THE DRFT–A ROTATION IN TIME–FREQUENCY SPACE

Balasubramaniam Santhanam *and* J. H. McClellan

School of Electrical and Computer Engineering
Georgia Institute of Technology
Atlanta, GA 30332-0250

ABSTRACT

The Continuous–time Angular Fourier Transformation (AFT) represents a rotation in Continuous time–frequency space and also serves as an orthonormal signal representation for chirp signals. In this paper we present a discrete version of the AFT (DRFT) that represents a rotation in Discrete time–frequency space and some properties of the transform that support its interpretation as a rotation. The transform is a generalization of the DFT. The Eigenvalue structure of the DFT is then exploited to develop an efficient algorithm for the computation of this transform.

1. INTRODUCTION

The Continuous–time Fourier Transformation (CTFT) is defined by the following pair

$$\begin{aligned} X(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)\exp(-j\omega t)dt \\ x(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X(\omega)\exp(j\omega t)d\omega \end{aligned} \tag{1}$$

The CTFT is a widely used time-frequency analysis tool and transforms a point $x(t)$ in $L^2(R)$ to a point $X(\omega)$ in $L^2(R)$. The CTFT, in effect treats the time and frequency axes as orthogonal axes. Recently signal representations (AFT) that represent a rotation on (t,ω) space and use "angularly coupled axes" have been developed [1, 2]. The AFT of a signal $x(t)$ is then defined as

$$X_\alpha(\omega) = K_\alpha \int_{-\infty}^{\infty} x(t)\exp(j(\frac{t^2+\omega^2}{2}\cot\alpha - t\omega\csc\alpha))dt \tag{2}$$

where $K_\alpha = \sqrt{\frac{1-j\cot\alpha}{2\pi}}$

In this paper we present a discrete version of the AFT which is an angular generalization of the DFT. If the DFT operator $\mathbf{W}$ is defined as an $N \times N$ matrix with entries

$$W_{nk} = \frac{1}{\sqrt{N}}\exp(-j\frac{2\pi}{N}nk) \tag{3}$$

then the DFT becomes a unitary operator with a set of N eigenvectors and four distinct eigenvalues $[1,-1,j,-j]$ [3]. The discrete version of the AFT is obtained by an angular generalization of these eigenvalues to a continuous angular parameter by using fractional powers of those eigenvalues. The unitary nature of the resultant operator is exploited in later sections of the paper to develop an algorithm for the computation of this transform.

This work was supported by the Joint Services Electronics Program under contract DAAH-04-93-G-0027.

2. INTERPRETATION AS A ROTATION

Successive application of the Fourier transformation $\mathbf{F}$ on $x(t)$ yields

$$\mathbf{F}^2[x(t)] = x(-t) \;\;,\;\; \mathbf{F}^3[x(t)] = X(-\omega) \;\;,\;\; \mathbf{F}^4[x(t)] = x(t) \tag{4}$$

In a similar fashion, the DFT operator defined in (3) when applied to $x[n]$, yields

$$\mathbf{W}^2[x] = x[((-n))_N]\,,\; \mathbf{W}^3[x] = X[((-k))_N]\,,\; \mathbf{W}^4[x] = x[n] \tag{5}$$

These equations lead us to interpret the CTFT as a 90° rotation operator in the (t,ω) plane and the DFT as a 90° rotation operator in (n,k) space. Application of the operator twice is a reflection or a 180° rotation; thrice is a reflection of the transform or a 270° rotation; and four times corresponds to the identity operation or a 360° rotation [1].

