Orthogonal Modes of Frequency Modulation and the Sturm–Liouville Frequency Modulation Model

Balu Santhanam, *Senior Member, IEEE*

***Abstract*—Sinusoidal signals and complex exponentials play a critical role in LTI system theory in that they are eigenfunctions of the LTI convolution operator. While processing frequency-modulated (FM) waveforms using LTI systems, restrictive assumptions must be placed on the system so that a quasi-eigenfunction approximation holds. Upon deviation from these assumptions, FM waveforms incur significant distortion.**

In this paper, a Sturm–Liouville (S-L) model for frequency modulation introduced by the author, is extended to a) study orthogonal modes of continuous and discrete frequency modulation and b) to develop system theoretical underpinnings for FM waveforms. These FM modes have the same special connection with respect to the FM S-L system operator, that complex exponentials have with LTI systems and the convolution operator. The finite S-L-FM spectrum or transform that measures the strength of the orthogonal FM modes present in a FM signal, analogous to the discrete Fourier spectrum for sinusoids, is introduced. Finally, similarities between the orthogonal S-L-FM modes and angular Mathieu functions are exposed, and a conjecture connecting the two is put forth.

***Index Terms*—Angular Mathieu functions, frequency modulation, instantaneous frequency response, orthogonal FM modes, Sturm–Liouville differential and difference equation, Sturm–Liouville FM spectrum.**

I. Introduction

SINUSOIDAL signals and complex exponentials play a crucial role in signal processing and spectral analysis of signals with stationary frequency content. They also enjoy a special connection with linear time-invariant (LTI) systems in that they are eigenfunctions of the convolution operation. However, they are unsuitable for analysis of signals such as speech, ECG, MEG, vibrations, and seismic waveforms, that are not stationary with respect to their frequency content.

Frequency-modulated (FM) signals in particular are a class of non-stationary signals, where the information resides in the instantaneous frequency (IF) of the signal. Traditional approaches for the analysis of these signals such as the spectrogram are based on assumptions of stationary frequency content over windowed segments of the signal and the sinusoidal model. Other time-frequency tools such as the fractional Fourier transform [12] are specific to just chirp signals.

Processing of FM signals using the AM-FM model and the quasi-eigenfunction approximation was described in [9], [8]. The energy separation algorithm (ESA) and its discrete version DESA were studied in [9] as a means for the demodulation of AM-FM signals. Recent work in [11] generalizes the AM-FM signal model to over-modulated AM-FM decompositions. In [8] it was shown that AM-FM signals can only be approximate eigenfunctions of LTI systems and consequently undergo harmonic distortion when subjected to LTI filtering. Constraints on the frequency response of a filter for minimizing the eigenfunction approximation error and bounds on the demodulation error for AM-FM signals were developed. However, when these constraints are not met, this approximation incurs significant demodulation error.

Quasi-orthogonal chirp signals are the ingredients of a spread spectrum communication system proposed for indoor wireless communications [3]. Orthogonal FM transforms based on simple permutations of the DFT phase were reported in [10] for the purpose of concentrating the energy of an image in a few transform domain coefficients. In recent work, a Sturm–Liouville (S-L) model for the analysis of FM signals was proposed [17]. Orthogonal modes of continuous and discrete FM were developed using the differential or difference equation satisfied by the FM signal. These FM modes are eigenfunctions of the underlying S-L operator and pass undistorted through the S-L operator.

In this paper, we consolidate and extend the S-L model for FM, by first studying the orthogonal S-L-FM modes in depth to develop a system theoretic framework for FM signals. We further introduce the notion of the finite S-L-FM spectrum or the S-L-FM transform for FM signals that is analogous to the Fourier spectrum for sinusoidal signals. This FM spectrum measures the strength of the S-L-FM modes present in a FM waveform. Finally, we point out the similarities between the orthogonal S-L-FM modes and angular Mathieu functions [23], [22] that arise in the process of describing vibrations of a membrane in elliptical coordinates. We conjecture a relationship between the two with the envisioned goal of a S-L-FM approach to non-stationary signal analysis.

II. Continuous Time FM

Let us first consider a sinusoidal signal of the form

$$x(t) = \cos(\omega_o t + \theta_o).$$

This fundamental signal satisfies the constant coefficient, homogenous, second-order differential equation of the classical harmonic oscillator:

$$\ddot{x} + \omega_o^2 x = 0.$$

Manuscript received August 26, 2011; revised December 23, 2011 and March 23, 2012; accepted March 31, 2012. Date of publication April 16, 2012; date of current version June 12, 2012. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Lawrence Carin. This work was supported by the U.S. Department of Energy (Award No. DE-FG52-08NA28782) and by the National Science Foundation (Award No. IIS-0813747).

The author is with the Department of Electrical and Computer Engineering, University of New Mexico, Albuquerque, NM 87131-0001 USA (e-mail: bsanthan@ece.unm.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSP.2012.2194709

Sinusoids are *eigenfunctions* of an LTI system operator and form the basis for LTI system theory:

$$L\left(\exp(j\omega_o t)\right) = H(j\omega_o)\exp(j\omega_o t)$$

where $H(j\omega_o)$ represents the complex eigenvalue. Now consider a FM signal of the form[1]

$$x(t) = \cos\left(\phi_i(t)\right) = \cos\left(\int_{-\infty}^{t} \omega_i(\tau)d\tau\right) \tag{1}$$

where $\omega_i(t)$ is the IF, and $\phi_i(t)$ is the instantaneous phase. This signal satisfies a second-order differential equation with time-varying coefficients of the form

$$\ddot{x} - \frac{\dot{\omega}_i(t)}{\omega_i(t)}\dot{x} + \omega_i^2(t)x = \left(\mathcal{D}^2 - \frac{\mathcal{D}\omega_i}{\omega_i}\mathcal{D} + \omega_i^2\right)x = 0 \tag{2}$$

where $\mathcal{D}$ denotes the derivative operator. In general, this system is a linear time-varying system. In the specific case of sinusoidal FM it becomes periodically time-varying. It is known that even in the simple case, where the message waveform is sinusoidal, the bandwidth of the FM signal is infinite and requires truncation. In fact, the Carson bandwidth of an FM signal retains only spectral components with an amplitude of at least 10% of the peak spectral amplitude [4].

A. Sturm–Liouville Differential Equation

The FM differential equation described in (2) does not correspond to a self-adjoint operator. The self-adjoint form of the FM differential equation is [1]

$$\mathcal{D}\left(\frac{1}{\omega_i(t)}\mathcal{D}x(t)\right) + \omega_i(t)x(t) = 0. \tag{3}$$

The self-adjoint form of the FM differential equation for the FM signal $x(t) = \cos(n\phi(t))$ is given by

$$\begin{aligned} -\left(\frac{1}{\omega_i}\mathcal{D}^2 - \frac{\mathcal{D}\omega_i}{\omega_i^2}\mathcal{D}\right)x &= n^2\omega_i x \\ \text{or}\quad \mathcal{H}(\omega_i)x &= n^2\omega_i x. \end{aligned} \tag{4}$$

Comparing this to the standard form of the S-L differential equation

$$\mathcal{D}\left(p(x)\mathcal{D}\left(y(x)\right)\right) + q(x)y(x) = \lambda w(x)y(x),$$

where λ is the eigenvalue and $w(x)$ is the weight function, we can see that (4) is a specific case of the S-L problem with the identification

$$\lambda_n = n^2, \quad p(t) = \frac{1}{\omega_i(t)}, \quad q(t) = 0, \quad w(t) = \omega_i(t)$$

where the weight function[2] is just the IF. Equation (4) can in turn be formulated as a Sturm–Liouville system with periodicity by periodic extension or as a extended S-L system through extrapolation of the IF, $\omega_i(t)$, at the boundaries of the duration of the solution for (4) without loss of generality.

