\mathbf{Q}\in\mathbf{R}^{N\times N}$ whose entries are given by

$$Q_{rr}=q[r]=\sqrt{\frac{2\pi}{N}}r,\quad |r|\leqslant\frac{(N-1)}{2}. \tag{3}$$

This operator constitutes the discrete equivalent of the position operator in finite dimensional space. We then define the DFT matrix and its centered version as

$$\{W_{oc}\}_{kn}=\frac{1}{\sqrt{N}}\exp\left(-\mathrm{j}\frac{2\pi}{N}nk\right),\quad 0\leqslant n,\ \ k\leqslant(N-1),$$
$$\{W_c\}_{kn}=\frac{1}{\sqrt{N}}\exp\left(-\mathrm{j}\frac{2\pi}{N}(n-c)(k-c)\right),$$

with $c=(N-1)/2$. For purposes of discussion both these matrices will be denoted by $\mathbf{W}$. The version discussed will be clear from the context. The matrix $\mathbf{P}$ is defined by combining the matrices $\mathbf{Q}$ and $\mathbf{W}$ as

$$\mathbf{P}=\mathbf{W}\mathbf{Q}\mathbf{W}^{\mathrm{H}}, \tag{4}$$

where $\mathbf{W}^{\mathrm{H}}$ denotes the Hermitian adjoint of $\mathbf{W}$. This matrix constitutes the discrete equivalent of the momentum operator in finite dimensions. Following the approach in the continuous case, the discrete Hamiltonian matrix $\mathbf{T}_1$ that commutes with the CDFT is then defined as

$$\mathbf{T}_1=\mathbf{P}^{\mathrm{H}}\mathbf{P}+\mathbf{Q}^{\mathrm{H}}\mathbf{Q}=\mathbf{P}^2+\mathbf{Q}^2. \tag{5}$$

We have shown in [14] that this matrix commutes with the CDFT and furnishes a basis of orthogonal CDFT eigenvectors as illustrated in Fig. 2. To further aid in our understanding of the commutation properties, we define a matrix $\mathbf{A}$ to be *centrosymmetric* or $\mathbf{J}$-symmetric if it satisfies

$$\mathbf{JAJ}=\mathbf{A},$$

where $\mathbf{J}=\mathbf{W}^2$ is the *exchange matrix* of the same dimensions as $\mathbf{A}$ with ones along the anti-diagonal. In a similar vein, a matrix $\mathbf{A}$ is defined to be $\mathbf{J}$-anti-symmetric if

$$\mathbf{JAJ}=-\mathbf{A}.$$

Since $q^2[r]=q^2[-r], 0\leqslant r\leqslant(N-1)$, from the form of $\mathbf{Q}$ defined in Eq. (3), the matrix $\mathbf{Q}^2$ is centrosymmetric:

$$\mathbf{JQ}^2\mathbf{J}=\mathbf{Q}^2\longleftrightarrow\mathbf{W}^2\mathbf{Q}^2\mathbf{W}^2=\mathbf{Q}^2. \tag{6}$$

We have also shown in [14] that $\mathbf{C}_1=[\mathbf{Q},\mathbf{P}]$ commutes with the CDFT sharing a basis of eigenvectors since

$$\mathbf{W}^2\mathbf{Q}\mathbf{W}^2=-\mathbf{Q} \tag{7}$$

or, in other words, the matrix $\mathbf{Q}$ is $\mathbf{J}=\mathbf{W}^2$-anti-symmetric. Eqs. (6) and (7) constitute constraints on the position matrix so that the resultant $\mathbf{T}_1$ and $\mathbf{C}_1$ both commute with the CDFT. Eq. (7) is the stricter condition since it implies the relation in Eq. (6). Although this analysis was for the CDFT, it applies to the conventional DFT also.

A generalized form of the commuting matrix is obtained by first defining a modified position matrix and the corresponding momentum matrix via

$$\{\mathbf{Q}_a\}_{mn}=\begin{cases}\{\mathbf{Q}\}_{mn}, & m=n, 0\leqslant m,n\leqslant N-1,\\ \mathrm{j}a, & m=0, n=N-1,\\ -\mathrm{j}a, & m=N-1, n=0,\end{cases}\qquad \mathbf{P}_a=\mathbf{W}\mathbf{Q}_a\mathbf{W}^{\mathrm{H}}. \tag{8}$$

The more general additive form[2] of the commuting matrix expressed in terms of these modified operators is

$$\mathbf{T}_2=(\mathbf{Q}_a-\mathrm{j}\mathbf{P}_a)(\mathbf{Q}_a+\mathrm{j}\mathbf{P}_a)+\mathbf{I}=\mathbf{P}_a^2+\mathbf{Q}_a^2+\mathrm{j}[\mathbf{Q}_a,\mathbf{P}_a]+\mathbf{I}. \tag{9}$$

This form of the commuting matrix invokes both terms $\mathbf{T}_1$ and $\mathbf{C}_1$ that commute with the CDFT. The scalar off-diagonal parameter[3] a removes the degeneracy present with a diagonal $\mathbf{Q}$. The motivation behind using a complex

[2] Product terms such as $(\mathbf{P}^2+\mathbf{Q}^2)[\mathbf{Q},\mathbf{P}]$ and other functions that also commute with $\mathbf{W}$ are not included. The intention is to retain just terms that are quadratic in $\mathbf{Q}$ and $\mathbf{P}$ analogous to the energy of the standard harmonic oscillator since we desire a second-order differential equation in the limit.

[3] While other forms of correcting the degeneracy are possible, this represents the simplest form of corner correction.

