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$\mathscr{H}_\infty$ filtering of time-varying systems with bounded rates of variation

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Abstract— **In this paper, the problem of robust filter design for time-varying discrete-time polytopic systems with bounded rates of variation is investigated. The design conditions are obtained by using a parameter-dependent Lyapunov function and the Finsler's Lemma. A robust filter, that minimizes an upper bound to the $\mathscr{H}_\infty$ performance of the estimation error, is obtained as the solution of an optimization problem. A more precise geometric representation of the parameter time variation was used in order to obtain less conservative design conditions. Robust filters for time-invariant, as well as arbitrarily time-varying, polytopic systems can be obtained as a particular case of the proposed method. Numerical examples illustrate the results.**

I. INTRODUCTION

In the last decades, the aim of the research within the filtering framework has been mainly concerned with uncertain systems, since the design of optimal filters for precisely known models is well characterized nowadays. In this context, the Lyapunov theory has been extensively applied as a tool to deal with synthesis of robust filters that guarantee the stability of the estimation error dynamic meanwhile assure a certain level of performance. As a result, many works dealing with robust filtering have appeared in print lately, for example [1–4].

Considering the class of uncertain linear systems, the results appeared so far in the literature have dealt with two main cases, time-invariant and time-varying parameters. In the first one, it can be mentioned the works [5–7], where a quadratic Lyapunov function was applied to provide sufficient conditions for $\mathscr{H}_2$ and $\mathscr{H}_\infty$ robust filtering and [8] in the context of parameter-dependent Lyapunov functions. In the second case, it can be cited, among others, the recent work [9] where $\mathscr{H}_\infty$ robust filtering of affine parameter-dependent systems with bounded rates of variation is considered.

When dealing with time-varying system where the parameters can be read online, linear parameter varying (LPV) techniques has also been used for filter design, as for instance [10, 11] where affine parameter varying filters, with limited rate of variation, are obtained, [12] in the context of parameter-dependent filters by means of nonlinear fractional transformation and quadratic stability, [13] concerned with LPV filtering for slowly varying systems and [14] where the LPV filtering for arbitrarily time-varying systems in polytopic domain is addressed.

This paper investigates the robust filtering of uncertain time-varying systems with bounds on the rate of variation. The Lyapunov theory is applied in order to obtain the design conditions of the filter. A parameter-dependent Lyapunov function is used to reduce the conservatism of the proposed method, resulting in a more general approach when compared to methods based on quadratic stability. The parameter variation modeling proposed in [15] is applied to give a better description of the uncertainty domain. As an index of performance it was used an $\mathscr{H}_\infty$ guaranteed cost. The $\mathscr{H}_\infty$ filtering limits the maximum possible variance of the error signal over all exogenous inputs with bounded variance [16], *i.e.* the $\mathscr{H}_\infty$ norm reflects the worst-case energy gain of the system and does not require statistical assumptions on the exogenous input, furthermore, it can provide robustness with respect to unmodeled uncertainties. Extra variables introduced by the Finsler's Lemma can be freely explored in the search for better $\mathscr{H}_\infty$ performance of the estimation error dynamic giving more flexibility to the design process. All the system matrices are assumed to be affected by the time-varying parameters, which are supposed to lie inside polytopic domains. The robust filter is then obtained by the solution of an optimization problem that minimizes an upper bound to the $\mathscr{H}_\infty$ index of performance subject to a finite number of constraints formulated only in terms of the vertices of a polytope. No grids in the parametric space are used. Robust filters for time-invariant and arbitrarily time-varying uncertain systems can be obtained as a particular case of the proposed method. Numerical examples illustrate the efficiency of the proposed results.

II. PRELIMINARIES AND PROBLEM STATEMENT

Consider the time-varying discrete-time system, for $k \geq 0$

$$\begin{aligned} x(k+1) &= A(\alpha(k))x(k) + B(\alpha(k))w(k) \\ z(k) &= C_1(\alpha(k))x(k) + D_1(\alpha(k))w(k) \\ y(k) &= C_2(\alpha(k))x(k) + D_2(\alpha(k))w(k) \end{aligned} \qquad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state space vector, $w(k) \in \mathbb{R}^m$ is the noise input belonging to $l_2[0,\infty)$, $z(k) \in \mathbb{R}^p$ is the signal to be estimated and $y(k) \in \mathbb{R}^q$ is the measured output. All matrices are real, with appropriate dimensions. The time-varying vector of parameters $\alpha(k)$ belongs to the unit simplex[1]

$$\mathscr{U} = \left\{ \alpha \in \mathbb{R}^N : \sum_{i=1}^{N} \alpha_i = 1,\ \alpha_i \geq 0\ ,\ i = 1,\ldots,N,\ \forall k \geq 0 \right\}$$

[1]The time dependence of α(k) will be omitted to lighten the notation.