B. Orthogonal FM Modes

The first consequence of the operator $\mathcal{H}$ being self-adjoint is that it has real and positive eigenvalues and a full set of orthogonal eigenfunctions: $\phi_n(t) = \cos\left(n\phi_i(t)\right)$ with respect to the weight function $\omega_i(t)$ over the interval $t \in [t_i, t_f]$:

$$\int_{t_{qi}}^{t_f} w(t)\phi_m(t)\phi_n(t)dt = 0, \quad m \neq n \tag{5}$$

where the instantaneous phase $\phi_i(t)$ satisfies the boundary conditions

$$\phi_i(t_i) = -\pi, \quad \phi_i(t_f) = \pi.$$

Another side-product of this orthogonality result is that the normalized sequence of functions

$$\begin{aligned} \gamma_n(t) &= \sqrt{\omega_i(t)}\cos\left(n\phi(t)\right) \\ \psi_n(t) &= \sqrt{\omega_i(t)}\sin\left(n\phi(t)\right) \end{aligned} \tag{6}$$

also form an orthogonal set. It is easily seen by a simple substitution of variables $u = \phi_i(t)$ that the basis defined in (6) indeed form an orthogonal sequence of functions [7]

$$\begin{aligned} \langle\gamma_m(t), \gamma_n(t)\rangle &= \int_{t_i}^{t_f} \omega_i(t)\gamma_m(t)\gamma_n(t)dt, \quad m \neq n \\ &= \int_{-\pi}^{\pi} \cos(mu)\cos(nu)du = 0. \end{aligned} \tag{7}$$

This result is consistent with earlier work on FAM-lets[3] [7], where this sequence of functions was studied for applications in speech/audio coding [7].

The second important implication is that complex exponential version of the FAM-lets given by

$$\begin{aligned} \alpha_n(t) &= \sqrt{\omega_i(t)}\exp\left(jn\phi(t)\right) \\ &= \sqrt{\omega_i(t)}\exp\left(jn\omega_c t + jn\omega_m\int_{-\infty}^{t}\omega_i(\tau)d\tau\right) \end{aligned} \tag{8}$$

is also an eigenvector of the S-L-FM system. This is an intuitively satisfying result in that it is analogous to the correspondence between complex exponentials and LTI systems.

Several other special functions encountered in the context of FM communications or quantum mechanics such as Legendre, Hermite, and Bessel functions satisfy the S-L framework for

[1]We are assuming here that $\omega_i(t)$ is a positive and differentiable function of time, such as a sum of cosines or sines.

[2]For the Sturm–Liouville framework to hold the weight function $\omega_i(t)$ should be strictly positive. This is not restrictive and is assumed in most FM systems.

[3]FAM-lets are called constant Q basis functions because both the carrier frequencies and frequency deviations of the FM modes scale linearly, so the ratio of their center-frequency to the Carson bandwidth is a constant. In the context of sinusoidally modulated FM signals and computer generated music this is called harmonic FM [4]. When the ratio of the carrier frequency to the frequency deviation is not rational it is called non-harmonic FM.

specific discrete values of the eigenvalue λ and the weight function $w(x)$ [18].

C. System Theoretic Implications

There are three important consequences of expressing the FM differential equation in the Sturm–Liouville form. The first implication is that if the FM signal $x(t)$ is input to the system $\mathcal{H}(\omega_i)$, then the output is just a scalar multiple of the input signal. In other words, the S-L system $\mathcal{H}$ does not introduce any IF distortion, i.e., the IF of the input signal $x(t)$ remains invariant:

$$\begin{aligned} \mathcal{H}\left(\sum_{k=0}^{\infty} a[k]\cos\left(k\phi(t)\right)\right) &= \sum_{k=0}^{\infty} a[k]\mathcal{H}\left(\cos\left(k\phi(t)\right)\right) \\ &= \omega_i(t)\sum_{k=0}^{\infty} b[k]\cos\left(k\phi(t)\right) \\ b[k] &= k^2 a[k] \end{aligned} \tag{9}$$

where we are assuming that the sequence $k^2a[k]$ is absolutely summable. The second implication is that results analogous to LTI systems and sinusoids such as a Fourier series and Fourier transforms can be developed for FM signals. With $\phi_k(t) = \cos(k\phi(t))$, we can now decompose a FM waveform in terms of the S-L-FM modes as [6]

$$\begin{aligned} x(t) &= \sum_{n=0}^{\infty} c[k]\phi_k(t) \\ c[k] &= \frac{\int_{t_i}^{t_f} x(t)\phi_k(t)\omega_i(t)dt}{\int_{t_i}^{t_f} \left|\phi_k(t)\right|^2 \omega_i(t)dt} \end{aligned} \tag{10}$$

where $c[k]$, the S-L coefficient, measures the strength of a particular FM mode in the signal, and $\phi(t_i) = -\pi, \phi(t_f) = \pi$. The third implication of the S-L framework is that sequence of S-L coefficients $c[k]$ is stationary in frequency content even though the underlying FM signal has non-stationary frequency content. This means that traditional signal processing concepts such as convolution and filtering can be applied to the S-L coefficients[4]:

$$\begin{aligned} b[k] &= a[k]\lambda_k, \quad \lambda_k = k^2 u[k] \\ B(z) &= \frac{1}{2\pi j}\left(A(z) * \Lambda(z)\right), \quad |z| > 1 \end{aligned} \tag{11}$$

where the notation $X(z)$ denotes the bilateral Z-transform of the sequence $x[n]$, and $*$ denotes the complex convolution contour integral. In this sense, the S-L coefficients constitute the stationary portion of the FM signal.

III. Discrete time FM

One could in theory substitute various discrete versions of the derivative operator in the definition of the continuous S-L operator $\mathcal{H}$ to yield different discrete versions of the S-L operator. However, they would only serve as discrete approximations of the continuous counterpart. Instead, the approach taken here for generating a discrete S-L-FM framework is to work directly with the difference equation satisfied by the FM signal.

[4]A causal eigenvalue indexing produces a outward region of convergence.

First, consider the sinusoidal sequence $s[n] = \cos(\Omega_o n)$ that satisfies the second-order difference equation

$$s[n] - 2\cos(\Omega_o)s[n-1] + s[n-2] = 0. \tag{12}$$

Now consider the discrete time FM sequence $x[n]$ given by

$$x[n] = \cos\left(\Theta[n]\right) = \cos\left(\sum_{k=0}^{n} \Omega_i[k] + \theta_o\right), \tag{13}$$

where the instantaneous phase $\Theta[n]$ and the IF $\Omega_i[n]$ are related via a first difference

$$\Theta[n] = \Theta[n-1] + \Omega_i[n].$$

It is easily seen that this satisfies a second-order generating difference equation of the form [16]

$$x[n] - c_1[n]x[n-1] + c_2[n]x[n-2] = 0 \tag{14}$$

where the time-varying coefficients are given by

$$\begin{aligned} c_1[n] &= \frac{\sin\left(\Omega_i[n] + \Omega_i[n-1]\right)}{\sin\left(\Omega_i[n-1]\right)} \\ c_2[n] &= \frac{\sin\left(\Omega_i[n]\right)}{\sin\left(\Omega_i[n-1]\right)}. \end{aligned} \tag{15}$$