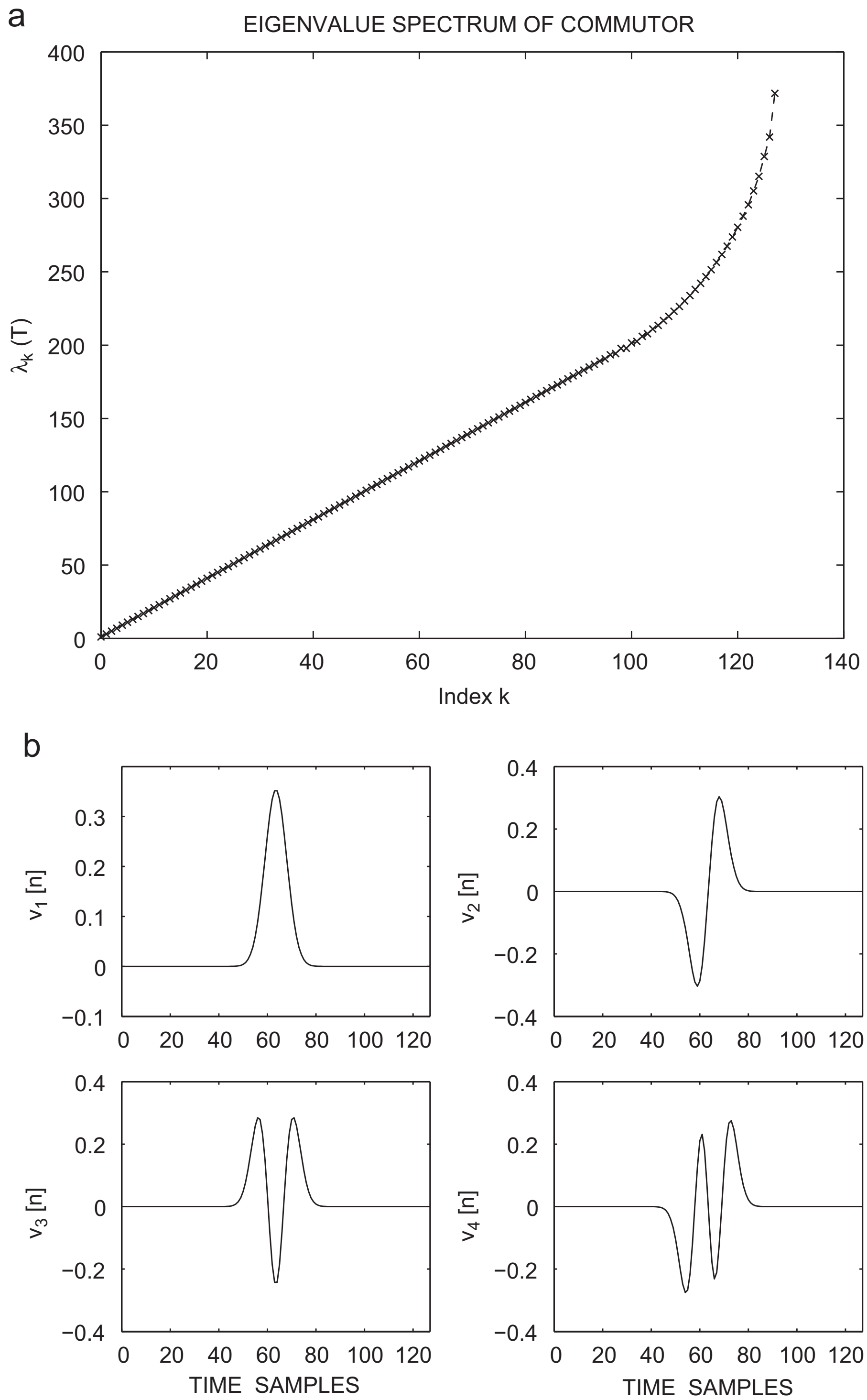

Fig. 2. Eigenvalue decomposition of the commutator for the CDFT: (a) sorted eigenvalues of the commutator $\mathbf{T}_1$ for $N=128$, (b) selected symmetric and skew-symmetric eigenvectors of the commutator for $k=0,1,2,3$.

$\mathbf{Q}$ matrix is so that the commutation conditions and Hermitian symmetry of $\mathbf{Q}$ are both satisfied. Fig. 3 depicts the eigenvalue spectrum of $\mathbf{T}_2$ for $N=129$ and $a=8$. Comparing the eigenvalue spectrum of $\mathbf{T}_2$ and $\mathbf{T}_1$ we see that the deviation from ideal linear spectrum is larger for $\mathbf{T}_2$ than for $\mathbf{T}_1$. Another key observation is the fact that the matrices $\mathbf{P}^2$, $\mathbf{QP}$, $\mathbf{PQ}$ are also $\mathbf{J}=\mathbf{W}^2$-symmetric since

$$\mathbf{JP}^2\mathbf{J}=\mathbf{W}^2\mathbf{WQ}^2\mathbf{W}^{\mathrm{H}}\mathbf{W}^2=\mathbf{W}^3\mathbf{Q}^2\mathbf{W}=\mathbf{WW}^2\mathbf{Q}^2\mathbf{W}^2\mathbf{W}^{\mathrm{H}}=\mathbf{P}^2,$$

$$\begin{aligned}\mathbf{JQPJ}&=\mathbf{W}^2\mathbf{QWQW}^{\mathrm{H}}\mathbf{W}^2=(\mathbf{W}^2\mathbf{QW}^2)\mathbf{W}(\mathbf{W}^2\mathbf{QW}^2)\mathbf{W}^{\mathrm{H}}\\&=\mathbf{QWQW}^{\mathrm{H}}=\mathbf{QP},\end{aligned}$$

$$\mathbf{JPQJ}=\mathbf{W}^2\mathbf{WQW}^{\mathrm{H}}\mathbf{QW}^2=-\mathbf{WQWQW}^2=\mathbf{WQW}^{\mathrm{H}}\mathbf{Q}=\mathbf{PQ}, \tag{10}$$

where we have used the fact that the position operator $\mathbf{Q}$ defined in Eq. (3) is $\mathbf{J}=\mathbf{W}^2$-anti-symmetric and $\mathbf{Q}^2$ is $\mathbf{J}=\mathbf{W}^2$-symmetric. A direct consequence of this observation