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with bounded rates of variation

$$-b \le \Delta\alpha_i \le b,\ b \in \mathbb{R},\ b \in [0,1] \tag{2}$$

where $\Delta\alpha_i = \alpha_i(k+1) - \alpha_i(k),\ i = 1,\dots,N$.

All matrices are real, with appropriate dimensions, belonging to the polytope

$$\mathscr{P} \triangleq \left\{ \left[\begin{array}{c|c} A(\alpha) & B(\alpha) \\ \hline C_1(\alpha) & D_1(\alpha) \\ \hline C_2(\alpha) & D_2(\alpha) \end{array}\right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c} A_i & B_i \\ \hline C_{1i} & D_{1i} \\ \hline C_{2i} & D_{2i} \end{array}\right] \right\} \tag{3}$$

More specifically, the system matrices are given, for any time $k \ge 0$, by the convex combination of the well-defined vertices of the polytope $\mathscr{P}$.

A full order proper robust filter is investigated here, being given by

$$\begin{aligned} \dot{x}_f(k+1) &= A_f x_f(k) + B_f y(k),\ x_f(0) = 0 \\ z_f(k) &= C_f x_f(k) + D_f y(k) \end{aligned} \tag{4}$$

where $x_f(t) \in \mathbb{R}^n$ is the filter state space vector and $z_f(t) \in \mathbb{R}^p$ the estimated signal. All filter matrices are real, with appropriate dimensions.

The estimation error dynamic is given by

$$\begin{aligned} \varsigma(k+1) &= \hat{A}(\alpha)\varsigma(k) + \hat{B}(\alpha)w(k), \varsigma(0) = 0 \\ e(k) &= \hat{C}(\alpha)\varsigma(k) + \hat{D}(\alpha)w(k) \end{aligned} \tag{5}$$

where $\varsigma(k) = [x(k)'\ \ x_f(k)']'$, $e(k) = z(k) - z_f(k)$ and

$$\hat{A}(\alpha) = \begin{bmatrix} A(\alpha) & \mathbf{0} \\ B_f C_2(\alpha) & A_f \end{bmatrix},\ \hat{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ B_f D_2(\alpha) \end{bmatrix}$$

$$\begin{aligned} \hat{C}(\alpha) &= \begin{bmatrix} C_1(\alpha) - D_f C_2(\alpha) & -C_f \end{bmatrix}, \\ \hat{D}(\alpha) &= \begin{bmatrix} D_1(\alpha) - D_f D_2(\alpha) \end{bmatrix} \end{aligned} \tag{6}$$

The whole of possible outcomes for the set (6) belongs to the polytope

$$\hat{\mathscr{P}} \triangleq \left\{ \left[\begin{array}{c|c} \hat{A}(\alpha) & \hat{B}(\alpha) \\ \hline \hat{C}(\alpha) & \hat{D}(\alpha) \end{array}\right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c} \hat{A}_i & \hat{B}_i \\ \hline \hat{C}_i & \hat{D}_i \end{array}\right] \right\} \tag{7}$$

The filtering problem to be dealt with can be stated as follows.

Problem 1: Find matrices $A_f \in \mathbb{R}^{n\times n}$, $B_f \in \mathbb{R}^{n\times q}$, $C_f \in \mathbb{R}^{p\times n}$ and $D_f \in \mathbb{R}^{p\times q}$ of the filter (4), such that the estimation error system (5) is asymptotically stable, and an upper bound γ to the $\mathscr{H}_\infty$ estimation error performance is minimized, that is, for all $k \ge 0$

$$\sup_{w(k)\neq 0} \frac{\|e(k)\|_2^2}{\|w(k)\|_2^2} < \gamma^2 \tag{8}$$

with $w(k) \in l_2[0,\infty)$.