First, note that the signal $y[n] = \sin\left(\Theta[n]\right)$ satisfies the same difference equation. This again is an intuitively satisfying result in that the complex exponential version of the FM signal given by $x[n] = \exp(j\Theta[n])$ will also be an eigenfunction of the S-L-FM operator. It can be verified that this difference equation will reduce to (12), when $\Omega_i[n] = \Omega_o$. The corresponding self-adjoint difference equation obtained by the S-L difference equation framework [13] is given by

$$\nabla_-\left(p[n]\Delta_+\left(x[n]\right)\right) + w[n]C[n]x[n] = 0 \tag{16}$$

where the weight function $w[n]$, $p[n]$, and $C[n]$ are given by

$$\begin{aligned} w[n] &= \prod_{r=0}^{n-1} \frac{\sin\left(\Omega_i[r]\right)}{\sin\left(\Omega_i[r+2]\right)} = \frac{\sin\left(\Omega_i[0]\right)\sin\left(\Omega_i[1]\right)}{\sin\left(\Omega_i[n]\right)\sin\left(\Omega_i[n+1]\right)} \\ p[n] &= \sin\left(\Omega_i[n]\right)w[n] = \frac{\sin\left(\Omega_i[0]\right)\sin\left(\Omega_i[1]\right)}{\sin\left(\Omega_i[n+1]\right)} \\ C[n] &= \sin\left(\Omega_i[n]\right) + \sin\left(\Omega_i[n+1]\right) \\ &\quad - \sin\left(\Omega_i[n+1] + \Omega_i[n]\right) \end{aligned} \tag{17}$$

and the symbols ∇_- and Δ_+ denote the one-sample backward and forward difference operators. As in the continuous case, the S-L operator is in general a linear time-varying system. It should be noted here that the form of the FM difference equation and as a result the self-adjoint S-L difference equation are sensitive to the form of discretization of the instantaneous phase $\Theta[n]$. As in the continuous case, the difference equation in (16) can be formulated as a extended/periodic S-L system by either a) periodic extension of the IF sequence $\Omega_i[n]$ at the boundaries [2], [19], which would correspond to a discrete Fourier series representation for the IF or b) extrapolation of the IF at the boundaries under the assumption that IF varies slowly, where the boundary values can be repeated [9]. The solution to the discrete S-L difference equation is then formulated as the solution to a weighted, tridiagonal eigenvalue problem of the form

$$\mathcal{L}(\mathbf{x}) = \lambda \mathbf{W}\mathbf{x} \tag{18}$$

where $\mathbf{W} = \mathrm{diag}(w[0], \ldots, w[N-1])$ is a diagonal matrix of the positive weights and λ is the eigenvalue.[5]

The symmetric, tridiagonal, weighted eigenvalue problem is encountered in the context of the theory of orthogonal polynomials, that satisfy an associated three term recursion. These polynomials are orthogonal with respect to a weighted inner product. In the limiting case of the S-L operator, where the S-L operator is circulant and Toeplitz-tridiagonal, its eigenvectors are sinusoids, the IFs of the eigenvectors would correspond to constants. The orthogonal polynomials associated with the asymptotic S-L operator are the Tchebychev polynomials of the second kind [20]. As the modulation depth decreases, the eigenvector approaches a sinusoid. Expressions for the eigenvectors in terms of the associated orthogonal polynomials and its roots can also be found in [20].

It should also be mentioned at this juncture that computation of the weight function $w[n]$ and the solution to the S-L eigenvalue problem are contingent upon knowledge of the IF sequence, $\Omega_i[n]$. When the exact IF is unknown, it needs to be estimated from the FM waveform, and consequently IF estimation becomes critical. Recent work on "syncrosqueezed wavelet transforms" [5] addresses this empirical IF estimation problem. While in theory any monocomponent demodulation algorithm could be applied, we use the ESA [9], due to its excellent time resolution and tracking capabilities.

A. Orthogonal FM Modes

As was seen in the continuous case, the eigenvectors of the S-L operator

$$\mathcal{L}\left(p[n]\right) = \nabla_{-} p[n] \Delta_{+} + w[n]C[n] \tag{19}$$

corresponding to distinct eigenvalues are orthogonal with respect to the positive weight function $w[n]$:

$$\langle v_p[n], v_q[n] \rangle = \sum_{n=0}^{N-1} w[n] v_p[n] v_q[n] = 0, \quad p \neq q. \tag{20}$$

The corresponding expansion of the discrete FM signal in terms of the eigenvectors $v_k[n]$ of the S-L operator is

$$\begin{aligned} x[n] &= \sum_{k=0}^{N-1} c[k] v_k[n] \\ c[k] &= \frac{\sum_{n=0}^{N-1} w[n] x[n] v_k[n]}{\sum_{n=0}^{N-1} w[n] \left| v_k[n] \right|^2}. \end{aligned} \tag{21}$$

[5]The S-L eigenvalue problem can be solved in various senses: exactly using the MATLAB functions `eig(A,B)` or `qz(A,B)` or in the min-norm sense using `svd.m` or `gsvd.m`. For situations where the signal of interest and consequently the estimate of the IF, $\Omega_i[n]$, are noisy, a generalized SVD version of (18) is employed.

As in the continuous case, the weight function can be absorbed into the orthogonal basis of eigenvectors to produce a normalized basis:

$$\gamma_k[n] = \sqrt{w[n]} v_k[n]. \tag{22}$$

These S-L-FM eigenvectors contain both AM and FM, and the IF of the eigenvectors of the matrix $\mathcal{L}$ furthermore have a form specified by the IF of the input signal, $\Omega_i[n]$. For example, if we use the tridiagonal formulation of the S-L operator with no corner correction, we obtain

$$\begin{aligned} v_k[n] &= \sqrt{\frac{2}{N+1}} a_k[n] \sin\left(\Omega_c^{(k)}(n+1) + \phi_k[n]\right) \\ \phi_k[n] &= \Omega_m^{(k)} \sum_{r=0}^{n} q[r], \quad |q[n]| < 1 \\ \Omega_c^{(k)} &= \frac{\pi}{N+1}(k+1), \quad 0 \leq k \leq N-1 \end{aligned} \tag{23}$$

where $q[n]$ is the normalized message signal. Note that, in the limiting case, where the input signal is a sinusoid, the eigenvector basis $v_k[n]$ reduces to the discrete sine transform (DST) basis, specifically the symmetric version of DST-I [14]:

$$\Upsilon_k[n] = \sqrt{\frac{2}{N+1}} \sin\left(\Omega_c^{(k)}(n+1)\right). \tag{24}$$

Note that, unlike the continuous case, the corresponding cosine version of the sequence, i.e., the DCT-I sequence does not constitute eigenvectors of the same operator.

If, however, we use an alternative set of boundary conditions that correspond to the addition/subtraction of 1 at the diagonal corners of the tridiagonal form of the S-L operator, via the use the "Toeplitz plus near Hankel," framework described in [15] we obtain the DCT-4/DST-4 versions of the S-L-FM modes:

$$\Upsilon_k[n] = \sqrt{\frac{2}{N}} a_k[n] \sin\left(\frac{\pi}{N}(n+0.5)(k+0.5) + \phi_k[n]\right). \tag{25}$$

However, DCT/DST versions are obtained from two distinct operators. This difference being attributed to the truncation of the infinite dimensional S-L operator to finite dimensions.

The inner-product of two spectrally distinct S-L DST-I based FM modes is expressed as

$$\begin{aligned} \langle v_p[n], v_q[n] \rangle &= T_1 + T_2 \\ T_1 &= -\sum_{n=0}^{N-1} \frac{w[n] a_p[n] a_q[n]}{N+1} \\ &\quad \times \cos\left(\Omega_c^{(n)}(p+q+2) + \phi_p[n] + \phi_q[n]\right) \\ T_2 &= \sum_{n=0}^{N-1} \frac{w[n] a_p[n] a_q[n]}{N+1} \\ &\quad \times \cos\left(\Omega_c^{(n)}(p-q) + \phi_p[n] - \phi_q[n]\right) \end{aligned} \tag{26}$$

where we are assuming that the mode indices q, p are well separated, i.e., $|q-p| > 40$ so that the corresponding modes $v_p[n]$ and $v_q[n]$ are spectrally distinct as depicted in Fig. 5(a). Both terms in the expression above are the inner-product of a lowpass waveform $w[n]a_p[n]a_q[n]$, with spectral content around DC, with a bandpass waveform, whose spectral content is around a

