# The Discrete Rotational Fourier Transform

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*Abstract*—**In this correspondence, we define a discrete version of the angular Fourier transform and present the properties of the transform that show it to be a rotation in time-frequency space. This new transform is a generalization of the DFT. Efficient algorithms for its computation can then be based on the FFT and the eigenstructure of the DFT.**

## I. Introduction

The continuous time Fourier transform (CTFT), which is defined by the following pair:

$$X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) \exp(-j\omega t)\, dt \leftrightarrow$$
$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(\omega) \exp(j\omega t)\, d\omega \qquad (1)$$

transforms a point $x(t)$ in $L^2(R)$ to a point $X(\omega)$ in $L^2(R)$. The CTFT also treats the time and frequency axes as orthogonal axes, which amounts to the assumption that the signal of interest has stationary frequency content. However, signal representations using intermediate "angularly coupled axes" hold some promise for analyzing signals with time-frequency coupling, e.g., linear-FM. Angular generalizations of the CTFT, called the angular Fourier transforms or fractional Fourier transforms (AFT or FRFT), which are controlled by a single continuous angular parameter $\alpha$, have been developed in [1] and [5]. These transformations, when evaluated for parameter values of $\alpha = 0°$ and $360°$, yield the identity transformation and with $\alpha = 90°$, they yield the CTFT. As envisaged in [2], the AFT is useful in the study of frequency swept filters as well as for solving certain classes of differential equations that appear in quantum mechanics and optics. In this correspondence, we present a discrete version of the AFT called the discrete rotational Fourier transform (DRFT). It is an angular generalization of the DFT operator. The DFT matrix $\mathbf{W}$ in its unitary form is the $N \times N$ matrix $\mathbf{W}$ with entries

$$W_{nk} = \frac{1}{\sqrt{N}} \exp\left(-j\frac{2\pi}{N} nk\right) \qquad (2)$$

which has $N$ orthogonal eigenvectors and four distinct eigenvalues $[1, -1, j, -j]$ [4]. The DRFT is derived from the fractional powers of these eigenvalues and amounts to a rotation of the discrete $(n, k)$ axis through a continuous angular parameter $\alpha$. The unitary nature of the resultant operator is exploited in Section V to develop an algorithm for the computation of the DRFT.

## II. Interpretation as a Rotation

Successive application of the Fourier transformation $\mathbf{F}$ on $x(t)$ yields

$$\mathbf{F}^2[x(t)] = x(-t),$$
$$\mathbf{F}^3[x(t)] = X(-\omega)$$

and

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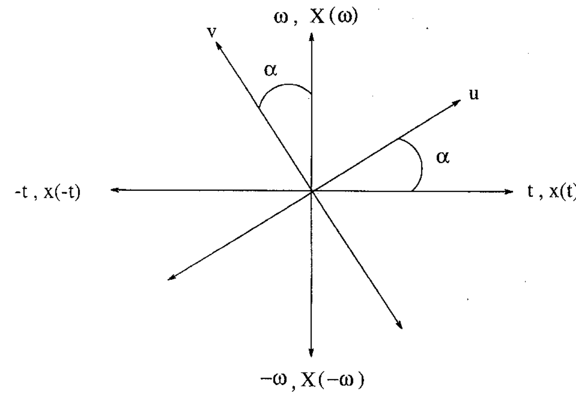


Fig. 1. AFT as a rotation in $(t, \omega)$ space.

