On a Matrix Framework for the Teager-Kaiser Energy Operator

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***Abstract*—The non-linear, second-order, *Teager-Kaiser energy operator* (TKEO) finds applications in numerous areas such as speech processing, image processing, communications, biomedical problems, and abrupt event detection. In quantum mechanics, the Hamiltonian is considered the energy operator whose eigenvalue is the scalar energy. In this work, we provide a matrix framework for the Teager-Kaiser operator and several of its generalizations, interpreting the eigenvalues of the presented energy matrices as the square-root of the TKEO or its generalization.**

***Index Terms*—Keywords: Teager-Kaiser energy operator, higher-order energy operators, energy matrices, eigenvalues, conditional positivity.**

I. Introduction

The classical Teager-Kaiser energy operator in the continuous and discrete cases is defined via [1], [2]:

$$\begin{aligned}\psi_c(x) &= [\dot{x}(t)]^2 - x(t)\ddot{x}(t)\\ \psi_d(x) &= x^2[n] - x[n+1]x[n-1].\end{aligned} \tag{1}$$

Originally used to describe nonlinear phenomena in speech production, the operator has found numerous applications in abrupt event detection [12], image texture analysis [11], communications [14], time-frequency analysis [4], and speech processing [8], [4], and several other areas. The operator derives its name from the fact that it computes the energy of a classical harmonic oscillator normalized by half unit mass. The output of the operator further tracks the energy of a frequency and/or amplitude modulated waveform and the TKEO constitutes the key ingredient of the related energy separation algorithm [7].

Prior efforts at generalization of this operator by Kumaresan et. al. into a matrix framework have focussed on interpreting the TKEO as the determinant of a Toeplitz matrix containing the signal and its derivatives [3], [5]:

$$\mathbf{A}_x^{(c)} = \begin{pmatrix} \dot{x}(t) & x(t) \\ \ddot{x}(t) & \dot{x}(t) \end{pmatrix} \tag{2}$$

or in the discrete case, as the determinant of the signal matrix:

$$\mathbf{A}_x^{(d)} = \begin{pmatrix} x[n] & x[n-1] \\ x[n+1] & x[n] \end{pmatrix} \tag{3}$$

Higher-order extensions of the TKEO were developed in [9] and measure the higher-order energies of the classical harmonic oscillator:

$$\begin{aligned}\Upsilon_k^{(c)}(x) &= \dot{x}x^{(k-1)} - xx^{(k)}\\ \Upsilon_k^{(d)}(x) &= x[n]x[n+k-2] - x[n-1]x[n+k-1].\end{aligned} \tag{4}$$

For example for $k = 2$, these reduce to the TKEO in both the continuous and discrete cases. For $k = 3$, the output of the higher-order energy operator has dimensions of energy-velocity. These higher-order energy operators have been used to extend the energy separation and demodulation approach to two-component AM–FM signals in [17].

II. Quantum Mechanical Energy Operator

The notion of an energy operator arising in quantum mechanics and the quantum oscillator revolves around the Hamiltonian operator $\mathbf{H}$ whose eigenvalue is the scalar energy of oscillator [6]:

$$\mathbf{H}\phi = E\phi, \tag{5}$$

where E denotes the "observable" scalar energy eigenvalue, and ϕ denotes the eigenfunctions of the Hamiltonian. The Hamiltonian is for most physical problems, a self-adjoint operator, and consequently the eigenvalues E, corresponding to the energy, are purely real. For the specific case of the quantum harmonic oscillator, the Hamiltonian takes the form:

$$\mathcal{H} = \frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}Ky^2, \tag{6}$$

where K denotes the spring constant, m is the mass associated with the quantum oscillator, and $\hbar$ denotes the normalized Plank's constant. In finite dimensions, the Hamiltonian reduces to a Hermitian symmetric matrix [15].

In quantum mechanics [6] the measured quantities such as the position, momentum, and energy are the eigenvalues of their corresponding linear Hermitian operators. The TKEO, is a nonlinear operator from classical mechanics, that however, also claims to be a energy operator. The goal of this paper is to provide a matrix framework for the TKEO, interpreting its output as the "measured" energy corresponding to the eigenvalue of its underlying energy matrix.