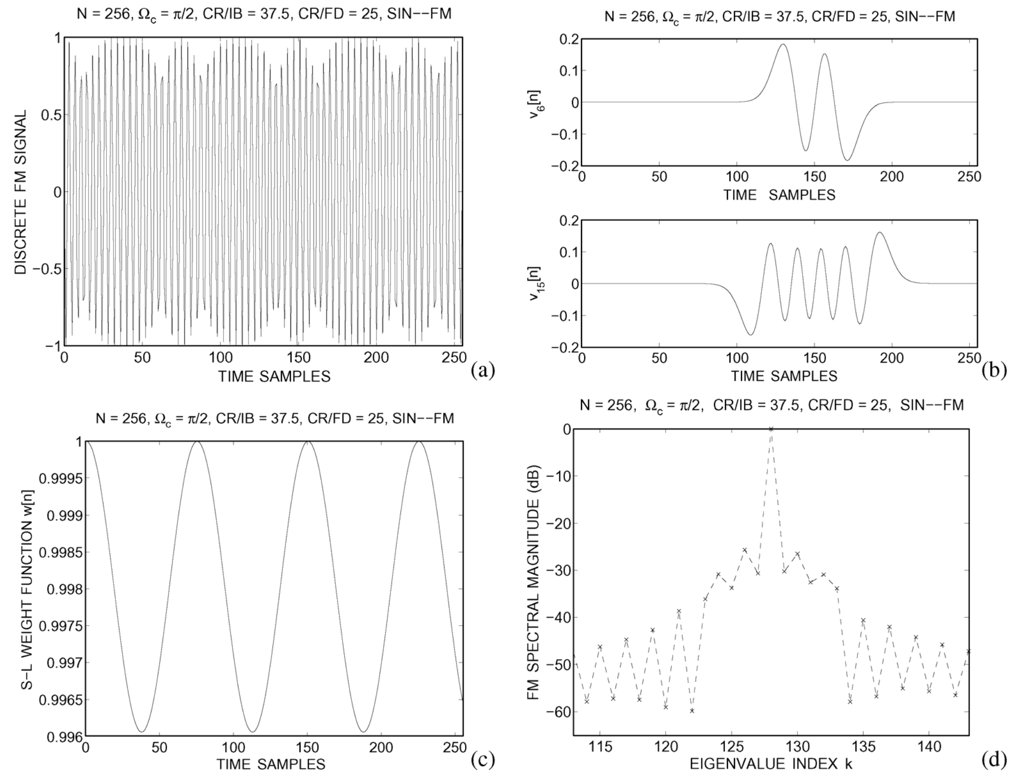

Fig. 1. Discrete S-L problem, sinusoidal-FM: (a) sinusoidal FM signal, (b) selected S-L-FM modes using the MATLAB function `eig(A,B)` depicting different number of zero crossings, (c) weight function $w[n]$ of the discrete S-L problem, (d) S-L-FM spectrum of the input FM signal using the S-L-FM modes and weight function $w[n]$ depicting a single FM mode that most resembles the input FM signal in the weighted inner-product sense. Note further that most of the signal energy resides in a few S-L coefficients around the peak.

much higher carrier frequency as embodied in the cosine term. The lowest possible carrier frequency of T_1 with the $(p+q+2)$ factor is approximately $\frac{45\pi}{N+1}$. Consequently by Parseval's theorem, T_1 is approximately zero:

$$T_1 = \sum_{n=0}^{N-1} w[n]a_p[n]a_q[n]\cos\Big(\Omega_c^{(n)}(p+q+2)+\phi_p[n]+\phi_q[n]\Big) \approx 0. \tag{27}$$

The inner product can then be approximated via T_2:

$$\begin{aligned}\langle v_p[n], v_q[n]\rangle &\approx \sum_{n=0}^{N-1} \frac{w[n]a_p[n]a_q[n]}{N+1} \\ &\quad\times \cos\Big(\Omega_c^{(n)}(p-q)+\phi_p[n]-\phi_q[n]\Big) \\ &= 0, \quad p \neq q\end{aligned} \tag{28}$$

where the last result follows from the same observation that there is no spectral overlap between the lowpass term $w[n]a_p[n]a_q[n]$ and the bandpass term with carrier frequency $\frac{\pi(p-q)}{N+1} \approx \frac{41\pi}{N+1}$. The lowpass approximation employed here is a common assumption in narrowband communications systems, where the carrier frequency is much larger than the message bandwidth [21]. For closely placed, cochannel S-L-FM modes depicted in Fig. 5(b), the claim of orthogonality follows from the fact that they are solutions to a S-L difference equation and results of S-L theory apply [13].

B. Discrete Time System Theoretic Implications

The response of the discrete S-L operator to the particular FM mode $z_k[n]$ is

$$y_k[n] = \mathcal{L}\left(z_k[n]\right) = \lambda_k w[n] z_k[n]. \tag{29}$$

Suppose the input to the S-L system is a superposition of these FM modes, then the corresponding output is

$$\begin{aligned}\mathcal{L}\left(\sum_{k=0}^{N-1}\alpha[k]z_k[n]\right) &= \sum_{k=0}^{N-1}\alpha[k]\mathcal{L}\left(z_k[n]\right) \\ &= \sum_{k=0}^{N-1}\lambda_k\alpha[k]w[n]z_k[n] \\ &= \sum_{k=0}^{N-1}\beta[k]w[n]z_k[n].\end{aligned} \tag{30}$$

Specifically, the IF modes present in the output of the S-L operator are the same IF modes present in the input to the S-L operator. The ratio of the S-L coefficients is the instantaneous frequency response (IFR) analogous to the conventional frequency response of LTI systems:

$$\lambda_k = H[k] = \frac{\beta[k]}{\alpha[k]} = \frac{(\mathbf{V}^T\mathbf{W}\mathbf{D}\mathbf{y})_k}{(\mathbf{V}^T\mathbf{W}\mathbf{D}\mathbf{x})_k} \tag{31}$$

where $\mathbf{D}$ denotes the unit sample advance operator, $\mathbf{V}$ is the matrix of S-L eigenvectors, and $\mathbf{W}$ is a diagonal matrix of S-L