$$\mathbf{F}^4[x(t)] = x(t). \qquad (3)$$

Since $\mathbf{F}^2$ corresponds to a reflection or a $180°$ rotation, these equations lead us to interpret the CTFT as a $90°$ rotation operator in the $(t, \omega)$ plane. Similarly, $\mathbf{F}^3$ corresponds to a reflection of the CTFT or a $270°$ rotation, and $\mathbf{F}^4$ corresponds to the identity operation or a $360°$ rotation as described in Fig. 1. A generalization of the CTFT is obtained if we consider a rotation through an arbitrary angle $\alpha$ in the $(t, \omega)$ plane. This is the AFT of a signal $x(t)$, which is defined as

$$X_\alpha(u) = \sqrt{\frac{1 - j\cot\alpha}{2\pi}} \int_{-\infty}^{\infty} x(t) \exp \cdot \left[j\left(\frac{t^2 + u^2}{2} \cot\alpha - tu \operatorname{cosec}\alpha\right)\right] dt \qquad (4)$$

where the variables $(u, v)$ in Fig. 1 are defined through the unitary transformation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} t \\ \omega \end{pmatrix}.$$

The AFT represents a rotation of the signal $x(t)$ in the $(t, \omega)$ space through an angle "$\alpha$" and serves as an orthonormal signal representation in terms of chirp signals [2]. The AFT can also be related to other time-frequency transformations like the STFT and the Wigner distribution. The properties of the AFT have been analyzed in detail in [1] and [2]. A discussion of the AFT for a smaller range of parameter values through the Weil representation of the group $SL_2(R)$[1] can be found in [5].

## III. Discrete Version of the AFT

The discrete Fourier transform of a point $x[n]$ in the vector space of finite-norm, length-$N$ sequences [6] is defined by the following DFT pair:

$$X[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x[n] \exp\left(-j\frac{2\pi}{N} nk\right) \leftrightarrow$$
$$x[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X[k] \exp\left(j\frac{2\pi}{N} nk\right).$$

[1] $SL_2(R)$ is the group of real $2 \times 2$ matrices with determinant 1.

Unfortunately, when creating an angular extension to the DFT operator, we cannot follow the same approach as in continuous time because a discrete $(n, k)$ grid does not permit rotational transformations, such as (4) with noninteger entries. Instead, we direct our attention to fractional powers of the DFT operator

$$\mathbf{A}_\alpha(x) = \mathbf{W}^{2\alpha/\pi}(x). \tag{5}$$

When the operator $\mathbf{A}_\alpha$ is evaluated for parameter values of $\alpha = 0$ and $2\pi$, one obtains

$$\mathbf{A}_{2\pi} = \mathbf{W}^4 = \mathbf{I} = \mathbf{W}^0 = \mathbf{A}_0. \tag{6}$$

When evaluated for a parameter value of $\alpha = \pi/2$, one obtains the DFT operator

$$\mathbf{A}_{\pi/2} = \mathbf{W}^1 = \mathbf{W}. \tag{7}$$

Upon evaluation for $\alpha = \pi$, one obtains the cyclic flip matrix

$$\mathbf{A}_\pi = \mathbf{W}^2. \tag{8}$$

These relations provide us with the motivation to interpret the operator $\mathbf{A}_\alpha$ as a rotation in the $(n, k)$ plane. A Taylor series expansion of the operator $\mathbf{A}_\alpha$ followed by the application of the Cayley–Hamilton theorem and regrouping of terms suggests that we could write

$$\begin{aligned}\mathbf{A}_\alpha &= a_0(\alpha)\mathbf{I} + a_1(\alpha)\mathbf{W} \\ &\quad + a_2(\alpha)\mathbf{W}^2 + a_3(\alpha)\mathbf{W}^3.\end{aligned} \tag{9}$$

This equation gives an interesting interpretation of subspace mixing, where the coefficients $[a_i(\alpha), i = 0, 1, 2, 3]$ generate an angular mixture of the basis matrices $(\mathbf{W}^i, i = 0, 1, 2, 3)$. Explicit expressions for the coefficients will be obtained in Section V. Since the operator $\mathbf{W}$ defined in (2) is unitary and has a basis of $N$ orthonormal eigenvectors $\vec{v}_i$ [6], its eigendecomposition is given by

$$\begin{aligned}\mathbf{W} &= \sum_{i\in N_1} \vec{v}_i\vec{v}_i^H - \sum_{i\in N_2} \vec{v}_i\vec{v}_i^H \\ &\quad + j\left(\sum_{i\in N_3} \vec{v}_i\vec{v}_i^H - \sum_{i\in N_4} \vec{v}_i\vec{v}_i^H\right)\end{aligned} \tag{10}$$