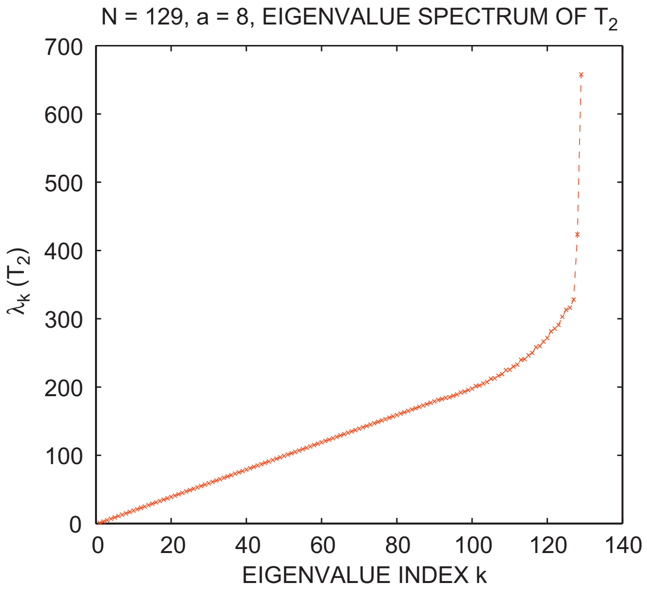

Fig. 3. Decomposition of $\mathbf{T}_2$: eigenvalue spectrum of the more general form of the commuting matrix $\mathbf{T}_2$ for $N = 129$ and $a = 8$. Truncation effects manifest as the deviation from the linear spectrum seen at the end.

is the result that the commuting matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are both symmetric $\mathbf{J} = \mathbf{W}^2$-symmetric matrices. In turn the framework from [15] on centrosymmetric matrices can be applied to furnish orthonormal symmetric or skew-symmetric eigenvectors.

3.2. Convergence to the G–H operator

Here we show that the commutor matrix $\mathbf{T}_1$ converges in the continuous limit to the operator $\mathscr{H}$. This can be demonstrated by looking at

$$\{\mathbf{P}(\mathbf{x})\}_r = \sum_{s=0}^{N-1} P_{rs}x[s],$$

$$0 \leqslant r \leqslant N-1 = \sum_{s=0}^{N-1} \sum_{m=-(N-1)/2}^{(N-1)/2} \sqrt{\frac{2\pi}{N}}\frac{m}{N} \exp\left(\mathrm{j}\frac{2\pi}{N}m(r-s)\right)x[s].$$

To enable the passage to the continuous limit, we can define the following quantities

$$q = \sqrt{\frac{2\pi}{N}}r, \quad \tilde{q} = \sqrt{\frac{2\pi}{N}}s,$$

$$p = \sqrt{\frac{2\pi}{N}}m \Rightarrow \mathrm{d}p = \mathrm{d}q = \mathrm{d}\tilde{q} = \sqrt{\frac{2\pi}{N}}.$$

Consequently, we can write the sum as

$$\begin{aligned}\lim_{N\to\infty} \{\mathbf{Px}\}_r &= \sum_{s=0}^{N-1} \sum_{m=-(N-1)/2}^{(N-1)/2} \sqrt{\frac{2\pi}{N}}\frac{m}{N} \exp\left(\mathrm{j}\frac{2\pi}{N}m(r-s)\right)x[s] \\ &= \int_{-\infty}^{\infty} \mathrm{d}\tilde{q}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} p\exp(\mathrm{j}p(q-\tilde{q}))\,\mathrm{d}p\,x(\tilde{q}) \\ &= -\mathrm{j}\frac{\mathrm{d}}{\mathrm{d}q}x(q).\end{aligned}$$

Therefore, the commuting matrix $\mathbf{T}_1$ in the limit can be written as

$$\lim_{N\to\infty} \mathbf{Tx} = \lim_{N\to\infty} (\mathbf{P}^2 + \mathbf{Q}^2)x(q) = -\frac{\mathrm{d}^2}{\mathrm{d}q^2}x(q) + q^2x(q) = -\mathscr{H}(x(q)).$$

If we now consider the commutator that constitutes the second part of the commuting matrix $\mathbf{T}_2$ we have

$$\begin{aligned}\lim_{N\to\infty} [\mathbf{Q},\mathbf{P}]\mathbf{x} &= [\hat{\mathbf{q}},\hat{\mathbf{p}}]x(q) = (\hat{\mathbf{q}}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{q}})x(q) \\ &= \hat{\mathbf{q}}\left(-\mathrm{j}\frac{\mathrm{d}x}{\mathrm{d}q}\right) + \mathrm{j}\frac{\mathrm{d}}{\mathrm{d}q}(qx(q)) = jx(q). \qquad (11)\end{aligned}$$

This in turn implies that the commutator in the limit converges[4] to a multiple of identity, a result that is consistent with Eq. (1). Since the form of $\mathbf{T}_2$ contains both $\mathbf{T}_1$ and $\mathbf{C}_1$ the commuting matrix $\mathbf{T}_2$ also converges to $\mathscr{H}$.