3. DISCRETE VERSION OF THE AFT

Direct discretization of the AFT defined in (2) by replacing t with n and ω by k does not produce a discrete rotation operator because a discrete (n,k) grid does not permit transformations with non-integer entries instead , we define the Disrete Rotational Fourier Transform (DRFT) as

$$\mathbf{A}_\alpha(x) = \mathbf{W}^{\frac{2\alpha}{\pi}}(x) \tag{6}$$

When the operator is evaluated for parameter values $\alpha = 0, 2\pi$ one obtains

$$\mathbf{A}_{2\pi} = \mathbf{W}^4 = \mathbf{I} = \mathbf{W}^0 = \mathbf{A}_0 \tag{7}$$

When evaluated for a parameter value of $\alpha = \frac{\pi}{2}$, one obtains the DFT operator.

$$\mathbf{A}_{\frac{\pi}{2}} = \mathbf{W}^1 = \mathbf{W} \tag{8}$$

Upon evaluation for $\alpha = \pi$ one obtains the cyclic flip matrix

$$\mathbf{A}_\pi = \mathbf{W}^2 \tag{9}$$

These relations provide us with motivation to interpret the operator $\mathbf{A}_\alpha$ as a rotation through "α" in (n,k) plane. A Taylor series expansion of the matrix operator $\mathbf{A}_\alpha$ followed by application of the Cayley-Hamilton theorem, which states that every matrix satisfies its characteristic equation suggests that we could write

$$\mathbf{A}_\alpha = a_0(\alpha)\mathbf{I} + a_1(\alpha)\mathbf{W} + a_2(\alpha)\mathbf{W}^2 + a_3(\alpha)\mathbf{W}^3. \quad (10)$$

where the coefficients $a_i(\alpha)$ are obtained on regrouping of terms containing $\mathbf{I}, \mathbf{W}, \mathbf{W}^2$ and $\mathbf{W}^3$ from the Taylor series expansion. This equation gives an interesting interpretation of subspace mixing, where the coefficients $a_0(\alpha)$, $a_1(\alpha)$, $a_2(\alpha)$ and $a_3(\alpha)$ generate an angular mixture of the basis matrices $\mathbf{I}, \mathbf{W}, \mathbf{W}^2$ and $\mathbf{W}^3$ [5].

4. PROPERTIES OF THE OPERATOR

Most of the properties of the DRFT are inherited from the DFT and the fact that $\mathbf{A}_\alpha$ commutes with $\mathbf{W}$

1) Unitary: The operator is unitary, hence

$$\mathbf{A}_\alpha^H = \mathbf{A}_\alpha^{-1} = \mathbf{A}_{-\alpha} \quad (11)$$

As a consequence of this property one obtains

$$\mathbf{A}_{-\alpha} = a_0^*(\alpha)\mathbf{I} + a_3^*(\alpha)\mathbf{W} + a_2^*(\alpha)\mathbf{W}^2 + a_1^*(\alpha)\mathbf{W}^3 \quad (12)$$

Since $\mathbf{A}_\alpha\mathbf{A}_{-\alpha} = \mathbf{I}$, the corresponding relations between the coefficients are

$$\begin{aligned}
\sum_{i,j=0,1,2,3} a_i(\alpha)a_j(-\alpha)\delta[((i+j))_4] &= 1 \\
\sum_{i,j=0,1,2,3} a_i(\alpha)a_j(-\alpha)\delta[((i+j-1))_4] &= 0 \\
\sum_{i,j=0,1,2,3} a_i(\alpha)a_j(-\alpha)\delta[((i+j-2))_4] &= 0 \\
\sum_{i,j=0,1,2,3} a_i(\alpha)a_j(-\alpha)\delta[((i+j-3))_4] &= 0 \quad (13)
\end{aligned}$$

1a) Another consequence of the unitary nature of the operator is the norm preserving property [Parseval's relation].

$$\|\mathbf{A}_\alpha[x]\| = \|x\| \quad (14)$$

2)Angle Additivity: Application of the operator with angular parameter "α" followed by an application of the operator with angular parameter "β" is equivalent to the application of the operator with angular parameter "$\alpha+\beta$".

$$\mathbf{A}_\alpha\mathbf{A}_\beta = \mathbf{A}_{\alpha+\beta} \quad (15)$$

2a) Multiple applications of the operator to a signal is equivalent to one rotation through a larger angle in the (n,k) plane

$$\mathbf{A}_\alpha^m = \mathbf{A}_{m\alpha} \quad (16)$$

The corresponding relations between the coefficients are

$$\begin{aligned}
(a_0(\alpha) + a_1(\alpha) + a_2(\alpha) + a_3(\alpha))^n &= \\
a_0(n\alpha) + a_1(n\alpha) + a_2(n\alpha) + a_3(n\alpha)
\end{aligned}$$

3)DFT $\leftrightarrow$ 90° rotation: Application of DFT operator is equivalent to a 90° rotation.

$$\mathbf{A}_{\alpha+\frac{\pi}{2}} = \mathbf{W}\mathbf{A}_\alpha \quad (17)$$

The corresponding relation between the coefficients are

$$a_i(\alpha + \frac{\pi}{2}) = a_{i-1}(\alpha) \quad (18)$$

where $a_{-1} = a_3$.