where $\mathcal{N}_1$ is the set of indices for eigenvectors belonging to $\lambda = 1$, $\mathcal{N}_2$ for $\lambda = -1$, and so on. The operator $\mathbf{A}_\alpha$, being a matrix power of $\mathbf{W}$, is also unitary and has an eigendecomposition defined by taking a fractional power of the eigenvalues $(\lambda_i)^{2\alpha/\pi}$.

$$\begin{aligned}\mathbf{A}_\alpha = &\sum_{i\in N_1} \exp(j4k_1\alpha)\vec{v}_i\vec{v}_i^H \\ &+ \sum_{i\in N_2} \exp[j(4k_2+2)\alpha]\vec{v}_i\vec{v}_i^H \\ &+ \sum_{i\in N_3} \exp[j(4k_3+1)\alpha]\vec{v}_i\vec{v}_i^H \\ &+ \sum_{i\in N_4} \exp[j(4k_4-1)\alpha]\vec{v}_i\vec{v}_i^H\end{aligned}$$

where $k_i \in \mathbf{Z}$. It should be noted that because of the ambiguity present in the fractional power operation, the operator $\mathbf{A}_\alpha$ defined in (5) is not unique. However, we take $k_i = 0$ to define the "principal value."

## IV. Properties of the Operator

### A. Unitary

The operator is unitary; hence

$$\mathbf{A}_\alpha^H = \mathbf{A}_\alpha^{-1} = \mathbf{A}_{-\alpha}. \tag{11}$$

As a consequence of this property, one obtains the IDRFT relation

$$\mathbf{A}_{-\alpha} = a_0^*(\alpha)\mathbf{I} + a_3^*(\alpha)\mathbf{W} + a_2^*(\alpha)\mathbf{W}^2 + a_1^*(\alpha)\mathbf{W}^3. \tag{12}$$

Since $\mathbf{A}_\alpha\mathbf{A}_{-\alpha} = \mathbf{I}$, we have

$$\begin{aligned}&\sum_{i,j=0,1,2,3} a_i(\alpha)a_j(-\alpha)\delta[((i+j))_4] = 1 \\ &\sum_{i,j=0,1,2,3} a_i(\alpha)a_j(-\alpha)\delta[((i+j-1))_4] = 0 \\ &\sum_{i,j=0,1,2,3} a_i(\alpha)a_j(-\alpha)\delta[((i+j-2))_4] = 0 \\ &\sum_{i,j=0,1,2,3} a_i(\alpha)a_j(-\alpha)\delta[((i+j-3))_4] = 0.\end{aligned} \tag{13}$$

Again, since the operator $\mathbf{A}_\alpha$ is unitary, we can prove a "Parseval" theorem:

$$||\mathbf{A}_\alpha[x]||^2 = ||x||^2. \tag{14}$$

### B. Angle Additivity

Application of the operator with angular parameter "$\alpha$" followed by an application of the operator with angular parameter "$\beta$" is equivalent to the application of the operator with angular parameter "$\alpha + \beta$" as given by

$$\mathbf{A}_\alpha\mathbf{A}_\beta = \mathbf{A}_{\alpha+\beta}. \tag{15}$$

Multiple applications of the operator to a signal is equivalent to one rotation through a larger angle in the $(n, k)$ plane as expressed in

$$\mathbf{A}_\alpha^m = \mathbf{A}_{m\alpha}. \tag{16}$$

The corresponding relations between the coefficients are

$$\begin{aligned}&[a_0(\alpha) + a_1(\alpha) + a_2(\alpha) + a_3(\alpha)]^n = \\ &\quad a_0(n\alpha) + a_1(n\alpha) + a_2(n\alpha) + a_3(n\alpha).\end{aligned} \tag{17}$$