The ideal form of the discrete Hermite operator would be the logarithm of the CDFT [16]:

$$\mathbf{T} = -\frac{4\mathrm{j}}{\pi}\log \mathbf{W} = -\frac{4\mathrm{j}}{\pi}\mathbf{V}\log \mathbf{\Lambda}\mathbf{V}^{\mathrm{H}}.$$

The eigenvalue spectrum of this operator would be exactly the integer valued and uniformly spaced eigenvalue spectrum that we seek. However, the orthogonal eigenvectors needed to compute the matrix logarithm could still be any of the existing ones and consequently this approach is not unique either. Our approach here may be viewed as the best quadratic approximation to the DFT logarithm using a fixed basis. The commuting matrix $\mathbf{T}_2$, that is quadratic in $\mathbf{Q}$ and $\mathbf{P}$ and containing both the terms $\mathbf{T}_1$ and $\mathbf{C}_1$ that commute with the CDFT, constitutes a unique and computable approximation to the logarithm of the CDFT in that: (a) it converges to $\mathscr{H}$ in the limit, (b) its orthogonal eigenvectors resemble sampled versions of G–H functions, (c) its eigenvalue spectrum is close to that of $\mathscr{H}$ and the logarithm of the DFT.

4. The off-centered case

Eqs. (6) and (7) form the backbone of the approach for generating the eigenvectors of the CDFT. However, these equations do not hold for the conventional DFT. Specifically, here the cyclic flip matrix is given by

$$\mathbf{W}^2 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{J}_{N-1} \end{bmatrix} \neq \mathbf{J}_N,$$

where $\mathbf{J}_N$ refers to the exchange matrix of size $N \times N$. If we further explore the implications of Eq. (7) when $\mathbf{W}$ is the conventional DFT, we obtain the relation

$$\{Q\}_{((-m))_N,((-n))_N} = -\{Q\}_{mn}, \quad 0 \leqslant m,n \leqslant (N-1), \qquad (12)$$

where the notation $((.))_N$ denotes the modulo-N representation. This condition needs to be met for the earlier framework to be applied. Obviously, it is not met with the regular DFT and the centered position operator $\mathbf{Q}$ defined before in the case of the CDFT. If we further restrict ourselves to a diagonal $\mathbf{Q}$ then Eq. (12) becomes

$$q[r] = -q[((-r))_N], \quad 1 \leqslant r \leqslant (N-1), \quad q[0] = 0.$$

The implication of Eq. (7) in the context of the DFT and the cyclic flip matrix is that the ideal position operator $\mathbf{Q}$

[4] Convergence here is in the sense of the action of the matrices on a vector that converges to a operator acting on a function. This is consistent with the quantum-mechanical notion that it is the action of an operator and not the operator itself that is the observable.

needs to be *cyclo-centro-anti-symmetric* or $\mathbf{W}^2$-anti-symmetric (see the Appendix). The implication of Eq. (6) on the other hand is that the matrix $\mathbf{Q}^2$ needs to be *cyclo-centrosymmetric* (CCS). Our task there is therefore to define a modified position operator $\tilde{\mathbf{Q}}$ that satisfies Eq. (12). The only possible solution for a diagonal $\tilde{\mathbf{Q}}$ matrix in this case requires the main diagonal to be anti-symmetric after the exclusion of the first element and this would also produce a degenerate position state when N is even, i.e., when $q[N/2+1]=0$. This degenerate position state will manifest as a eigenvalue degeneracy in the commuting matrix when N is even. To remedy this degeneracy we present two approaches that involve modifying the position operator. In the first approach we still require Eq. (12) be satisfied, while in the second approach we only require Eq. (6) to hold.

(1) *First approach*: Let us now define the modified position matrix $\mathbf{Q}$ as

$$\tilde{\mathbf{Q}} = \begin{pmatrix} 0 & \mathbf{a} \\ \mathbf{a}^{\mathrm{T}} & \mathbf{Q}_{N-1} \end{pmatrix}, \tag{13}$$

where $\mathbf{Q}_{N-1}$ is the position matrix of order $(N-1)$ defined for the CDFT and $\mathbf{a}$ is any anti-symmetric vector. The discrete equivalent of the momentum operator is then given by $\tilde{\mathbf{P}} = \mathbf{W}\tilde{\mathbf{Q}}\mathbf{W}^{\mathrm{H}}$, where $\mathbf{W}$ now denotes the conventional DFT matrix. It is easily seen from Eq. (13) and the partitioned form of the cyclic flip matrix that both the conditions that are required of the ideal position operator are met because the sub-matrices $\mathbf{J}_{N-1}$ and $\mathbf{Q}_{N-1}$ satisfy the same relations seen in the case of the CDFT. The commutator of these matrices is given by $\tilde{\mathbf{C}} = [\tilde{\mathbf{Q}}, \tilde{\mathbf{P}}] = \tilde{\mathbf{Q}}\tilde{\mathbf{P}} - \tilde{\mathbf{P}}\tilde{\mathbf{Q}}$. The version of the commuting matrix for the conventional DFT is given by

$$\tilde{\mathbf{T}}_1 = \tilde{\mathbf{P}}^2 + \tilde{\mathbf{Q}}^2. \tag{14}$$

Similar to the case of the CDFT, the commuting matrix $\tilde{\mathbf{T}}_1$ is K-centrosymmetric matrix with $\mathbf{K} = \mathbf{W}^2$ (see the Appendix) since Eq. (6) is satisfied. Consequently, the eigenvectors that result from this matrix will not have the symmetries present in the G–H functions or the eigenvectors from the CDFT. The eigenvectors in particular are conjugate symmetric or anti-symmetric as seen in the Appendix. In addition the eigenvalue spectrum of the commutator exhibits deviations from the eigenvalue spectrum seen in the case of the CDFT at two regions as seen in Fig. 4(a) for the choice of $\mathbf{a} = \mathbf{a}_1 = \mathbf{0}$, due to the eigenvalue degeneracy. Another choice for the asymmetric vector

$$\mathbf{a}_2 = \mathrm{diag}(\mathbf{Q}_{N-1}) = [-(N-2)/2, \ldots, (N-2)/2]^{\mathrm{T}} \tag{15}$$

removes the eigenvalue deviations in the linear region while increasing the deviation at the end as seen in Fig. 4(b).