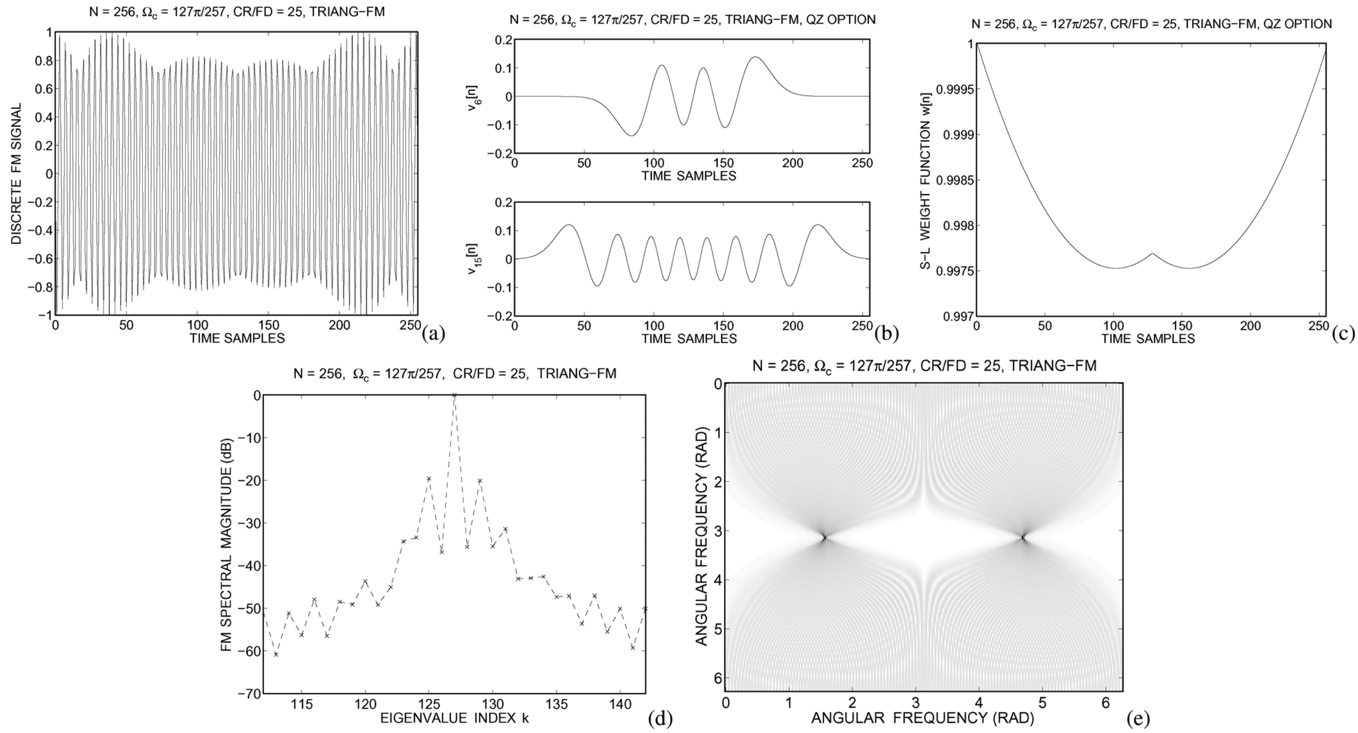

Fig. 2. Discrete S-L problem, triangular frequency modulation: (a) FM signal; (b) selected eigenvectors of the discrete S-L operator using a QZ decomposition for the generalized eigenvalue problem; (c) weight function associated with the discrete S-L operator; and (d),(e) S-L-FM spectrum computed using the orthogonal FM modes and weight function $w[n]$, and the corresponding discrete fractional Fourier spectrum of the FM signal.

weights. Furthermore, the generalized Fourier coefficient sequences $\beta[k]$ and $\alpha[k]$ are connected via circular convolution through

$$\begin{aligned}\mathcal{F}^{-1}(\beta[k]) &= \mathcal{F}^{-1}\left(\alpha[k]\right) * \mathcal{F}^{-1}(\lambda_k)\\ &= \mathcal{F}^{-1}\left(\alpha[k]\right) * h[k]\end{aligned} \tag{32}$$

where $\mathcal{F}^{-1}$ here denotes the inverse DFT matrix. This relationship is significant in that conventional LTI system theory can be applied to the generalized Fourier coefficients $\beta[k]$ and $\alpha[k]$, despite the underlying waveforms $y[n]$ and $x[n]$ being FM signals. Specifically, the quantity

$$\alpha[k] = (\mathbf{V}^T\mathbf{W}\mathbf{D}\mathbf{x})_k = \sum_{m=0}^{N-1} v_k[m]w[m]x[m+1] \tag{33}$$

is formally defined as the finite S-L-FM spectrum[6] or *FM transform* of the FM signal $x[n]$. This concept is analogous to the discrete Fourier spectrum for sinusoidal signals, except in this case the spectrum indicates the strength of a particular FM mode in the signal.

IV. Discussion of Results

Figs. 1, 2, and 4 describe the application of the discrete S-L approach to three different signals: (a) sinusoidally modulated FM signal with a carrier-to-frequency deviation (CR/FD) ratio of 25 and a carrier-to-information bandwidth (CR/IB) ratio of 37.5 and a duration of $N = 256$ samples, where the MATLAB function `eig(A,B)` is employed, (b) FM signal with a triangular IF, with $\frac{\mathrm{CR}}{\mathrm{FD}} = 25$, $N = 256$, and $\Omega_c = \frac{127\pi}{257}$, where the MATLAB function `qz.m` is employed, and (c) FM signal with a triangular IF in white Gaussian noise ($\mathrm{SNR} = 25$ dB), where the generalized SVD function in MATLAB `gsvd.m` is employed. Note that for the triangular IF example in Fig. 2, the carrier frequency of the input FM signal is chosen to be an integer multiple of $\frac{\pi}{N+1}$ so that it coincides with the carrier frequency of one of the normal FM modes. This translates to a sharp peak at the specific mode where the carrier frequencies match as described in the S-L-FM spectrum in Fig. 2(d). Fig. 2(e) compares the S-L FM spectrum which is a one dimensional spectrum to the MA-CDFRFT spectrum that is a two dimensional spectrum [12]. The distinguishing characteristic of the S-L approach from the discrete fractional Fourier transform based approaches is that the IF of the eigenvectors in the S-L approach are of the same form as the IF of the FM signal being analyzed and not specific to just chirps.

Fig. 3(a) and (b) describes the center-frequencies and the inverse normalized frequency deviations of the FM modes for sinusoidal-FM signal with a carrier frequency of $\Omega_c = \frac{\pi}{2}$, a CR/FD of 25, a CR/IB of 37.5, and duration $N = 256$ samples. Fig. 3(b) describes the relationship of the inverse normalized frequency deviations to the symmetric-DST expression $\sin(\Omega_c^{(k)})$, a result used later in Section V. Modulation indices larger than 1 are considered wideband, while indices smaller than 1 are considered narrowband. Fig. 3(c) describes the frequency modulation index for selected FM modes indicating the presence of both narrowband and wideband modes. S-L eigenvectors with more zero-crossings correspond to high-frequency FM modes, while the eigenvectors with fewer zero-crossings correspond to lowpass FM modes. While the carrier frequencies

[6]The non-causal definition of the spectrum is a direct result of the non-causal formulation of the S-L operator that is defined with a one sample noncausal shift.

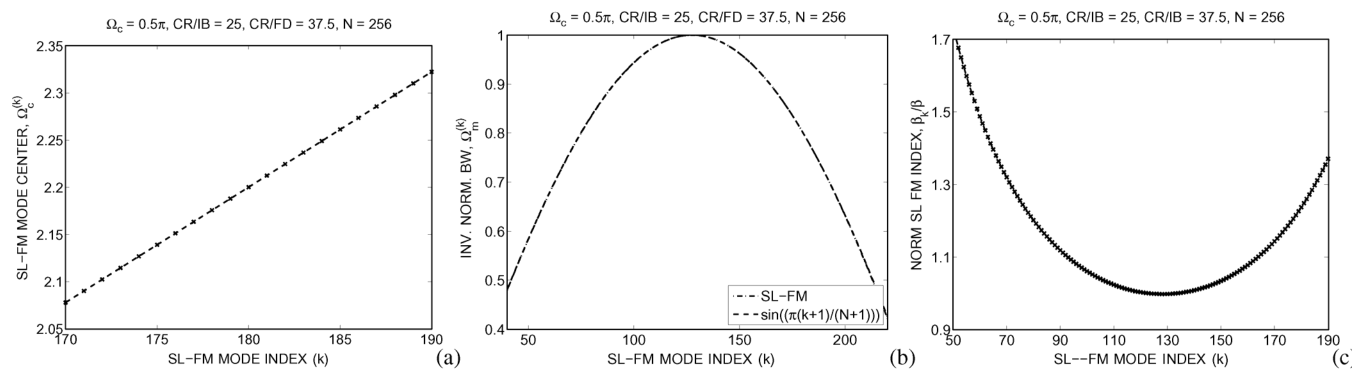

Fig. 3. Carrier spacing and spread properties of S-L-FM modes: (a) center frequency of selected modes of the discrete S-L-FM operator for the first sinusoidally modulated example indicating a symmetric-DST type linear spacing as indicated in (24); (b) inverse of the normalized frequency deviations of selected S-L-FM modes depicting a perfect symmetric-DST type sinusoidal fit as indicated in (38); (c) normalized frequency modulation indices for specific FM modes for the sinusoidal FM example depicting the presence of both narrowband and wideband FM modes relative to the input signal.