### C. DFT ≡ 90° Rotation

Application of DFT operator is equivalent to a 90° rotation as expressed in

$$\mathbf{A}_{\alpha+(\pi/2)} = \mathbf{W}\mathbf{A}_\alpha \tag{18}$$

while the corresponding relation between the coefficients are

$$a_i\left(\alpha + \frac{\pi}{2}\right) = a_{i-1}(\alpha), \quad \text{where } a_{-1} = a_3. \tag{19}$$

### D. Circular Flip Property

A circular flip transforms into a circular flip in the rotated frequency domain as given by

$$\mathbf{A}_\alpha\mathbf{W}^2(x[n]) = \mathbf{W}^2(\mathbf{A}_\alpha x[n]). \tag{20}$$

This property is a consequence of the fact that $\mathbf{A}_\alpha$ is a matrix function of $\mathbf{W}$ and hence commutes with $\mathbf{W}^2$.

### E. Periodicity

The operator is periodic in the parameter $\alpha$ with a fundamental period of $2\pi$ as expressed in

$$\mathbf{A}_\alpha = \mathbf{A}_{\alpha+2r\pi}. \tag{21}$$

This implies that the coefficients are periodic in the parameter $\alpha$ with period $2\pi$. These properties illustrate well the angular and rotational aspects of the transform.

## V. Computation of the DRFT

Direct computation of the transform for an angle $\alpha$ requires that we perform an eigendecomposition on the operator $\mathbf{W}$ and form the new matrix operator $\mathbf{A}_\alpha$. This can be an impractical thing to do for large orders. Instead, we can determine the coefficients in the expansion of (9) directly. Upon reproducing the eigenvalue expansion of the operator $\mathbf{A}_\alpha$, one obtains

$$\begin{aligned}\mathbf{A}_\alpha &= \mathbf{T}\mathbf{\Lambda}^{2\alpha/\pi}\mathbf{T}^H \\ &= \sum_{i=0}^{3} a_i(\alpha)\mathbf{W}^i\end{aligned} \tag{22}$$

where $\mathbf{T}$ is the matrix of eigenvectors of $\mathbf{W}$ as well as of $\mathbf{A}_\alpha$. Application of the operator $\mathbf{T}^H$ on the left and $\mathbf{T}$ on the right of (22) yields the following diagonal system of equations:

$$\mathbf{\Lambda}^{2\alpha/\pi} = \sum_{i=0}^{3} a_i(\alpha)\mathbf{\Lambda}_{W^i} \tag{23}$$

where $\mathbf{\Lambda}_{W^i} = \mathbf{T}^H\mathbf{W}^i\mathbf{T}$ is a diagonal matrix. This is a system of $N$ linear equations in four unknowns $[a_i(\alpha),\ i = 0, 1, 2, 3]$, where only four of these equations are independent. The others are repetitions of these four because of the repeated eigenvalues of the DFT matrix. When $N \le 4$, some of the eigenvalues are missing; therefore, the representation folds over to accommodate those deficiencies. Specifically, for the operator of dimension $N = 2$, we obtain

$$\mathbf{A}_{\alpha,2} = [a_0(\alpha) + a_2(\alpha)]\mathbf{I} + [a_1(\alpha) + a_3(\alpha)]\mathbf{W}. \tag{24}$$

For $N \ge 5$, DFT matrices have all four eigenvalues in their eigenvalue expansion, and the coefficients become independent of the order $N$ and can be determined explicitly from the diagonal system of (23). Since the domain of the fractional power operation needs to be restricted to make it a function, we choose the branch cut $\{k_i = 0,\ i = 1, 3, 4, 5\}$. In addition, an ordering must be imposed on the eigenvalues according to $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = j$, and $\lambda_4 = -j$ in order to make the solution unique. Upon substitution, one obtains the following system of equations for $N = 5$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \\ 1 & -j & -1 & j \end{pmatrix} \cdot \begin{pmatrix} a_0(\alpha) \\ a_1(\alpha) \\ a_2(\alpha) \\ a_3(\alpha) \end{pmatrix} = \begin{pmatrix} \exp(j4k_1\alpha) \\ \exp[j2\alpha(2k_3+1)] \\ \exp[j\alpha(4k_4+1)] \\ \exp[j\alpha(4k_5-1)] \end{pmatrix}. \tag{25}$$