As with the CDFT, the operator $\tilde{\mathbf{P}}^2 + \tilde{\mathbf{Q}}^2 + \mathrm{j}[\tilde{\mathbf{Q}}, \tilde{\mathbf{P}}]$ has eigenvalue degeneracy. To furnish a non-degenerate commuting matrix we combine the correction factors from both the CDFT in Eq. (8) and the DFT in Eq. (13) to define a modified position operator:

$$\{\tilde{\mathbf{Q}}_a\}_{mn} = \begin{cases} \{\tilde{\mathbf{Q}}\}_{mn}, & m = n, \\ \mathrm{j}\mathbf{a}_2, & m = 1, n = 2, 3, \ldots, N, \tilde{\mathbf{P}}_a = \mathbf{W}\tilde{\mathbf{Q}}_a\mathbf{W}^{\mathrm{H}}, \\ & \tilde{\mathbf{C}}_a = [\tilde{\mathbf{Q}}_a, \tilde{\mathbf{P}}_a], \\ -\mathrm{j}\mathbf{a}_2, & n = 1, m = 2, 3, \ldots, N, \end{cases} \tag{16}$$

where $\mathbf{a}_2$ is the asymmetric vector defined in Eq. (15). The corresponding commuting matrix is

$$\tilde{\mathbf{T}}_2 = \tilde{\mathbf{Q}}_a^2 + \tilde{\mathbf{P}}_a^2 + \mathrm{j}[\tilde{\mathbf{Q}}_a, \tilde{\mathbf{P}}_a] + \mathbf{I}. \tag{17}$$

This generalized commuting matrix, as with the CDFT, is also $\mathbf{W}^2$-symmetric and consequently can furnish orthogonal conjugate symmetric or anti-symmetric eigenvectors as shown in the Appendix. Fig. 5(a) depicts the eigenvalue spectrum of the matrix $\tilde{\mathbf{T}}_2$ for $N = 128$. Note that the deviation from the linear spectrum has been pushed to the end where the deviation now is larger than that seen for the commutator $\mathbf{T}_1$.

(2) *Second approach*: Suppose we relax the requirements so that the modified position operator just needs to satisfy Eq. (6). Then we can remedy the degeneracy by choosing a modified position operator of the form

$$\tilde{\mathbf{Q}} = \left(\sqrt{\frac{2\pi}{N}}\right)\mathbf{W}^2\,\mathrm{diag}(0, 1, 2, \ldots, N-1) = \sqrt{\frac{2\pi}{N}}\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{J}_{N-1}\boldsymbol{\Lambda} \end{pmatrix}, \tag{18}$$

where $\boldsymbol{\Lambda} = \mathrm{diag}(1, 2, 3, \ldots, N-1)$. It is easy to see that Eq. (6) is met with this form of the position operator:

$$\mathbf{J}_{N-1}\boldsymbol{\Lambda}\mathbf{J}_{N-1}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\mathbf{J}_{N-1}\boldsymbol{\Lambda}\mathbf{J}_{N-1}$$

and consequently the corresponding commuting matrix

$$\tilde{\mathbf{T}}_1 = N\pi\mathbf{I} - \tilde{\mathbf{P}}^2 - \tilde{\mathbf{Q}}^2 \tag{19}$$

commutes with the DFT matrix. The eigenvalues of the commutator are exactly the same as those seen for the CDFT as seen in Fig. 4(c). The eigenvectors of this commutator are $\mathbf{K}$-symmetric or $\mathbf{K}$-anti-symmetric with $\mathbf{K} = \mathbf{W}^2$ as described in the Appendix, where we outline an algorithm to construct $\mathbf{K}$-symmetric and $\mathbf{K}$-anti-symmetric eigenvectors from the even and odd eigen-subspaces of a generalized K-symmetric matrix. The symmetric involution matrix $\mathbf{K}$ used in the formulation of the $\mathbf{K}$-symmetric matrix framework for computing the eigenvectors of the commutator can be any cyclic permutation matrix that commutes with the DFT and does not need to be $\mathbf{W}^2$ [17].

5. Discrete fractional Fourier analysis

After obtaining an orthogonal basis of DFT eigenvectors, we can now define a *discrete fractional Fourier transform* (DFRFT) using the eigenvectors of the commuting matrices $\mathbf{T}_1$ or $\mathbf{T}_2$ via [18]

$$\mathbf{A}_\alpha = \mathbf{W}^{2\alpha/\pi} = \mathbf{V}\boldsymbol{\Lambda}^{2\alpha/\pi}\mathbf{V}^{\mathrm{H}}. \tag{20}$$

Furthermore, if we discretize the angle so that it can take only rational values, i.e., $\alpha_r = (2\pi/N)r$, $r = 0, 1, 2, \ldots, (N-1)$, we obtain a multiangle chirp-rate/frequency