III. On a Matrix Framework for the TKEO

The eigenvalues of the Toeplitz signal matrices $\mathbf{A}_x^{(c,d)}$ defined in the previous section in the continuous and discrete-cases are given by:

$$\begin{aligned}\rho_{1,2}^{(c)}(t) &= \dot{x}(t) \pm \sqrt{x(t)\ddot{x}(t)},\\ \rho_{1,2}^{(d)}[n] &= x[n] \pm \sqrt{x[n+1]x[n-1]}.\end{aligned} \tag{7}$$

Note that these are not directly connected to the energy operator and are in general not lowpass or slowly time-varying quantities. Instead we consider three classes of signal matrices in the continuous and discrete cases. The first group, class I, is defined via:

$$
\begin{aligned}
\boldsymbol{\Psi}_c &= \begin{pmatrix} \dot{x}(t) & -\ddot{x}(t) \\ x(t) & -\dot{x}(t) \end{pmatrix} \\
\boldsymbol{\Psi}_d &= \begin{pmatrix} x[n] & -x[n+1] \\ x[n-1] & -x[n] \end{pmatrix}
\end{aligned} \tag{8}
$$

Now let us consider class II of energy matrices that are defined via:

$$
\begin{aligned}
\boldsymbol{\Psi}_c &= \begin{pmatrix} \dot{x}(t) & \sqrt{x(t)\ddot{x}(t)} \\ -\sqrt{x(t)\ddot{x}(t)} & -\dot{x}(t) \end{pmatrix} \\
\boldsymbol{\Psi}_d &= \begin{pmatrix} x[n] & V[n] \\ -V[n] & -x[n] \end{pmatrix} \\
V[n] &= \sqrt{x[n+1]x[n-1]}.
\end{aligned} \tag{9}
$$

Class III of energy matrices are defined via:

$$
\begin{aligned}
\boldsymbol{\Psi}_c &= \begin{pmatrix} \dot{x}(t) & j\sqrt{x(t)\ddot{x}(t)} \\ j\sqrt{x(t)\ddot{x}(t)} & -\dot{x}(t) \end{pmatrix} \\
\boldsymbol{\Psi}_d &= \begin{pmatrix} x[n] & jV[n] \\ jV[n] & -x[n] \end{pmatrix}.
\end{aligned} \tag{10}
$$

The three classes presented correspond to the three different off-diagonal combinations of terms that produce the second term of the energy operator, i.e., $x(t)\ddot{x}(t)$ or $x[n+1]x[n-1]$. The diagonal terms of these matrices provide for the first term of the energy operator: $[\dot{x}(t)]^2$ or $x^2[n]$. Due to the fact that class III results in complex arithmetic, we will limit our discussions to class I and II.

It can be easily seen that the eigenvalues of the energy matrices in all the classes are given by:

$$
\begin{aligned}
\lambda^2 &- [\dot{x}(t)]^2 + x(t)\ddot{x}(t) = 0 \\
\lambda_{1,2}^{(c)} &= \pm\sqrt{\psi_c(x)} \\
\lambda^2 &- x^2[n] + x[n+1]x[n-1] = 0 \\
\lambda_{1,2}^{(d)} &= \pm\sqrt{\psi_d(x)}.
\end{aligned} \tag{11}
$$

In either case, the TKEO can be expressed as:

$$
\psi(x) = \frac{1}{2}(\lambda_1^2 + \lambda_2^2) = -\lambda_1\lambda_2. \tag{12}
$$

The corresponding matrix of eigenvectors in the discrete case for class I is given by:

$$
\mathbf{V} = \begin{pmatrix} x[n] + \sqrt{\psi_d(x)} & x[n-1] \\ x[n+1] & x[n] + \sqrt{\psi_d(x)} \end{pmatrix}. \tag{13}
$$

IV. Properties of Energy Operator Matrices

A. Trace and Determinant

At this point it is instructive to study some of the properties of the signal matrices $\boldsymbol{\Psi}_{c,d}(x)$. As a first point of observation, note that signal matrices described before are traceless with determinant given by:

$$
\begin{aligned}
\mathrm{Trace}(\boldsymbol{\Psi}_c) &= 0,\ \mathrm{Det}(\boldsymbol{\Psi}_c) = -\psi_c(x) \\
\mathrm{Trace}(\boldsymbol{\Psi}_d) &= 0,\ \mathrm{Det}(\boldsymbol{\Psi}_d) = -\psi_d(x).
\end{aligned} \tag{14}
$$

In the case, where the signal is a monocomponent AM–FM signal, this property results in the eigenvalues being the square root of the TKEO. As a second point of observation, the energy matrices in all three classes satisfy a second-order involution property:

$$
\begin{aligned}
\boldsymbol{\Psi}_c^2 &= \psi_c(x)\mathbf{I} \\
\boldsymbol{\Psi}_d^2 &= \psi_d(x)\mathbf{I}.
\end{aligned} \tag{15}
$$