The solution to (25) with $\{k_i = 0,\ i = 1, 3, 4, 5\}$, when simplified, is

$$\begin{aligned} a_0(\alpha) &= \tfrac{1}{2}(1 + e^{j\alpha})\cos\alpha \\ a_1(\alpha) &= \tfrac{1}{2}(1 - je^{j\alpha})\sin\alpha \\ a_2(\alpha) &= \tfrac{1}{2}(e^{j\alpha} - 1)\cos\alpha \\ a_3(\alpha) &= \tfrac{1}{2}(-1 - je^{j\alpha})\sin\alpha. \end{aligned} \tag{26}$$

These coefficients satisfy all the relations mentioned before and are periodic in the parameter $\alpha$ with period $2\pi$. They are also unique for the specific branch cut chosen.

Summarizing the algorithm for constructing $\mathbf{A}_\alpha$, one first generates the matrices $(\mathbf{W}^i,\ i = 0, 1, 2, 3)$ for a given dimension $N$, where $\mathbf{W}^2$ is the circular flip matrix, and $\mathbf{W}^3$ is a circularly flipped version of $\mathbf{W}$. The DRFT coefficients are then obtained via (25). The transform and its inverse are then constructed via the subspace projection equations (9) and (12). This algorithm has the advantage in that an eigendecomposition of $\mathbf{W}$ is not necessary, thereby reducing computational complexity. If the DRFT of $\mathbf{x}$ is to be computed, there are two possibilities: Multiply by the matrix $\mathbf{A}_\alpha$ or use (9) to combine $\mathbf{x}$, its DFT, and their circularly flipped versions. The second approach could exploit the FFT for faster computation.

## VI. Example

Consider a signal mixture $x[n]$ of a sinusoid and an impulse

$$x[n] = \cos\left(\frac{2\pi}{100}n\right) + \delta(n - 100).$$

Fig. 2(a)–(d) shows the magnitudes of the 210-pt DRFT for the above mixture at angles 15°, 30°, 45°, and 60°. As we rotate from 0–90°, the two components exchange roles by showing their peaks. In other words, the signal energy that remains a constant over different angles gets squeezed into different frequencies. The DRFT of a signal $x[n]$ is therefore equivalent to an "angular mixture" of the signal, its DFT, the signal circularly flipped, and its DFT circularly flipped. In the case of signals with symmetry (real signals, even and odd symmetric signals, etc.), the DRFT expansion of (9) folds over to accommodate those symmetries. In general, for a signal that lacks any kind of symmetry, the DRFT is a complete orthonormal signal expansion.[2]

The DRFT is, however, different from other time–frequency transformations in that it transforms a signal $x[n]$ into a signal $X(\alpha, k]$ as opposed to a signal $X[n, k]$ in time and frequency. Fractional powers and functions of the DFT have been investigated in [3] using Lagrange interpolation polynomials. Cryptographic applications for the removal of effects of pseudonoise sequences given a "key" angle, and transform coding applications for data compression purposes have also been suggested [3]. Note that the DRFT is able to analyze a signal along intermediate axes indexed by $(n, k, \alpha)$ as opposed to the DFT that only allows analysis along $(n, k, 0)$, suggesting applications akin to time–frequency multiplexing.