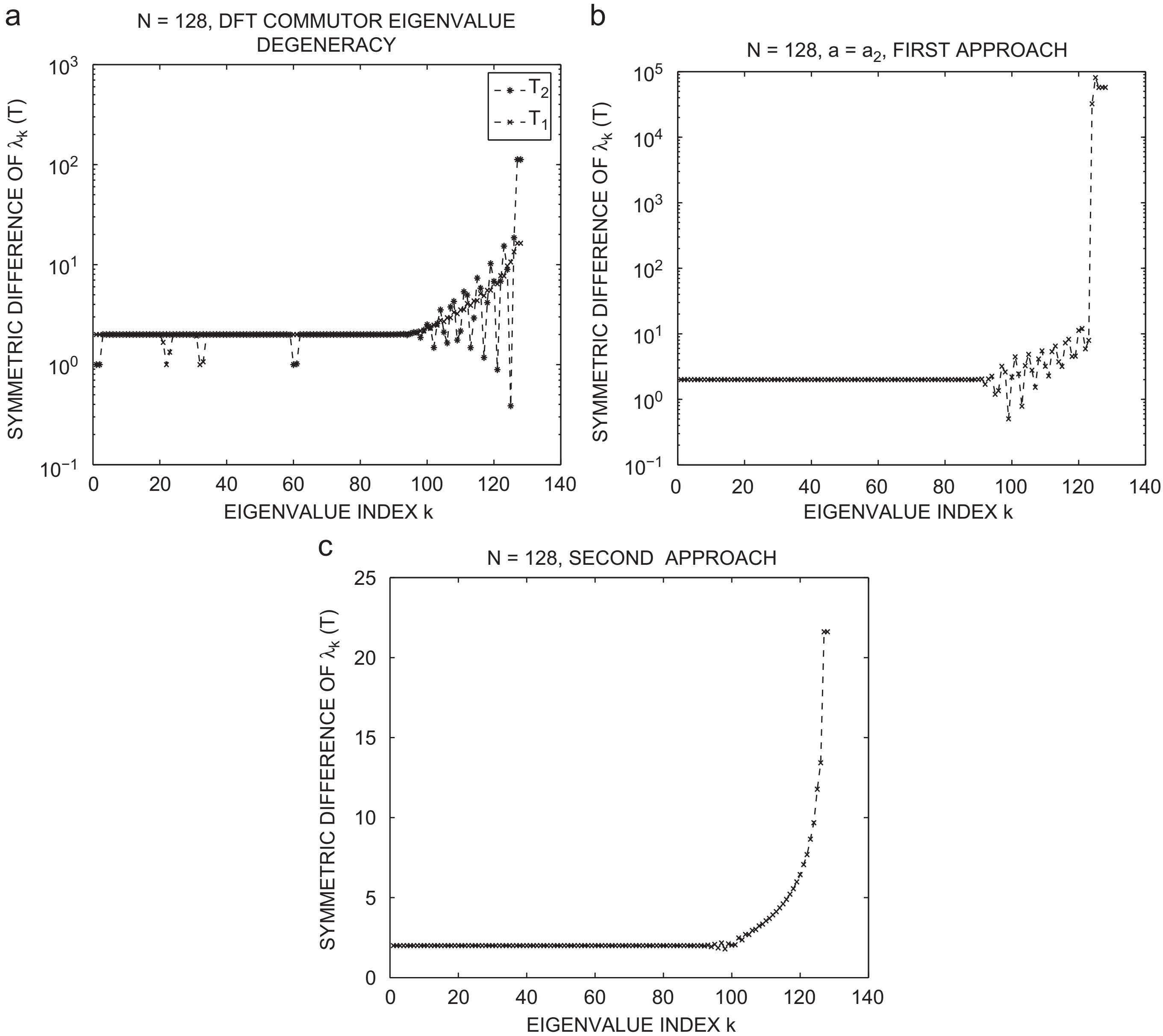

Fig. 4. Eigenvalue spectrum of the commutator $\tilde{\mathbf{T}}_1$ for the DFT: (a) symmetric difference of the eigenvalue spectrum of the commutator depicting eigenvalue degeneracy, (b) symmetric difference of the eigenvalue spectrum for $N = 128$ using $\mathbf{a}_2$ for the asymmetric vector, (c) the corresponding symmetric difference spectrum when just Eq. (6) is satisfied.

representation for the DFRFT computed via the FFT [19]:

$$X_r[k] = \sum_{p=0}^{N-1} v_{kp} \sum_{n=0}^{N-1} x[n] v_{np} \mathrm{e}^{-\mathrm{j}(2\pi/N)pr}, \tag{21}$$

where the notation v_{kp} denotes the pth component of the kth CDFT eigenvector. This transform, referred to as the *multiangle centered discrete fractional Fourier transform* (MA-CDFRFT), computed using the eigenvectors derived from the Grünbaum matrix was shown to be able to concentrate a chirp signal in a few transform coefficients [19]. We now use the same multiangle transform framework but instead use the CDFT eigenvectors derived from $\mathbf{T}_1$.

Consider the example in Fig. 6(a) that describes a monocomponent cosinusoidal chirp signal with zero average frequency of the form: $x[n] = \cos(c_r(n-(N-1)/2)^2)$, $0 \leqslant n \leqslant N-1$, where $c_r = 2\pi/2048 = 0.003068$ is the chirp rate of the signal. Fig. 6(b) depicts the MA-CDFRFT spectrum of the signal for $N = 512$ using a half-spectral zoom implemented via the *chirp Zee transform* (CZT) version of the FFT [20], thereby enabling a zoom into a desired region of the MA-CDFRFT spectrum. The two distinct peaks observed in the MA-CDFRFT spectrum correspond to identical but opposite chirp rates present in the real signal. From the location of the peaks at $r = 180, 332$, and using the approach presented in [19], we can estimate the chirp rates of the peaks to be $\hat{c}_r = \pm 0.003088$ that are very close to the actual chirp rates. The eigenvectors derived from the proposed approach together with the MA-CDFRFT framework are therefore a powerful tool for multicomponent chirp demodulation and for cochannel signal separation and demodulation as seen with the Grünbaum eigenvectors [21].