4)Circular flip property: A circular flip transforms into a circular flip in the rotated frequency domain.

$$\mathbf{A}_\alpha\mathbf{W}^2(x[n]) = \mathbf{W}^2[\mathbf{A}_\alpha x[n]] \quad (19)$$

5)Periodicity: The operator is periodic in the parameter α with a fundamental period of 2π

$$\mathbf{A}_\alpha = \mathbf{A}_{\alpha+2r\pi} \quad (20)$$

This implies that the coefficients $a_i(\alpha)$ are periodic in the parameter α with period 2π. These above mentioned properties illustrate well the angular and rotational aspects of the transform.

5. COMPUTATION OF THE DRFT

The operator $\mathbf{W}$ defined in (3) is a unitary operator and consequently has a set of N orthonormal eigenvectors $\vec{v}_i$ [4]. The operator $\mathbf{W}$ has an eigen-decomposition given by

$$\begin{aligned}
\mathbf{W} &= \sum_i \lambda_i \vec{v}_i\vec{v}_i^H \\
&= \sum_{i\in N_1} \vec{v}_i\vec{v}_i^H - \sum_{i\in N_2} \vec{v}_i\vec{v}_i^H + j(\sum_{i\in N_3} \vec{v}_i\vec{v}_i^H - \sum_{i\in N_4} \vec{v}_i\vec{v}_i^H)
\end{aligned} \quad (21)$$

where N_1 is the set of indices for eigenvectors belonging to $\lambda = 1$, N_2 for $\lambda = -1$ and so on. $\mathbf{A}_\alpha$, being a matrix function of $\mathbf{W}$ is also unitary and has an eigen-decomposition defined by taking a fractional power of the eigenvalues $(\lambda_i)^{\frac{2\alpha}{\pi}}$.

$$\begin{aligned}
\mathbf{A}_\alpha = \sum_{i\in N_1} \exp(j4k_1\alpha)\vec{v}_i\vec{v}_i^H + \sum_{i\in N_2} \exp(j(4k_2+2)\alpha)\vec{v}_i\vec{v}_i^H \\
+ \sum_{i\in N_3} \exp(j(4k_3+1)\alpha)\vec{v}_i\vec{v}_i^H + \sum_{i\in N_4} \exp(j(4k_4-1)\alpha)\vec{v}_i\vec{v}_i^H
\end{aligned}$$

where k_1, k_2, k_3 and k_4 are integers. It should be noted that because of the ambiguity present in the fractional power operation, the operator $\mathbf{A}_\alpha$ defined by (6) is not unique. The branch cut $k_1 = 0$, $k_2 = 0$, $k_3 = 0$ and $k_4 = 0$ is imposed to make the operation unique. A specific ordering on the eigenvalues of $\mathbf{W}$, $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = j$ and $\lambda_4 = -j$ is also imposed to make the solution unique.

Computation of the transform directly for an angle α requires that we perform an eigen decomposition on the operator $\mathbf{W}$ and compute the new operator $\mathbf{A}_\alpha$. This would be an impractical thing to do for large orders. Instead, we first generate the matrices $\mathbf{I}, \mathbf{W}, \mathbf{W}^2$ and $\mathbf{W}^3$ for a given dimension N where $\mathbf{W}^2$ is the circular flip matrix and $\mathbf{W}^3$ is a circularly flipped version of $\mathbf{W}$. The transform and its inverse $\mathbf{A}_{-\alpha}$ are then computed through the subspace projection equations.