## VII. The DRFT as an Orthonormal Signal Expansion

The DRFT of a discrete signal $x[n]$ is written as

$$X_\alpha[k] = \sum_{n=0}^{N-1} K_\alpha[n, k]x[n]$$

where the kernel of the transformation corresponding to the operator $\mathbf{A}_\alpha$ is given by

$$\begin{aligned} K_\alpha[n, k] = {}& a_0(\alpha)\delta[n-k] + \frac{a_1(\alpha)}{\sqrt{N}}\exp\left(-j\frac{2\pi}{N}nk\right) \\ &+ a_2(\alpha)\delta[((n+k))_N] + \frac{a_3(\alpha)}{\sqrt{N}}\exp\left(j\frac{2\pi}{N}nk\right) \end{aligned} \tag{27}$$

while the signal is recovered through the IDRFT relation. Some significant properties of the kernel are given in the following:

1) The kernel is symmetric in $n$ and $k$

$$K_\alpha[n, k] = K_\alpha[k, n]$$

[2] This operator is a unitary group representation of the additive group of real numbers.

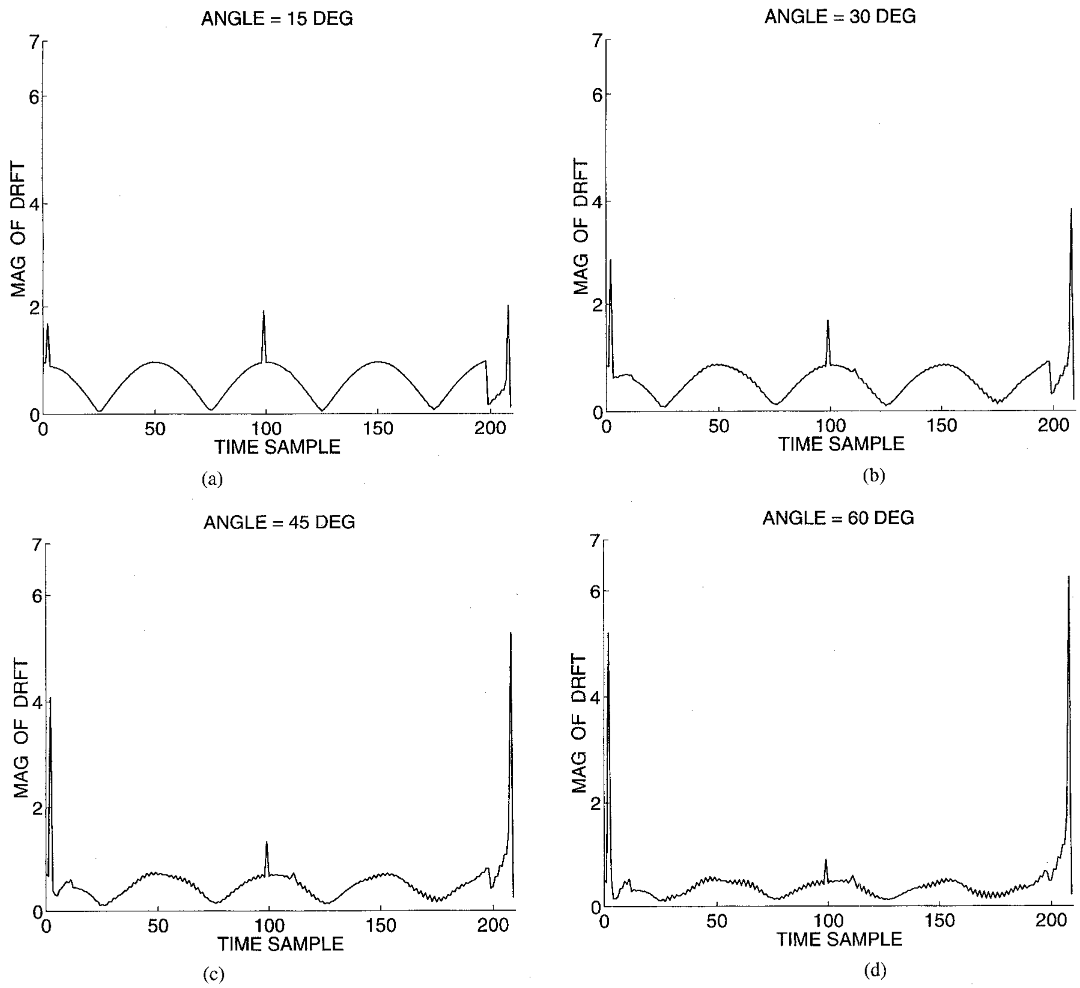


Fig. 2. Magnitude of the DRFT of the example signal for angles (a) $\alpha = 15^\circ$, (b) $\alpha = 30^\circ$, (c) $\alpha = 45^\circ$, and (d) $\alpha = 60^\circ$.

which signifies the fact that the roles of the time and frequency variables are interchangeable.