Before proceeding to the solution of Problem 1, some previous results are needed.

As shown in [15], the vector $\Delta\alpha = [\Delta\alpha_1\ \Delta\alpha_2\ \dots\ \Delta\alpha_N]$ can be assumed to belong to the compact set, $\forall k \ge 0$,

$$\Gamma_b = \left\{\delta \in \mathbb{R}^N : \delta = co\{h^1,\dots,h^M\}, \sum_{i=1}^{M} h_i^j = 0, j = 1,\dots,M\right\}$$

defined as the convex combination of vectors $h^j,\ j = 1,\dots,N$ given *a priori*. As a consequence, the linear constraint

$$\sum_{i=1}^{N} \Delta\alpha_i = 0, \tag{9}$$

imposed by $\alpha \in \mathscr{U} \Rightarrow \sum_{i=1}^{N} \alpha_i = 1$, is always satisfied. However, the use of Γ_b to describe vector $\Delta\alpha$ introduces conservativeness by not taking into account the dependence between bound b and α_i, leading to infeasible values of ($\Delta\alpha_i$, α_i). In order to consider only feasible values, inequality (2) must be rewritten as

$$-b\alpha_i \le \Delta\alpha_i \le b(1-\alpha_i),\quad i = 1,\dots,N \tag{10}$$

producing the set $\Gamma_{b\alpha}$ with vectors $h^j,\ j = 1,\dots,M$ given by

$$\begin{bmatrix} h^1 & h^2 & \cdots & h^M \end{bmatrix} = b \begin{bmatrix} 1-\alpha_1 & -\alpha_1 & -\alpha_1 & \cdots \\ -\alpha_2 & 1-\alpha_2 & -\alpha_2 & \cdots \\ \vdots & \vdots & \ddots & \cdots \\ -\alpha_N & -\alpha_N & -\alpha_N & 1-\alpha_N \end{bmatrix}$$

Note that the convex combination of vectors h^j of $\Gamma_{b\alpha}$ gives the following expression

$$\Delta\alpha_j = b(\beta_j - \alpha_j(\beta_1 + \cdots + \beta_M)) = b(\beta_j - \alpha_j) \tag{11}$$

since vector β belongs to a unit simplex (*i. e.*, $\sum_{j=1}^{M} \beta_j = 1, \forall\beta$) and $M = N$. Note that M would be different from N if distinct bounds b_i were considered. For more details about the parameter variation modeling see [15].

Lemma 1: (*Finsler*) Let $\xi \in \mathbb{R}^a$, $\mathscr{Q} = \mathscr{Q}' \in \mathbb{R}^{a\times a}$, $\mathscr{B} \in \mathbb{R}^{b\times a}$ with $rank(\mathscr{B}) < a$, and $\mathscr{B}^\perp$ a basis for the null-space of $\mathscr{B}$ (*i.e.* $\mathscr{B}\mathscr{B}^\perp = 0$). The following statements are equivalent.

i) $\xi'\mathscr{Q}\xi < 0,\ \forall \mathscr{B}\xi = 0,\ \xi \neq 0$;
ii) $\mathscr{B}^{\perp\prime}\mathscr{Q}\mathscr{B}^\perp < 0$;
iii) $\exists\ \mu \in \mathbb{R} : \mathscr{Q} - \mu\mathscr{B}'\mathscr{B} < 0$;
iv) $\exists\ \mathscr{X} \in \mathbb{R}^{a\times b} : \mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}'\mathscr{X}' < 0$.

Proof: See [17]. ■

By applying the Bounded Real Lemma [18], combined with the Finsler's Lemma (1), condition (8) can be guaranteed as follows.