6. Conclusions

We have presented in this paper a commuting matrix framework, utilizing concepts from quantum mechanics in finite dimensions, for both the centered and conventional DFT. This framework is simultaneously able to furnish a fully orthogonal basis of eigenvectors for both versions of the DFT and a discrete computable version of the G–H differential operator. We have shown that the commuting matrices developed converge in the limit of a large matrix size to the G–H operator. Furthermore, their quadratic form is analogous to the Hamiltonian of the conventional harmonic oscillator. The proposed commuting matrices were shown to produce a distinct linear spectrum with a uniform eigenvalue spacing (except at the end due to truncation effects), analogous to what is seen with the G–H operator. We have demonstrated that the eigenvectors produced from this approach possess the same properties of concentrating chirp signals that the Harper or Grünbaum matrix approaches have. The proposed commuting matrices can also be interpreted as discrete versions of the logarithm of the DFT for a specified basis of eigenvectors.

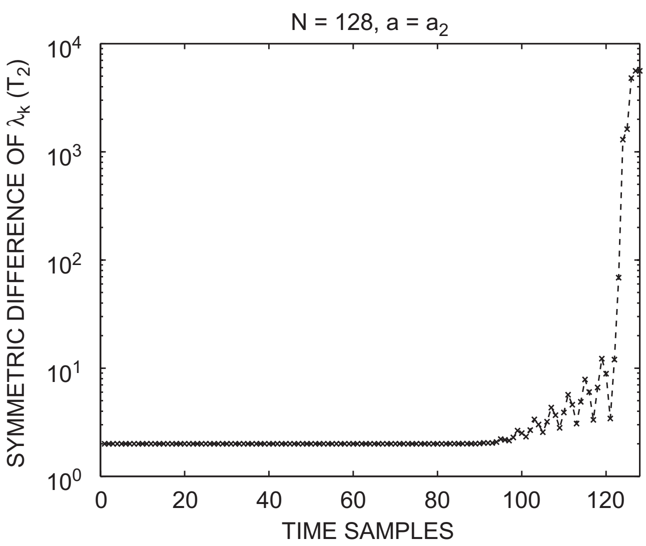

Fig. 5. Eigenvalue decomposition of $\tilde{\mathbf{T}}_2$: symmetric difference of the eigenvalue spectrum of the generalized commutator for $N = 128$. The observed symmetric difference of two for a significant portion of the spectrum is indicative of the uniform integer spacing between eigenvalues a unique feature of the proposed approach.

Appendix A

The algorithm for generating the eigenvectors for both the DFT and and CDFT can be formulated as special cases of the eigenvalue problem for the broader class of generalized centrosymmetric matrices [13]. A matrix $\mathbf{M} \in \mathbf{R}^{N\times N}$ is said to be CCS or centrosymmetric with respect to $\mathbf{K} = \mathbf{W}^2$ if

$$\mathbf{KMK} = \mathbf{M} \quad \text{or} \quad \mathbf{W}^2\mathbf{M}\mathbf{W}^2 = \mathbf{M}.$$

Correspondingly, a matrix $\mathbf{M}$ is *cyclo-centro-anti-symmetric* or anti-symmetric with respect to $\mathbf{K} = \mathbf{W}^2$ if

$$\mathbf{KMK} = -\mathbf{M} \quad \text{or} \quad \mathbf{W}^2\mathbf{M}\mathbf{W}^2 = -\mathbf{M}.$$

More specifically in terms of its elements: $\{M\}_{((-i))_N,((-j))_N} = \pm\{M\}_{ij},\ \ 0 \leqslant i,j \leqslant (N-1)$. If the matrix $\mathbf{M}$ is symmetric and CCS then we can express it in block matrix form as

$$\mathbf{M} = \begin{pmatrix} k & \mathbf{c}^{\mathrm{T}} & \mathbf{c}^{\mathrm{T}}\mathbf{J}_M \\ \mathbf{c} & \mathbf{R}_M & \mathbf{S}_M \\ \mathbf{J}_M\mathbf{c} & \mathbf{J}_M\mathbf{S}_M\mathbf{J}_M & \mathbf{J}_M\mathbf{R}_M\mathbf{J}_M \end{pmatrix}$$