2) The kernel is periodic in the parameter $\alpha$, which is analogous to a rotation operation as is evident from

$$K_{\alpha+2\pi}[n, k] = K_\alpha[n, k].$$

3) The kernel corresponds to the basis function of an orthonormal representation

$$\sum_{n=0}^{N-1} K_\alpha[n, k]K_\alpha^*[n, k'] = \delta[k - k']. \tag{28}$$

4) The inverse operation of a rotation through an angle "$\alpha$" is a rotation through "$-\alpha$"

$$K_{-\alpha}[n, k] = K_\alpha^*[n, k].$$

5) A rotation through "$\alpha$" followed by a rotation through "$\beta$" is a rotation through "$\alpha + \beta$"

$$\sum_{k'=0}^{N-1} K_\alpha[n, k']K_\beta[k', k] = K_{\alpha+\beta}[n, k]. \tag{29}$$

6) The kernel of the transformation is an angular mixture of four basic kernels

$$\begin{aligned} K_\alpha[n, k] =& a_0(\alpha)K_0[n, k] + a_1(\alpha)K_{\pi/2}[n, k] \\ &+ a_2(\alpha)K_\pi[n, k] + a_3(\alpha)K_{3\pi/2}[n, k]. \end{aligned}$$

## VIII. Conclusion

The DRFT, which is an angular generalization of the DFT and represents a rotation in $(n, k)$ space, has been presented. The DRFT operator when evaluated for angular parameter values of $0^\circ$ and $360^\circ$ becomes the identity operation, where on evaluation for $90^\circ$ yields the DFT operator. The DRFT of a signal is equivalent to an angular mixture of the signal, its DFT, a circularly flipped version of the signal, and a circularly flipped version of its DFT. An efficient algorithm that avoids eigenvalue decomposition has also been presented.

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## Localization of the Complex Spectrum: The $S$ Transform

R. G. Stockwell, L. Mansinha, and R. P. Lowe

***Abstract*—The $S$ transform, which is introduced in this correspondence, is an extension of the ideas of the continuous wavelet transform (CWT) and is based on a moving and scalable localizing Gaussian window. It is shown here to have some desirable characteristics that are absent in the continuous wavelet transform. The $S$ transform is unique in that it provides frequency-dependent resolution while maintaining a direct relationship with the Fourier spectrum. These advantages of the $S$ transform are due to the fact that the modulating sinusoids are fixed with respect to the time axis, whereas the localizing scalable Gaussian window dilates and translates.**

### I. Introduction

In geophysical data analysis and in many other disciplines, the concept of a stationary time series is a mathematical idealization that is never realized and is not particularly useful in the detection of signal arrivals. Although the Fourier transform of the entire time series does contain information about the spectral components in a time series, for a large class of practical applications, this information is inadequate. An example from seismology is an earthquake seismogram. The first signal to arrive from an earthquake is the $P$ (primary) wave followed by other $P$ waves traveling along different paths. The $P$ arrivals are followed by the $S$ (secondary) wave and by higher amplitude dispersive surface waves. The amplitude of these oscillations can increase by more than two orders of magnitude within a few minutes of the arrival of the $P$. The spectral components of such a time series clearly have a strong dependence on time. It would be desirable to have a joint time-frequency representation (TFR). This correspondence proposes a new transform (called the $S$ transform) that provides a TFR with frequency-dependent resolution while, at the same time, maintaining the direct relationship, through time-averaging, with the Fourier spectrum. Several techniques of examining the time-varying nature of the spectrum have been proposed in the past; among them are the Gabor transform [7], the related short-time Fourier transforms, the continuous wavelet transform (CWT) [8], and the bilinear class of time–frequency distributions known as Cohen's class [4], of which the Wigner distribution [9] is a member.