Lemma 2: For a given γ, if there exists a parameter-dependent matrix $P(\alpha)' = P(\alpha) > 0$ such that the statements of Lemma 1 are satisfied with

$$\mathscr{Q} = \begin{bmatrix} P(\alpha_+) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -P(\alpha) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma^{-1}\hat{B}(\alpha)\hat{B}(\alpha)' & \gamma^{-1}\hat{B}(\alpha)\hat{D}(\alpha)' \\ \mathbf{0} & \gamma^{-1}\hat{D}(\alpha)\hat{B}(\alpha)' & \gamma^{-1}\hat{D}(\alpha)\hat{D}(\alpha)' - \gamma\mathbf{I} \end{bmatrix}$$

$$\mathscr{B} = \begin{bmatrix} -\mathbf{I} & \hat{A}(\alpha)' & \hat{C}(\alpha)' \end{bmatrix},$$

$$\mathscr{B}^\perp = \begin{bmatrix} \hat{A}(\alpha)' & \hat{C}(\alpha)' \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},\ \xi = \begin{bmatrix} \varsigma(k+1)' & \varsigma(k)' & w(k)' \end{bmatrix}'$$

for all $\alpha \in \mathscr{U}$ and $\Delta\alpha \in \Gamma_{b\alpha}$, where $\alpha_+ = \alpha(k+1)$, then the error dynamic (5) is asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.

Proof: Let $v(k) = \varsigma(k)'P(\alpha)\varsigma(k)$ be a parameter-dependent Lyapunov function. Considering the dual system (*i. e.* $\hat{A} = \hat{A}'$, $\hat{B} = \hat{C}'$, $\hat{C} = \hat{B}'$ and $\hat{D} = \hat{D}'$), it is straightforward from statement *i)* of Lemma 1 that Lemma 2 ensures $v(k) > 0$ and

$$\Delta v(k) < -\gamma^{-1}e(k)'e(k) + \gamma w(k)'w(k)$$

with the choice $\xi = [\varsigma(k+1)' \ \varsigma(k)' \ w(k)']'$. The last inequality comes from $\Delta v(k) < 0$ and

$$e(k)'e(k) - \gamma^2 w(k)'w(k) < 0$$

by applying the Bounded Real Lemma. Therefore, system (5) has an upper bound γ to the $\mathscr{H}_\infty$ performance and, from the Lyapunov theory [19], is asymptotically stable. ∎

The conditions of Lemma 2 appear as nonlinearities that must be tested at all points of the simplex $\mathscr{U}$, *i.e.*, at an infinite number of points. Hence, the main goal hereafter is to obtain finite-dimensional LMI conditions in terms of the vertices of the polytope $\mathscr{P}$ to solve Problem 1. Using Schur complement, change of variables and exploring the extra variables provided by Lemma 1, finite-dimensional LMIs assuring the existence of such filters are given in the next section.

III. Main Results

Theorem 1: ($\mathscr{H}_\infty$ Filtering) Given the system (1), if there exist matrices $Z, Y, R, Q \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times q}$, $J \in \mathbb{R}^{p\times n}$, $\tilde{D}_f \in \mathbb{R}^{p\times q}$, G_i, $M_i = M_i' > 0 \in \mathbb{R}^{2n\times 2n}$, $H_i \in \mathbb{R}^{p\times 2n}$ $i = 1,\ldots,N$ and a scalar $\gamma > 0$ such that[2]

$$\Xi_{ij} \triangleq \begin{bmatrix} \mathscr{F}_{11} & \mathscr{F}_{12} & \hat{F}_{3i} - \hat{F}_1' H_j' & \mathbf{0} \\ (\star) & \mathscr{F}_{22} & G_j \hat{F}_{3i} + \hat{F}_{2i}' H_j' & \hat{F}_{4i} \\ (\star) & (\star) & H_j \hat{F}_{3i} + \hat{F}_{3i}' H_j' - \gamma \mathbf{I} & \mathscr{F}_{34} \\ (\star) & (\star) & (\star) & -\gamma \mathbf{I} \end{bmatrix} < 0 \quad (12)$$

$$i = 1,\ldots,N, \ j = 1,\ldots,N$$
$$\mathscr{F}_{11} = (1-b)M_i + bM_j - \hat{F}_1 - \hat{F}_1', \ \mathscr{F}_{12} = \hat{F}_{2i} - \hat{F}_1' G_j'$$
$$\mathscr{F}_{22} = G_j \hat{F}_{2i} + \hat{F}_{2i}' G_j' - M_i, \ \mathscr{F}_{34} = D_{1i} - \tilde{D}_f D_{2i}$$