B. Non-uniqueness

An important consequence of the trace and determinant property of the energy operator matrices is that in both the continuous and discrete cases, the signal matrices and their transpose are also energy operator matrices:

$$
\begin{aligned}
(\boldsymbol{\Psi}_c^T)^2 &= \psi_c(x)\mathbf{I} \\
(\boldsymbol{\Psi}_d^T)^2 &= \psi_d(x)\mathbf{I}
\end{aligned} \tag{16}
$$

If fact, the energy matrices $\boldsymbol{\Psi}$ in each of the three classes are arbitrary up to a similarity or unitary transformation $\mathbf{M}$. This follows from the fact that a similarity or unitary transformation would not change the characteristic polynomial:

$$
\det(\lambda\mathbf{I} - \boldsymbol{\Psi}) = \det(\lambda\mathbf{I} - \mathbf{M}\boldsymbol{\Psi}\mathbf{M}^{-1}) = 0. \tag{17}
$$

C. Conditional Positivity

Another consequence of the energy matrix framework is that Teager-Kaiser energy operator is only conditionally positive since the eigenvalues of the energy matrix $\boldsymbol{\Psi}$, $\lambda_{1,2}$ could be either: (a) purely real (opposite sign), or (b) complex conjugates of each other (purely complex):

$$
\begin{aligned}
\psi_{c,d}(x) &= 0.5\left(\lambda_1^2 + \lambda_2^2\right) > 0 \iff \lambda_{1,2} \in \mathbf{R} \\
\psi_{c,d}(x) &= 0.5\left(\lambda_1^2 + \lambda_2^2\right) < 0 \iff \lambda_{1,2} \in \mathbf{C}
\end{aligned} \tag{18}
$$

Fig. (1)(a-d) illustrate this connection between the conditional positivity of the output of the TKEO and the imaginary part of the eigenvalues of the TKEO matrix. This conditional positivity is a consequence of the energy matrices not being self-adjoint, and consistent with the work in [10], where conditions for the positivity of the TKEO were explored. Positivity of the energy operator output is further, a explicit requirement for the applicability of the *energy separation algorithm* (ESA) and its discrete version, the DESA, for monocomponent AM–FM demodulation [7], [8].

V. Other TKEO Generalizations

Lastly, of the three classes of energy matrices, class II corresponds better with the extension of the energy operator to complex-valued signals [13]. For a complex-valued signal, $x(t) = x_r(t) + jx_i(t)$, and a complex sequence $x[n] = x_r[n] + jx_i[n]$, the energy matrices would generalize as:

$$
\begin{aligned}
\boldsymbol{\Psi}_c &= \begin{pmatrix} \sqrt{\dot{x}_r^2(t) + \dot{x}_i^2(t)} & C(t) \\ -C(t) & -\sqrt{\dot{x}_r^2(t) + \dot{x}_i^2(t)} \end{pmatrix} \\
C(t) &= \sqrt{x_r(t)\ddot{x}_r(t) + x_i(t)\ddot{x}_i(t)} \\
\boldsymbol{\Psi}_d &= \begin{pmatrix} \sqrt{x_r^2[n] + x_i^2[n]} & C[n] \\ -C[n] & -\sqrt{x_r^2[n] + x_i^2[n]} \end{pmatrix}, \\
C[n] &= \sqrt{x_r[n-1]x_r[n+1] + x_i[n-1]x_i[n+1]}
\end{aligned} \tag{19}
$$

It can be verified here that the eigenvalues of the energy matrices listed above are given by:

$$\lambda_{1,2} = \pm\sqrt{\psi_{c,d}(x)},$$

where $\psi_{c,d}(x)$ here denotes the complex version of the energy operator [13]:

$$\begin{aligned} \psi_c(x) &= |\dot{x}(t)|^2 - \Re(x^*(t)\ddot{x}(t)) \\ &= \dot{x}_r^2(t) + \dot{x}_i^2(t) - x_r(t)\ddot{x}_r(t) - x_i(t)\ddot{x}_i(t) \\ &= \psi_c(x_r(t)) + \psi_c(x_i(t)) \\ \psi_d(x) &= |x[n]|^2 - \Re(x^*[n-1]x[n+1]). \end{aligned} \tag{20}$$

Note that in the specific case, where the input signal is purely real, the framework reduces to the real TKEO matrix framework described before.