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### II. The $S$ Transform

There are several methods of arriving at the $S$ transform. We consider it illuminating to derive the $S$ transform as the "phase correction" of the CWT. The CWT $W(\tau, d)$ of a function $h(t)$ is defined by

$$W(\tau, d) = \int_{-\infty}^{\infty} h(t) w(t-\tau, d)\, dt \tag{1}$$

where $w(t, d)$ is a scaled replica of the fundamental mother wavelet. The dilation $d$ determines the "width" of the wavelet $w(t, d)$ and thus controls the resolution. Along with (1), there exists an admissibility condition on the mother wavelet $w(t, d)$ [5] that $w(t, d)$ must have zero mean. Refer to Rioul and Vetterli [10] and Young [11] for reviews of the literature.

The $S$ transform of a function $h(t)$ is defined as a CWT with a specific mother wavelet multiplied by the phase factor

$$S(\tau, f) = e^{i2\pi f\tau} W(\tau, d) \tag{2}$$

where the mother wavelet is defined as

$$w(t, f) = \frac{|f|}{\sqrt{2\pi}} e^{-\frac{t^2 f^2}{2}} e^{-i2\pi f t}. \tag{3}$$

Note that the dilation factor $d$ is the inverse of the frequency $f$.

The wavelet in (3) does not satisfy the condition of zero mean for an admissible wavelet; therefore, (2) is not strictly a CWT. Written out explicitly, the $S$ transform is

$$S(\tau, f) = \int_{-\infty}^{\infty} h(t) \frac{|f|}{\sqrt{2\pi}} e^{-\frac{(\tau - t)^2 f^2}{2}} e^{-i2\pi f t} dt. \tag{4}$$

If the $S$ transform is indeed a representation of the local spectrum, one would expect a simple operation of averaging the local spectra over time to give the Fourier spectrum. It is easy to show that

$$\int_{-\infty}^{\infty} S(\tau, f) d\tau = H(f) \tag{5}$$

(where $H(f)$ is the Fourier transform of $h(t)$). It follows that $h(t)$ is exactly recoverable from $S(\tau, f)$. Thus

$$h(t) = \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} S(\tau, f) d\tau \right\} e^{i2\pi f t} df. \tag{6}$$

This is clearly distinct from the concepts of the CWT.

The $S$ transform provides an extension of *instantaneous frequency* (IF) [2] to broadband signals. The 1-D function of the variable $\tau$ and fixed parameter $f_1$ defined by $S(\tau, f_1)$ is called a voice (as with the CWT). The voice for a particular frequency $f_1$ can be written as

$$S(\tau, f_1) = A(\tau, f_1) e^{i\Phi(\tau, f_1)}. \tag{7}$$

Since a voice isolates a specific component, one may use the phase in (7) to determine the IF as defined by Bracewell [2].

$$\mathrm{IF}(\tau, f_1) = \frac{1}{2\pi} \frac{\partial}{\partial t} \{2\pi f_1 \tau + \Phi(\tau, f_1)\}. \tag{8}$$

Thus, the absolutely referenced phase information leads to a generalization of the IF of Bracewell to broadband signals. The validity of (8) can easily be seen for the simple case of $h(t) = \cos(2\pi w t)$, where the function $\Phi(\tau, f) = 2\pi(w - f)\tau$.

The linear property of the $S$ transform ensures that for the case of additive noise (where one can model the data as $\mathrm{data}(t) = \mathrm{signal}(t) + \mathrm{noise}(t)$), the $S$ transform gives

$$S\{\mathrm{data}\} = S\{\mathrm{signal}\} + S\{\mathrm{noise}\}.$$