$$\hat{F}_1 = \begin{bmatrix} Z & Y' + R' \\ Z & Y' \end{bmatrix}, \ \hat{F}_{2i} = \begin{bmatrix} A_i' Z & A_i' Y' + C_{2i}' L' + Q' \\ A_i' Z & A_i' Y' + C_{2i}' L' \end{bmatrix}$$
$$\hat{F}_{3i} = \begin{bmatrix} C_{1i}' - C_{2i}' \tilde{D}_f' - J' \\ C_{1i}' - C_{2i}' \tilde{D}_f' \end{bmatrix}, \ \hat{F}_{4i} = \begin{bmatrix} Z' B_i \\ Y B_i + L D_{2i} \end{bmatrix}$$

then there exists a proper robust filter in the form of (4), ensuring the asymptotic stability of the estimation error dynamic (5) and an $\mathscr{H}_\infty$ guaranteed cost γ, for all $\alpha \in \mathscr{U}$ and $\Delta\alpha \in \Gamma_{b\alpha}$, with matrices given by

$$A_f = \hat{V}^{-1} Q (UZ)^{-1}, \ B_f = \hat{V}^{-1} L$$
$$C_f = J (UZ)^{-1}, \ D_f = \tilde{D}_f \quad (13)$$

where $U \in \mathbb{R}^{n\times n}$ and $\hat{V} \in \mathbb{R}^{n\times n}$ are matrices arbitrarily chosen such that $R = \hat{V} U Z$.

[2]The term $(\star)$ indicates symmetric blocks in the LMIs.

Proof: Firstly, note that

$$\begin{aligned} M(\alpha_+) &= \alpha_{+1} M_1 + \cdots + \alpha_{+N} M_N \\ &= (\alpha_1 + \Delta\alpha_1) M_1 + \cdots + (\alpha_N + \Delta\alpha_N) M_N \\ &= \sum_{i=1}^N \alpha_i M_i + \sum_{j=1}^N \Delta\alpha_j M_j \\ &= \sum_{i=1}^N \alpha_i M_i + \sum_{j=1}^N b(\beta_j - \alpha_j) M_j \end{aligned}$$

where the last equality comes from (11) leading to

$$M(\alpha_+) = \sum_{i=1}^N \alpha_i (1-b) M_i + \sum_{j=1}^N \beta_j b M_j.$$

Secondly, applying the following operation

$$\Xi(\alpha) = \sum_{j=1}^N \beta_j \left\{ \sum_{i=1}^N \alpha_i \Xi_{ij} \right\} \quad (14)$$

in the BMI (12) one gets

$$\Xi(\alpha) = \begin{bmatrix} \mathscr{F}_{11}(\alpha) & \mathscr{F}_{12}(\alpha) & \mathscr{F}_{13}(\alpha) & \mathbf{0} \\ (\star) & \mathscr{F}_{22}(\alpha) & \mathscr{F}_{23}(\alpha) & \hat{F}_4(\alpha) \\ (\star) & (\star) & \mathscr{F}_{33}(\alpha) & \mathscr{F}_{34}(\alpha) \\ (\star) & (\star) & (\star) & -\gamma \mathbf{I} \end{bmatrix} < 0 \quad (15)$$

$$\mathscr{F}_{11}(\alpha) = M(\alpha_+) - \hat{F}_1 - \hat{F}_1',$$
$$\mathscr{F}_{12}(\alpha) = \hat{F}_2(\alpha) - \hat{F}_1' G(\alpha_+)'$$
$$\mathscr{F}_{13}(\alpha) = \hat{F}_3(\alpha) - \hat{F}_1' H(\alpha_+)'$$
$$\mathscr{F}_{22}(\alpha) = G(\alpha_+) \hat{F}_2(\alpha) + \hat{F}_2(\alpha)' G(\alpha_+)' - M(\alpha),$$
$$\mathscr{F}_{23}(\alpha) = G(\alpha_+) \hat{F}_3(\alpha) + \hat{F}_2(\alpha)' H(\alpha_+)'$$
$$\mathscr{F}_{33}(\alpha) = H(\alpha_+) \hat{F}_3(\alpha) + \hat{F}_3(\alpha)' H(\alpha_+)' - \gamma \mathbf{I}$$
$$\mathscr{F}_{34}(\alpha) = D_1(\alpha) - \tilde{D}_f D_2(\alpha)$$

where

$$\hat{F}_2(\alpha) = \begin{bmatrix} A(\alpha)' Z & A(\alpha)' Y' + C_2(\alpha)' L' + Q' \\ A(\alpha)' Z & A(\alpha)' Y' + C_2(\alpha)' L' \end{bmatrix}$$
$$\hat{F}_3(\alpha)' = \begin{bmatrix} C_1(\alpha) - \tilde{D}_f C_2(\alpha) - J & C_1(\alpha) - \tilde{D}_f C_2(\alpha) \end{bmatrix},$$
$$\hat{F}_4(\alpha)' = \begin{bmatrix} B(\alpha)' Z & B(\alpha)' Y' + D_2(\alpha)' L' \end{bmatrix}$$