$$\begin{aligned}
\mathbf{A}_\alpha &= a_0(\alpha)\mathbf{I} + a_1(\alpha)\mathbf{W} + a_2(\alpha)\mathbf{W}^2 + a_3(\alpha)\mathbf{W}^3 \\
\mathbf{A}_{-\alpha} &= a_0^*(\alpha)\mathbf{I} + a_3^*(\alpha)\mathbf{W} + a_2^*(\alpha)\mathbf{W}^2 + a_1^*(\alpha)\mathbf{W}^3
\end{aligned}$$

where the coefficients $a_0(\alpha)$, $a_1(\alpha)$, $a_2(\alpha)$ and $a_3(\alpha)$ in the expansion are given by

$$\begin{aligned}
a_0(\alpha) &= \frac{1}{2}(1 + e^{j\alpha})\cos\alpha \\
a_1(\alpha) &= \frac{1}{2}(1 - je^{j\alpha})\sin\alpha \\
a_2(\alpha) &= \frac{1}{2}(e^{j\alpha} - 1)\cos\alpha \\
a_3(\alpha) &= \frac{1}{2}(-1 - je^{j\alpha})\sin\alpha
\end{aligned} \tag{22}$$

6. THE DRFT AS AN ORTHONORMAL SIGNAL EXAPANSION

The DRFT of a signal $x[n]$ can also be written as

$$X_\alpha[k] = \sum_{n=0}^{N-1} K_\alpha[n,k]x[n] \tag{23}$$

where the kernel of the transformation corresponding to the operator $\mathbf{A}_\alpha$ is given by

$$\begin{aligned}
K_\alpha[n,k] = a_0(\alpha)\delta[n-k] + \frac{a_1(\alpha)}{\sqrt{N}}\exp\left(-j\frac{2\pi}{N}nk\right) \\
+ a_2(\alpha)\delta[((n+k))_N] + \frac{a_3(\alpha)}{\sqrt{N}}\exp\left(j\frac{2\pi}{N}nk\right)
\end{aligned}$$

The signal is recovered through the IDRFT relation

$$x[n] = \sum_{k=0}^{N-1} K_{-\alpha}[n,k]X_\alpha[k] \tag{24}$$

6.1. PROPERTIES OF THE KERNEL

Some significant properties of the kernel are given below
1) The kernel is symmetric in n, k

$$K_\alpha[n,k] = K_\alpha[k,n] \tag{25}$$

which signifies the fact that the roles of the time and frequency variables are interchangable.
2) The Kernel is periodic in the parameter α analogous to a rotation operation

$$K_{\alpha+2\pi}[n,k] = K_\alpha[n,k] \tag{26}$$

3) The Kernel corresponds to the basis function of an orthonormal representation

$$\sum_{n=0}^{N-1} K_\alpha[n,k]K_\alpha^*[n,k'] = \delta[k-k'] \tag{27}$$

4) The inverse operation of a rotation through an angle α is a rotation through $-\alpha$

$$K_{-\alpha}[n,k] = K_\alpha^*[n,k] \tag{28}$$

5) A rotation through "α" followed by a rotation through "β" is a rotation through "$\alpha+\beta$"

$$\sum_{k'=0}^{N-1} K_\alpha[n,k']K_\beta[k',k] = K_{\alpha+\beta}[n,k] \tag{29}$$

6) The kernel of the transformation is an angular mixture of four basic kernels

$$\begin{aligned}
K_\alpha[n,k] &= a_0(\alpha)K_0[n,k] + a_1(\alpha)K_{\frac{\pi}{2}}[n,k] + \\
&\quad a_2(\alpha)K_\pi[n,k] + a_3(\alpha)K_{\frac{3\pi}{2}}[n,k]
\end{aligned}$$

7. EXAMPLE OF THE DRFT

Consider a signal mixture $x[n]$ of a sinusoid and an impulse.

$$x[n] = \cos\left(\frac{2\pi}{100}n\right) + \delta(n-100) \tag{30}$$

Figs. 2a,2b,2c and 2d are the magnitudes of the 210-pt DRFT for the above mixture at angles 15°, 30°, 45° and 60° respectively.