Other generalizations of the TKEO such as the variable length energy operator (VTEO) [16] can also be put into the same matrix framework:

$$\mathbf{\Psi}_i(x[n]) = \begin{pmatrix} x[n] & x[n-i] \\ -x[n+i] & -x[n] \end{pmatrix}. \tag{21}$$

It is easy to verify that the eigenvalues of VTEO matrix are given by:

$$\lambda_{1,2} = \pm\sqrt{\psi_i(x[n])}$$

where the VTEO of the modulated sequence $x[n]$ is given by:

$$\psi_i(x[n]) = x^2[n] - x[n+i]x[n-i]. \tag{22}$$

The higher-order energy operators introduced in Eq. (4) can be formulated in this framework as:

$$\begin{aligned} \mathbf{\Upsilon}_c^{(k)}(x) &= \begin{pmatrix} \sqrt{\dot{x}(t)x^{(k-1)}(t)} & -\sqrt{x(t)x^{(k)}(t)} \\ \sqrt{x(t)x^{(k)}(t)} & -\sqrt{\dot{x}(t)x^{(k-1)}(t)} \end{pmatrix} \\ \mathbf{\Upsilon}_d^{(k)}(x) &= \begin{pmatrix} A_k[n] & -B_k[n] \\ B_k[n] & -A_k[n] \end{pmatrix} \\ A_k[n] &= \sqrt{x[n]x[n+k-2]} \\ B_k[n] &= \sqrt{[x[n-1]x[n+k-1]}. \end{aligned} \tag{23}$$

It is again easily verified that for both these matrices, the eigenvalues of the these matrices are:

$$\lambda_{1,2} = \pm\sqrt{\Upsilon_k(x)}.$$

Note that unlike the scalars $\psi_{c,d}(x)$ and $\psi_i(x)$, that are in general positive, the higher-order energy operators can in general be negative. Consequently, using the matrix framework we will only be able to extract the magnitude of the operator from the eigenvalues for odd orders according to:

$$\frac{1}{2}(\lambda_1^2 + \lambda_2^2) = \begin{cases} \Upsilon_k(x), & k = 2, 6, 10, \ldots \\ -\Upsilon_k(x), & k = 4, 8, 12, \ldots \\ |\Upsilon_k(x)|, & k \text{ odd} \end{cases} \tag{24}$$

VI. Energy Separation Algorithms

Now that a matrix framework has been established for the TKEO, we can now formulate the related energy separation algorithms in terms of the eigenvalues of the underlying energy matrices. For the ESA in continuous-time, we can formulate the instantaneous frequency and envelope estimates as:

$$\begin{aligned} \omega_i(t) &\approx \frac{\sqrt{\psi_c(\dot{x})}}{\sqrt{\psi_c(x)}} = \frac{\lambda(\mathbf{\Psi}_c(\dot{x}))}{\lambda(\mathbf{\Psi}_c(x))} \\ |a(t)| &\approx \frac{\psi_c(\dot{x})}{\sqrt{\psi_c(x)}} = \frac{\lambda^2(\mathbf{\Psi}_c(\dot{x}))}{\lambda(\mathbf{\Psi}_c(x))}. \end{aligned} \tag{25}$$

In discrete-time, specifically for the DESA-1 algorithm we can formulate the instantaneous frequency and envelope estimates for $y[n] = x[n] - x[n-1]$ and $z[n] = x[n+1] - x[n]$ as:

$$\begin{aligned} G[n] &= \frac{\lambda^2(y[n]) + \lambda^2(z[n])}{4\lambda^2(x[n])} \\ \Omega_i[n] &\approx \cos^{-1}(1 - G[n]) \\ |a[n]| &\approx \frac{\lambda(x[n])}{\sqrt{1-(1-G[n])^2}}, \end{aligned} \tag{26}$$

where the underlying energy matrices could be of either type that was described earlier.

VII. Conclusion

In this paper, we have explored a matrix framework for the popular Teager-Kaiser energy operator as the square root of the eigenvalues of traceless energy matrices in both the continuous and discrete cases. Conditions regarding the positivity of the energy operator were translated to the condition that the eigenvalues of the underlying energy matrices be real.

This matrix construct for the TKEO provides a framework, where the "measured" energy of the TKEO can be related to the square of the eigenvalue of its underlying energy matrix, a notion analogous to that seen in quantum mechanics. Several other generalizations of the energy operator such as the higher-order energy operators, the energy operator for complex-valued signals, and the variable length energy operator were also incorporated into this matrix framework. Finally, the energy separation algorithms that are based on the energy operator, were also cast into the developed matrix framework.