Then, define the partitioned matrices [20]

$$F = \begin{bmatrix} X' & U' \\ \hat{U}' & \hat{X}' \end{bmatrix}, \ F^{-1} = \begin{bmatrix} Y & \hat{V} \\ V & \hat{Y} \end{bmatrix}, \ T = \begin{bmatrix} X^{-1} & Y' \\ \mathbf{0} & \hat{V}' \end{bmatrix}$$

together with the following variable transformation

$$\begin{bmatrix} Q & L \\ J & \tilde{D}_f \end{bmatrix} = \begin{bmatrix} \hat{V} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \begin{bmatrix} UZ & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \ R = \hat{V} U Z \quad (16)$$

where $Z = X^{-1}$. Using the above change of variable, multiply the inequality (15) to the left by S' and to the right by S with

$$S = \begin{bmatrix} \mathscr{S} & \mathbf{0} \\ \star & \mathscr{I} \end{bmatrix}, \ \mathscr{S} = \begin{bmatrix} T^{-1} & \mathbf{0} \\ \star & T^{-1} \end{bmatrix}, \ \mathscr{I} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \star & \mathbf{I} \end{bmatrix}$$

yielding the following inequality

$$\begin{bmatrix} P(\alpha_+) - F - F' & \mathscr{L}_{12}(\alpha) & \mathscr{L}_{13}(\alpha) & \mathbf{0} \\ (\star) & \mathscr{L}_{22}(\alpha) & \mathscr{L}_{23}(\alpha) & \hat{B}(\alpha) \\ (\star) & (\star) & \mathscr{L}_{33}(\alpha) & \hat{D}(\alpha) \\ (\star) & (\star) & (\star) & -\gamma \mathbf{I} \end{bmatrix} < 0 \quad (17)$$

$$\mathscr{L}_{12}(\alpha) = F\hat{A}(\alpha)' - F'TG(\alpha_+)'T^{-1}$$
$$\mathscr{L}_{13}(\alpha) = F\hat{C}(\alpha)' - F'TH(\alpha_+)'$$
$$\mathscr{L}_{22}(\alpha) = (T')^{-1}G(\alpha_+)T'F\hat{A}(\alpha)' + \hat{A}(\alpha)F'TG(\alpha_+)'T^{-1} - P(\alpha),$$
$$\mathscr{L}_{23}(\alpha) = (T')^{-1}G(\alpha_+)T'F\hat{C}(\alpha)' + \hat{A}(\alpha)F'TH(\alpha_+)'$$
$$\mathscr{L}_{33}(\alpha) = H(\alpha_+)T'F\hat{C}(\alpha)' + \hat{C}(\alpha)F'TH(\alpha_+)' - \gamma\mathbf{I}$$

where $P(\alpha) = (T')^{-1}M(\alpha)T^{-1}$. Applying Schur complement, inequality (17) can be rewritten as follows

$$\begin{bmatrix} P(\alpha_+) - F - F' & \mathscr{L}_{12}(\alpha) & \mathscr{L}_{13}(\alpha) \\ (\star) & \mathscr{L}_{22}(\alpha) & \mathscr{L}_{23}(\alpha) \\ (\star) & (\star) & \mathscr{L}_{33}(\alpha) \end{bmatrix} + \zeta'\gamma^{-1}\zeta < 0 \tag{18}$$

where

$$\zeta = \begin{bmatrix} \mathbf{0} & \hat{B}(\alpha)' & \hat{D}(\alpha)' \end{bmatrix}.$$

Defining $\mathscr{X} = [F' \quad F'TG(\alpha_+)'T^{-1} \quad F'TH(\alpha_+)']'$ inequality (18) yields statement *iv)* of Lemma 1 with $\mathscr{Q}$, $\mathscr{B}$ and ξ given by Lemma 2. Lastly, the filter matrices are obtained by the change of variables (16), what concludes the proof. ■