- Over different angles the signal energy is squeezed into different frequencies.
- For angles between 0° and 90°, the impulses produced by the sinusoid grow in amplitude and finally swamp out the original impulse.
- The transform for other rotation angles can be obtained from the transform values for angles between 0° and 90° using the symmetry present in the signal of the example.
- In general, for a signal without symmetry in it the DRFT expansion of (10) is a complete orthonormal representation.

8. CONCLUSION

The DRFT which is an angular generalization of the DFT and represents a rotation in (n, k) space has been presented. The DRFT operator when evaluated for angular parameter values of $0°$ and $360°$ becomes the identity operation, while on evaluation for $90°$ yields the DFT operator. The DRFT of a signal is equivalent to an angular mixture of the signal, its DFT, a circularly flipped version of the signal and a circularly flipped version of its DFT. An efficient algorithm that avoids eigenvalue decomposition has also been presented.

9. REFERENCES

[1] L. B. Almeida, "An Introduction to the Angular Fourier Transformation," *Proceedings of ICASSP 93.*, Vol 3, pp. 257-260.

[2] J. Sadowsky, "Applications of Weyl Representation Theory to Signal Processing," *Proceedings of the* 23^{rd} *Asilomar Conference on Circuits, Systems and computers.*, pp. 628-632, 1989.

[3] J. H. McClellan and T.W. Parks, "Eigenvalue and Eigenvector Decomposition of the Discrete Fourier Transform," *IEEE Transactions on Audio and Electroacoustics.*, AU-20, pp. 66-74, March 1972.

[4] G. Strang. *Linear Algebra and its applications*. Academic Press, New York, NY, 1976.

[5] B. Santhanam and J. H. McClellan, "The Discrete Rotational Fourier Transformation," *Submitted to IEEE Transactions on SP, correspondence.*

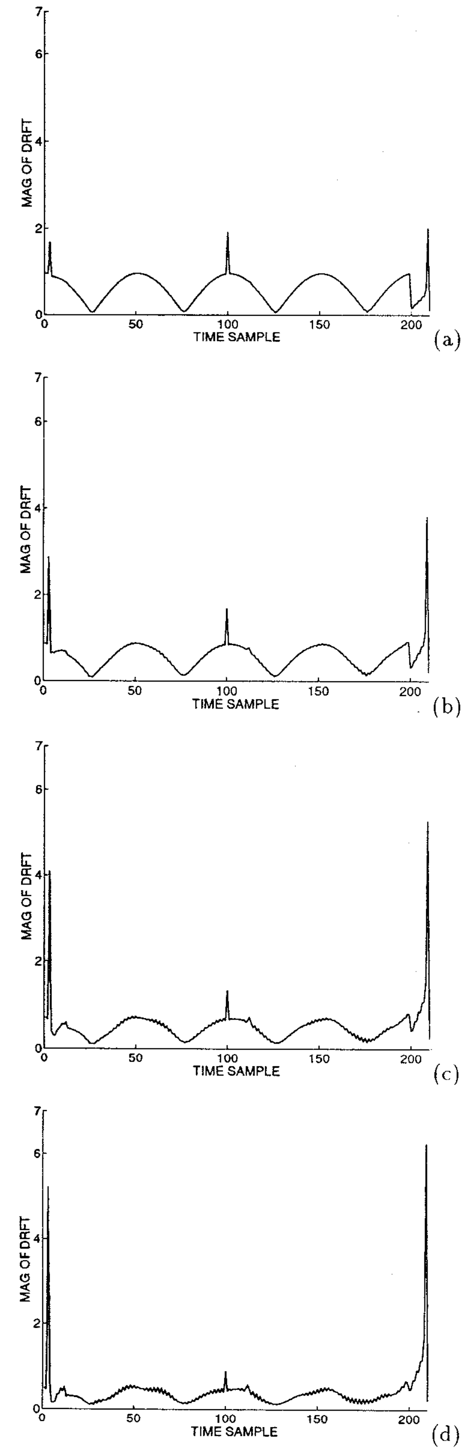

Figure 1: Magnitude of the DRFT of the example signal for angles (a)$\alpha = 15°$, (b)$\alpha = 30°$, (c)$\alpha = 45°$ and (d)$\alpha = 60°$.