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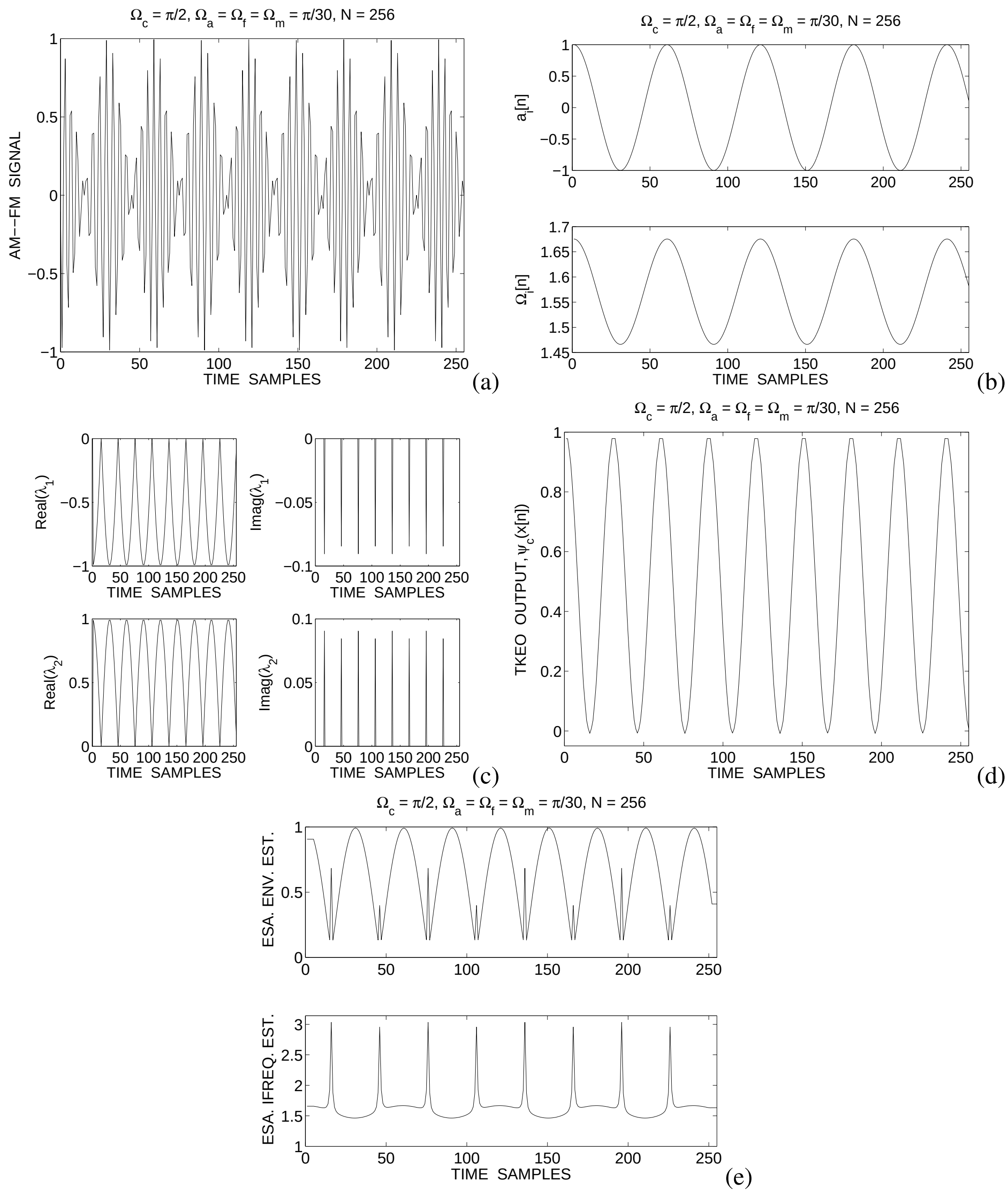

Fig. 1. Conditional positivity of TKEO: (a) monocomponent AM–FM signal, (b) instantaneous frequency and amplitude of the AM-FM signal depicting an envelope that becomes negative, (c) real and imaginary part of the eigenvalues of the TKEO matrix indicating complex eigenvalues, (d) output of the TKEO, further depicting the several instants where the imaginary part of the eigenvalues are non-zero and the exactly at same instants where the energy operator output becomes negative, and (e) ESA IF and IA estimates depicting singularities at the points where the TKEO output is negative.

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