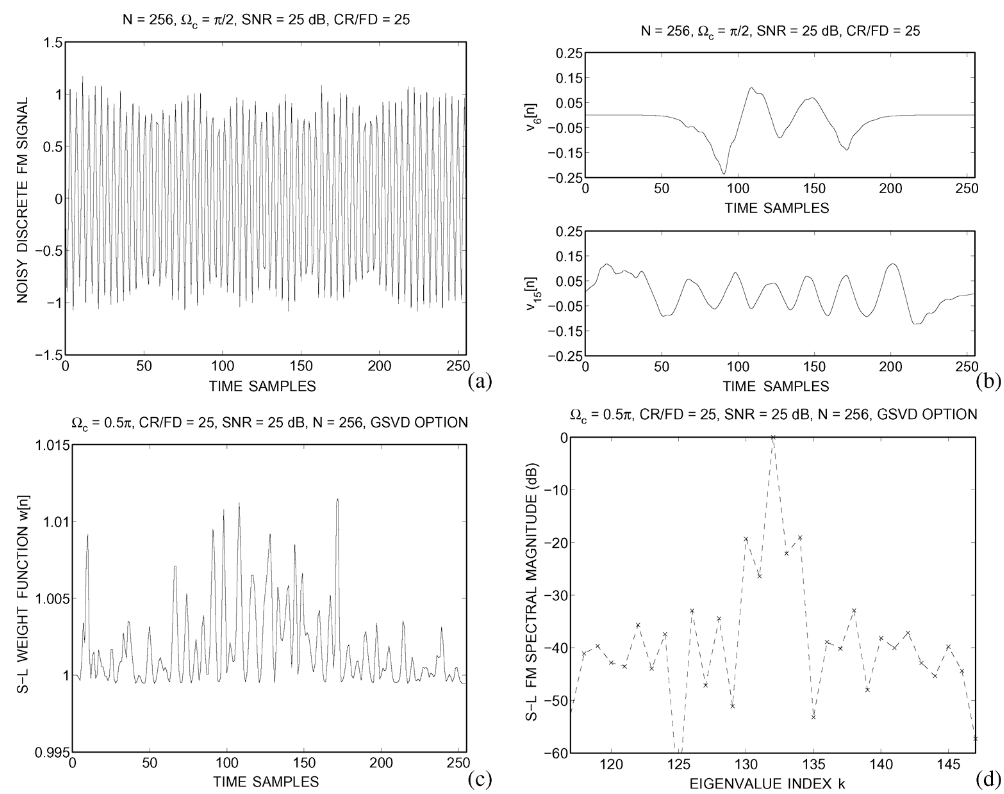

Fig. 4. S-L-FM mode decomposition in AWGN using the generalized SVD version of (18): (a) noisy FM signal with SNR $= 25$ dB; (b) selected S-L-FM modes of the generalized SVD solution; (c) discrete S-L weight function; and (d) S-L-FM spectrum of the noisy FM signal depicting the presence of a few dominant modes.

of the FM modes are linearly spaced apart as with the FAM-lets, the frequency deviations of the modes are not linearly spaced apart, but are symmetric about a central mode as depicted in Fig. 3(b) and (c).

Fig. 4 describes application of the S-L-FM approach to a noisy version (AWGN) of the triangular FM example in Fig. 2 at a SNR of 25 dB. The S-L-FM modes are obtained from the generalized SVD version of the tridiagonal system. Fig. 4(b) depicts two of the S-L-FM modes obtained from the SVD system, while Fig. 4(c) depicts the corresponding weight function $w[n]$. Fig. 4(d) depicts the S-L FM spectrum computed using the computed modes and $w[n]$. Notice the difference between the largest peak and its nearest neighbor is around 30 dB, indicating that the S-L-FM modes provide significant energy compaction in just a few modes. The operation of truncation, i.e., retaining just the S-L coefficients above a certain power threshold dependent on the SNR, therefore is equivalent to applying a bandpass filter with a ideal brick-wall IFR on the noisy S-L coefficients. This result further implies that a sufficient reliability of the initial IF estimate is required for a reliable estimation of the S-L-FM modes.

Fig. 6(a) studies the ESA frequency demodulation error between the IF of the mode corresponding to the S-L spectral peak and the IF of the input FM signal versus the S-L system size for the sinusoidal example. As the size of the S-L system increases, the error decreases indicating that one of the FM modes will eventually capture the input FM signal. Fig. 6(b) depicts the frequency demodulation error when the input signal is one of the

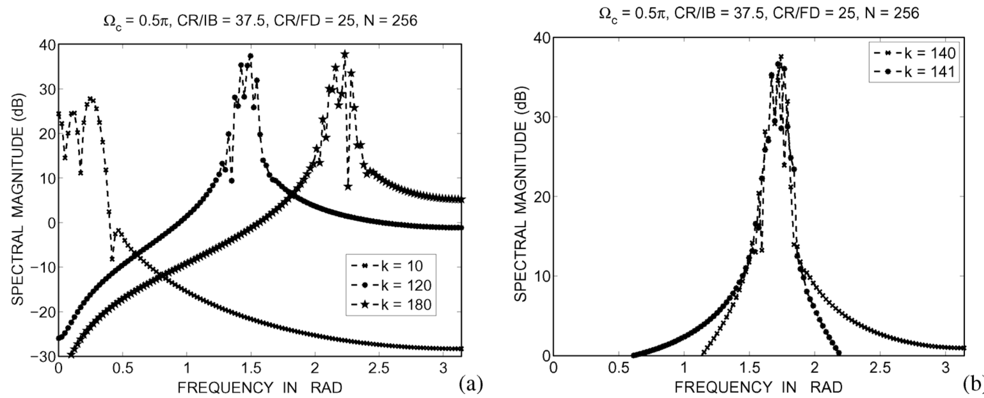

Fig. 5. Spectra of S-L-FM modes from sinusoidal-FM example: (a) Fourier spectra of well separated FM modes (k = 20, 120, 200) depicting non-overlapping spectral support and orthogonality implied by Parseval's theorem and (b) spectra of adjacent S-L-FM modes with indices (k = 140, 141) depicting overlapping spectral support and yet from S-L theory these S-L-FM modes are orthogonal with respect to the weighted inner-product.

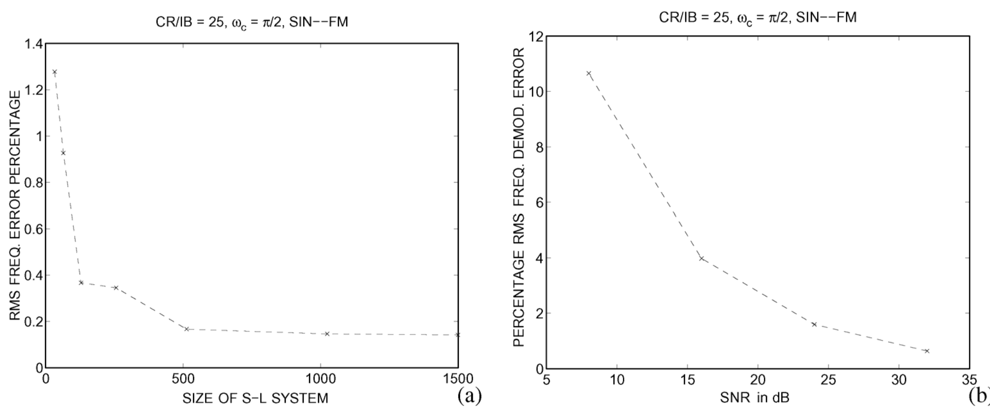

Fig. 6. Effect of parameters: (a) RMS ESA frequency demodulation error percentage for different S-L operator sizes and (b) RMS frequency demodulation error percentage for different SNRs in AWGN averaged over 100 experiments for a S-L operator size of $N = 256$.

S-L-FM modes averaged over 100 experiments, indicating that a fair amount of SNR is needed for reliable estimation of the S-L parameters.

V. Orthogonal FM Modes and Angular Mathieu Functions

Angular Mathieu functions (AMF) denoted $\mathrm{ce}_n(z,q)$ and $\mathrm{se}_n(z,q)$ are solutions of the angular part of the Helmoltz differential equation in elliptical coordinates [23], [22]

$$\frac{d^2\Phi}{dz^2} + (a - 2q\cos 2z)\Phi = 0. \tag{34}$$

The elliptical variable z takes the range of $[0, 2\pi]$ and plays the role of a angular frequency [23], while the q parameter is connected to the deviation from the classical sinusoidal harmonic oscillator. In this section, since analytical expressions for the AMFs are not available, we take the approach of pointing out the similarities between the S-L-FM modes and the AMFs in the context of a sinusoidally modulated FM signal of duration $N = 256$ samples, with a carrier frequency $\Omega_c = \frac{\pi}{2}$, and with $\frac{\mathrm{CR}}{\mathrm{FD}} = 50$, $\frac{\mathrm{CR}}{\mathrm{IB}} = 64$, as illustrated in Fig. 7. The envisioned goal behind studying AMFs and their connection to the S-L-FM modes is to enable development of a general S-L-FM analysis approach that does not require knowledge of the input signals IF.