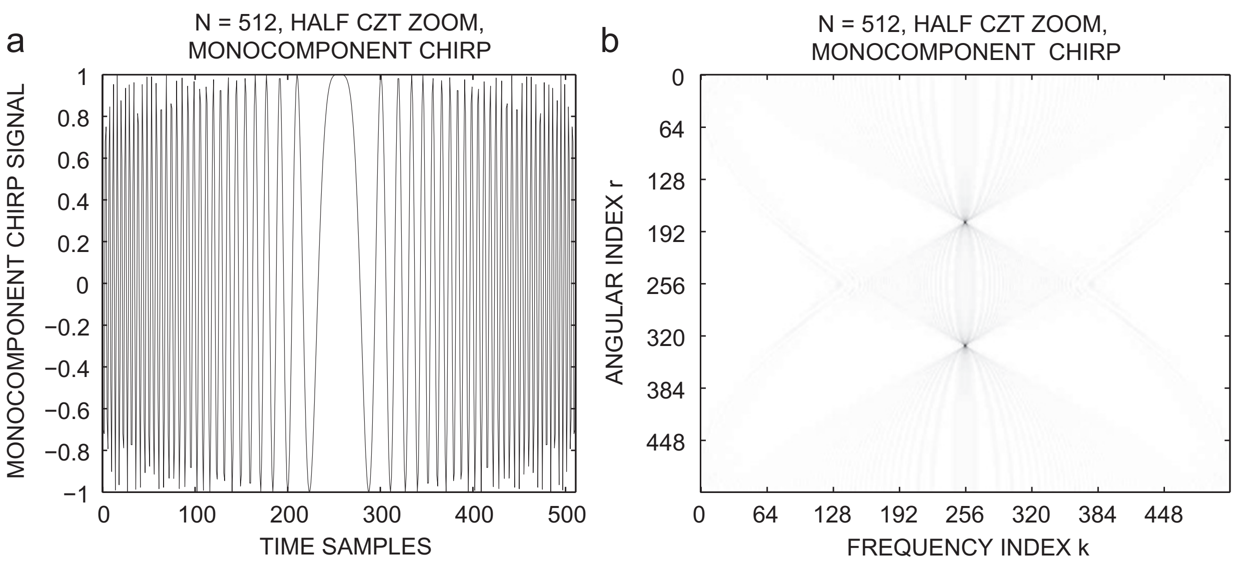

Fig. 6. Concentrating a chirp: (a) monocomponent real chirp signal, (b) cross-sectional view showing peak locations in the MA-CDFRFT spectrum. The CZT version of the FFT algorithm was used to implement the multiangle transform.

for $N=2M+1$ and

$$\mathbf{M}=\begin{pmatrix} k_1 & \mathbf{c}_1^{\mathrm{T}} & k_2 & \mathbf{c}_1^{\mathrm{T}}\mathbf{J}_L \\ \mathbf{c}_1 & \mathbf{R}_L & \mathbf{c}_2 & \mathbf{S}_L \\ k_2 & \mathbf{c}_2^{\mathrm{T}} & k_3 & \mathbf{c}_2^{\mathrm{T}}\mathbf{J}_L \\ \mathbf{J}_L\mathbf{c}_1 & \mathbf{J}_L\mathbf{S}_L\mathbf{J}_L & \mathbf{J}_L\mathbf{c}_2 & \mathbf{J}_L\mathbf{R}_L\mathbf{J}_L \end{pmatrix} \tag{22}$$

for $N=2L+2$, where $\mathbf{R}$ and $\mathbf{S}$ are symmetric sub-matrices of the appropriate order. It is easy to show that the orthogonal similarity transformation $\mathbf{U}$ given by

$$\mathbf{U}=\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_M & \mathbf{J}_M \\ \mathbf{0} & \mathbf{J}_M & -\mathbf{I}_M \end{pmatrix} \quad \text{for } N=2M+1,$$

$$\mathbf{U}=\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_L & \mathbf{0} & \mathbf{J}_L \\ 0 & \mathbf{0} & \sqrt{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_L & \mathbf{0} & -\mathbf{I}_L \end{pmatrix} \quad \text{for } N=2L+2 \tag{23}$$

block diagonalizes $\mathbf{M}$. The eigenvalues of the transformed matrix $\mathbf{D}=\mathbf{U}^{-1}\mathbf{MU}$ and $\mathbf{M}$ are identical, while the eigenvectors of $\mathbf{D}$ are related to the eigenvectors of $\mathbf{M}$ through the matrix $\mathbf{U}$ which is further a symmetric involution, i.e., $\mathbf{U}=\mathbf{U}^{-1}=\mathbf{U}^{\mathrm{T}}$. We can now furnish orthogonal eigenvectors for $\mathbf{M}$ by patching together the eigenvectors of $\mathbf{D}_1$ and $\mathbf{D}_2$. Specifically, if we solve the smaller eigenvalue problems for when N is odd we obtain

$$\begin{pmatrix} k & \sqrt{2}\mathbf{c}^{\mathrm{T}} \\ \sqrt{2}\mathbf{c} & \mathbf{R}_M+\mathbf{S}_M\mathbf{J}_M \end{pmatrix}\begin{pmatrix} \sqrt{2}w \\ \mathbf{u} \end{pmatrix}=\lambda\begin{pmatrix} \sqrt{2}w \\ \mathbf{u} \end{pmatrix},$$

$$\mathbf{D}_2\mathbf{v}=\rho\mathbf{v}.$$

We can construct $M+1$ K-symmetric and M K-anti-symmetric unit-norm, orthogonal eigenvectors for $\mathbf{M}$ via

$$\mathbf{x}=\frac{1}{\sqrt{2}}\begin{pmatrix} 2w \\ \mathbf{u} \\ \mathbf{Ju} \end{pmatrix}, \quad \mathbf{y}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \mathbf{Jv} \\ -\mathbf{v} \end{pmatrix}. \tag{24}$$

In a similar vein, for $N=2L+2$, if we represent a general eigenvector of the matrix $\mathbf{D}_1$ as $[\sqrt{2}w_1\ \mathbf{u}\ \sqrt{2}w_2]^{\mathrm{T}}$ and a general eigenvector of the matrix $\mathbf{D}_2$ as $\mathbf{v}$, we can generate $L+2$ K-symmetric and L K-anti-symmetric eigenvectors via

$$\mathbf{x}=\frac{1}{\sqrt{2}}\begin{pmatrix} 2w_1 \\ \mathbf{u} \\ 2w_2 \\ \mathbf{Ju} \end{pmatrix}, \quad \mathbf{y}=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \mathbf{Jv} \\ 0 \\ -\mathbf{v} \end{pmatrix}. \tag{25}$$

As a special case if $\mathbf{K}=\mathbf{J}$, the commutator becomes centrosymmetric with symmetric or anti-symmetric eigenvectors.

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