A. Negative q Parameter

For IFs with negative frequency deviations, the FM modes from the S-L framework exhibit the same symmetry or anti-symmetry about their mid-point that the cosine and sine elliptic AMFs do

$$\begin{aligned} \mathrm{ce}_n(z,-q) &= \pm\, \mathrm{ce}_n(\pi/2 - z, q) \\ \mathrm{se}_n(z,-q) &= \pm\, \mathrm{se}_n(\pi/2 - z, q). \end{aligned} \tag{35}$$

This is illustrated in Fig. 7(a) and (b), where the S-L-FM modes for negative frequency deviation values are specific modes are plotted. This symmetry or anti-symmetry for a negative frequency deviation parameter translates to S-L-FM modes with a IF with the same carrier frequency but a negative frequency deviation as illustrated in Fig. 7(c). This result is consistent with the framework in [23] and the results in Fig. 7(g) that imply that the parameter q of the AMFs is a odd function of frequency deviation of the modes.

B. Asymptotic Properties

The S-L-FM modes also satisfy the same asymptotic behavior as the AMFs in that in the limit as the frequency deviation goes to zero, we obtain sinusoids

$$\begin{aligned} \lim_{q\to 0} ce_n(z,q) &= \cos(nz) \\ \lim_{q\to 0} se_n(z,q) &= \sin(nz). \end{aligned} \tag{36}$$

This property is illustrated in Fig. 7(f), where the first FM mode is plotted for different frequency deviation parameters, depicting the change from a FM signal to a sinusoid.

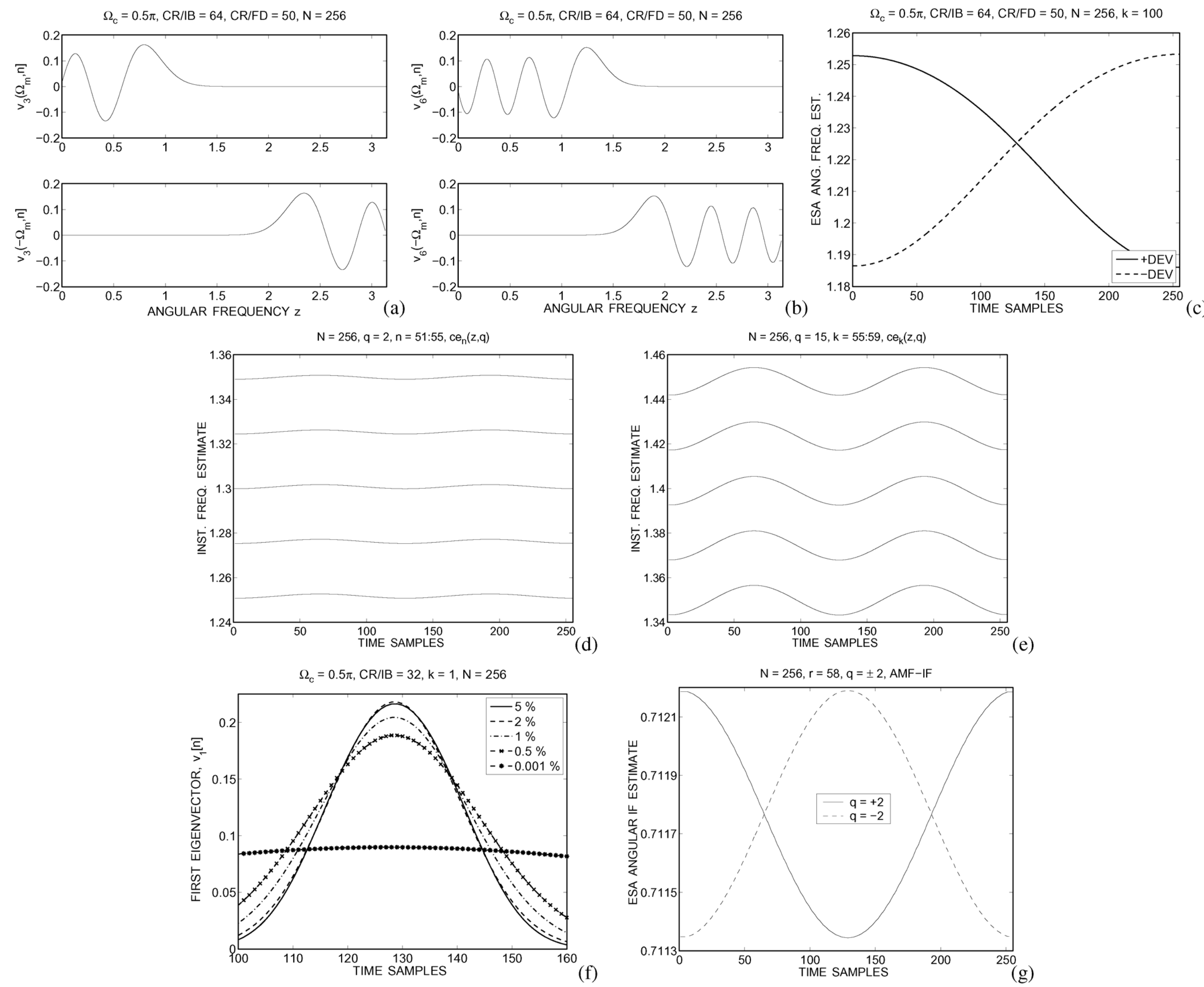

Fig. 7. S-L-FM modes and AMFs: (a),(b),(c) orthogonal FM modes from the S-L system for positive and negative frequency deviation parameters and the corresponding ESA IF estimates of the $k = 100$th mode for both positive and negative frequency deviations, depicting properties identical with angular Mathieu functions $ce_n(z,q)$ or $se_n(z,q)$ for negative q parameters; (d),(e) ESA IF estimates of AMFs evaluated using the framework and MATLAB functions in [23] for different q parameters depicting sinusoidal FM; (f) first orthogonal FM mode for different frequency deviation depicting that asymptotically as the frequency deviation goes to zero, the FM modes become purely sinusoidal, as is the case with Mathieu functions; and (g) ESA IF estimates of the $k = 58$th mode for both positive and negative values of q.

C. Sinusoidal Frequency Modulation

Fig. 7(c) and (d) depicts the IFs of selected AMFs obtained using the approach in [23] and the associated MATLAB functions for different values of the q parameter. Notice that the AMFs are sinusoidally modulated and that a change in the parameter q results in a increase in the frequency deviation of the underlying IFs. This result is very similar to results seen with the S-L-FM modes, where the modes are also frequency modulated with an IF of the same form as the input FM signal. Furthermore, the AMFs obtained through the framework in [23] also have the same linear spacing of mode center-frequencies of $\frac{\pi}{N}$ as depicted in the ESA IF estimates of selected AMFs in Fig. 7(d) and (e), a result similar to that seen in the S-L-FM modes. In addition, the S-L-FM modes exhibit simultaneous AM and FM, a property seen in the approximate solutions to the Mathieu differential equation [22].

D. Conjecture Relating S-L-FM Modes and AMFs

Effectively these similarities imply that the carrier-spacing of the FM modes is related to the parameter z and the frequency deviation of the FM modes is related to the parameter q of the AMFs. These striking similarities combined with the results from Fig. 3(b) lead us to the conjecture that the S-L-FM modes are contained in the span of a finite dimensional subset of AMFs for specific discrete values of the parameters:

$$v_k[n] = \sum_{r=0}^{N-1} p_k[r]\mathrm{ce}_n(z_k, q_r) + q_k[r]\mathrm{se}_n(z_k, q_r). \tag{37}$$

For the symmetric DST based S-L-FM modes, this becomes

$$\begin{aligned} z_k &= \frac{(k+1)\pi}{N+1} = \Omega_c^{(k)}, \quad 0 \le k \le N-1 \\ \Omega_m^{(r)} &= C\csc(z_r), \quad q_r = \sinh\left(\Omega_m^{(r)}\right). \end{aligned} \tag{38}$$

The result that the AMFs are sinusoidally modulated, i.e., have a sinusoidal IF, and yet are able to represent a general FM waveform as depicted in Fig. 7(c) and (d) is consistent with the ESA framework in [8] and [9], which allows for any IF that can be represented through a finite Fourier series of cosines/sines:

$$\Omega_i[n] = \Omega_o + \sum_{k=1}^{N} \alpha[k] \cos(\Omega_k n + \Theta_k), \tag{39}$$

except at discontinuity points, where the estimated IF goes through the mid-point of the discontinuity.

VI. Conclusion

We have extended and expanded on the S-L framework for continuous and discrete FM introduced in [17]. Orthogonal FM modes arising from the eigenfunctions or eigenvectors of the S-L-FM operator are shown to undergo no IF distortion when subjected to the S-L-FM system. A generalized Fourier series representation of a FM waveform in terms of the S-L-FM modes was presented and the notion of the finite S-L-FM spectrum that describes the strength of the FM modes prevalent in a modulated signal was defined. Simulation results indicate that the S-L-FM modes provide significant energy compaction by representing a FM waveform with a few transform coefficients.

These S-L FM modes furthermore, reduce to the standard Fourier basis or the symmetric sine basis, asymptotically when the modulation strength becomes negligible. More significantly it was also shown that S-L coefficients of a FM signal with respect to the S-L-FM modes are stationary even though the underlying signal has nonstationary frequency content. The implication is that standard system theory results such as convolution, filtering, and the DTFT can be applied to the S-L coefficients. In the continuous case, the S-L-FM modes reduce to the better known FAM-let basis, while in the discrete case, the striking similarities between the S-L-FM modes and AMFs were examined and it was conjectured that the S-L-FM modes lie in the span of a finite dimensional subset of the AMFs for specific discrete values of the underlying parameters. Sliding window versions of the S-L-FM approach suitable for general non stationary signal analysis can now be envisioned and are being developed.

References

[1] D. T. Hess, "FM differential equation," *Proc. IEEE*, vol. 54, no. 8, p. 1089, Aug. 1966.

[2] F. A. Grunbaum, "A property of the Legendre differential equation and its discretization," *SIAM J. Algorithms Discrete Methods*, vol. 7, pp. 358–361, 1986.

[3] A. Springer, W. Gugler, M. Huemen, L. Reindl, C. C. W. Ruppel, and R. Weigel, "Spread spectrum communication using chirp signals," in *Proc. Eurocomm 2000*, pp. 166–170.

[4] J. M. Chowning, "The synthesis of complex audio spectra by means of frequency modulation," *J. Audio Eng. Soc.*, vol. 21, no. 7, pp. 526–534, Sep. 1973.

[5] I. Daubechies, J. Lu, and H.-T. Wu, "Synchrosqueezed wavelet transforms: An empirical mode decomposition-like tool," *Appl. Comput. Harmon. Anal.*, vol. 30, no. 2, pp. 243–261, Mar. 2011.

[6] J. N. Pandey and R. S. Pathak, "Eigenfunction expansion of generalized functions," *Nagoya Math. J.*, vol. 72, pp. 1–25, 1978.

[7] U. K. Laine and T. Altosaar, "An orthogonal set of frequency and amplitude modulated (FAM) functions for variable resolution signal analysis," in *Proc. Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, 1990, vol. 3, pp. 1615–1618.

[8] A. C. Bovik, P. Maragos, and T. F. Quatieri, "AM-FM energy detection and separation in noise using multiband energy operators," *IEEE Trans. Signal Process.*, vol. 41, no. 12, pp. 3245–3265.

[9] P. Maragos, J. F. Kaiser, and T. F. Quatieri, "On amplitude and frequency demodulation using energy operators," *IEEE Trans. Signal Process.*, vol. 41, pp. 1532–1550, Apr. 1993.

[10] M. S. Pattichis, A. C. Bovik, J. W. Havalicek, and N. D. Sidiropoulos, "Multidimensional orthogonal FM transforms," *IEEE Trans. Image Process.*, vol. 10, no. 3, pp. 448–464, 2001.

[11] Q. Li and L. E. Atlas, "Over-modulated AM-FM decomposition," in *Proc. SPIE*, 2004, vol. 5559, pp. 172–183.

[12] J. G. Vargas and B. Santhanam, "On the multiangle centered discrete fractional Fourier transform," *IEEE Signal Process. Lett.*, vol. 12, no. 4, pp. 273–276, Apr. 2005.

[13] A. Jirari, "Second-order sturm-liouville difference equations and orthogonal polynomials," *Memoirs Amer. Math. Soc.*, vol. 113, no. 542, Jan., 1995.

[14] S. A. Martucci, "Symmetric convolution and the discrete sine and cosine transforms," *IEEE Trans. Signal Process.*, vol. 42, pp. 1038–1051, 1994.

[15] G. Strang, "The discrete cosine transform," *SIAM Rev.*, vol. 41, no. 1, pp. 135–147, 1999.

[16] A. S. Kayhan, "Difference equation representation of chirp signals and instantaneous frequency/amplitude estimation," *IEEE Trans. Signal Process.*, vol. 44, no. 12, pp. 2948–2958, 1996.

[17] B. Santhanam, "On a Sturm-Liouville framework for discrete and continuous and discrete modulation," in *Proc. Asilomar Conf. Signals, Syst., Comput.*, Nov. 2009, pp. 1747–1751.

[18] G. Arfken, *Mathematical Methods for Physicists*, 3rd ed. New York: Academic, 1985.

[19] F. B. Hildebrand, *Finite Difference Equations and Simulations*. Englewood Cliffs, NJ: Prentice-Hall, 1968.

[20] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*. Oxford, U.K.: Oxford Univ. Press, 2004.

[21] J. G. Proakis, *Digital Communications*, 5th ed. New York: McGraw-Hill, 2007.

[22] N. W. McLachlan, *Theory and Application of Mathieu Functions*. New York: Dover, 1964.

[23] J. C. Gutierez-Vega and R. M. Rodriguez-Dagnino, "Mathieu functions, A visual approach," *Amer. J. Phys.*, vol. 71, no. 3, pp. 233–242, 2003.

[24] A. Zayezdny and I. Druckmann, "A new method of signal description and its applications to signal processing," *Signal Process.*, vol. 22, pp. 153–178, 1991.

Balu Santhanam (S'92–M'98–SM'05), received the B.S. degree in electrical engineering from Saint Louis University, Saint Louis, MO, in 1992 and the M.S. and Ph.D. degrees in electrical engineering from the Georgia Institute of Technology, Atlanta, in 1994 and 1998 respectively.

He completed a year as a Lecturer and Postdoctoral Researcher in the Department of Electrical and Computer Engineering (ECE) at the University of California Davis, from 1998 to 1999. In 1999, he joined the Department of ECE at the University of New Mexico, Albuquerque, where he is currently an Associate Professor of electrical engineering. His current research interests include discrete fractional Fourier analysis and applications, multicomponent AM-FM models and demodulation, signal representations for modulated signals, hybrid ICA-SVM systems, and their application to pattern recognition problems.

Dr. Santhanam served as the Technical Area Chair for adaptive systems for the Asilomar Conference on Signals, Systems, and Computers, 2004. He served on the international program committee for the IEEE international Conference on Systems, Man, and Cybernetics in 2005 and 2006, and the 2011 IEEE DSP Workshop. He is the recipient of the UNM ECE Department's 2000 and 2005 distinguished teacher award. He is currently the Chair of the Albuquerque Chapter of the IEEE Signal Processing and Communication societies.