Corollary 1: The minimum γ attainable by the conditions of Theorem 1 is given by the optimization problem

$$\min \gamma \quad \text{s.t. } (12) \tag{19}$$

Remark 1: Although the main goal in this work was stated as to obtain LMI conditions to solve Problem 1, Theorem 1 is presented in terms of BMI constraints. This follows from the use of statement *iv)* in Lemma 1 with multipliers defined as in Lemma 2 and $\mathscr{X} = [F' \quad F'TG(\alpha_+,\alpha_{++})'T^{-1} \quad F'TH(\alpha_+,\alpha_{++})']'$. The advantages of this approach is due to the extra variables that can be used in the search for better performance of the closed-loop system. As for instance, a lower $\mathscr{H}_\infty$ guaranteed cost may be obtained exploring the new variables $G(\alpha_+)$ and $H(\alpha_+)$. Nevertheless, by choosing $G(\alpha_+) = \mathbf{0}$ and $H(\alpha_+) = \mathbf{0}$ the conditions of Theorem 1 reduce to LMIs. As a consequence, Corollary 1 becomes a convex optimization problem that can be handled by Semi-Definite Programming (SDP) algorithms, as for example SeDuMi [21] and YALMIP [22] within the Matlab environment.

Remark 2: If $b = 0$, Problem 1 corresponds to the robust filtering problem of time-invariant uncertain systems. In this case, Theorem 1 provides sufficient conditions to design robust filters for uncertain discrete-time systems in polytopic domains. Furthermore, by setting $G(\alpha_+) = \mathbf{0}$ and $H(\alpha_+) = \mathbf{0}$, the conditions of Theorem 1 reduce to the ones similar to [8, Theorem 5.1] but in the $\mathscr{H}_\infty$ framework. The case $b = 1$, *i.e.* the parameters may vary arbitrarily inside the unit simplex $\mathscr{U}$, the conditions of Theorem 1 encompass the ones provided in [14, Corollary 4] leading to less conservative results when contrasted with robust filters designed by using quadratic Lyapunov functions.

Remark 3: In order to reduce the number of BMIs and the computational time required to solve the optimization problem (19), the conditions of Theorem 1 are written with variables $G(\alpha_+)$ and $H(\alpha_+)$ at time $k+1$ ($\alpha_+ = \alpha(k+1)$). Consequently, all products between parameter-dependent matrices appeared at the BMIs (12) are done in different instants of time. If $G(\cdot)$ and $H(\cdot)$ were written at time k a more sophisticated procedure, as the one proposed in [23], should be applied in order to get the BMI conditions expressed just in terms of the vertices of the polytope, resulting in a larger number of BMIs. Note, however, that $G(\alpha_+)$ and $H(\alpha_+)$ do not follow the structure of $M(\alpha_+)$. Since $G(\alpha_+)$ and $H(\alpha_+)$ are arbitrary extra variables, introduced to provide more freedom during the solution of Theorem 1, they can be chosen, for instance, as

$$G(\alpha_+) = \sum_{j=1}^{N} \beta_j G_j \ , \ H(\alpha_+) = \sum_{j=1}^{N} \beta_j H_j$$

with $\beta \in \mathscr{U}$, providing a different set of sufficient conditions.

Remark 4: Lastly, many methods appeared so far in the literature could be applied in the solution of Theorem 1. Nevertheless, the following algorithms are suggested. The first one is sometimes called an Alternating Semi-Definite Programming (or Gauss-Seidel) method [24] and consists of fixing some variables and solving for others in such a way that at each step a convex optimization problem is solved. The second one is called path-following method [25] and consists of linearizing the BMIs and then compute an increment that slightly improves the controller performance by solving an SDP problem. Although in both case there is no guarantee of convergence to local minimum, these methods are easy to implement and provide good results in many cases.

IV. Numerical Experiments

Example I: Consider the following uncertain time-varying discrete-time system borrowed from [9]

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0 & -0.5 \\ 1 & 1+\theta(k) \end{bmatrix} x(k) + \begin{bmatrix} -6 & 0 \\ 1 & 0 \end{bmatrix} w(k) \\ z(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \\ y(k) &= \begin{bmatrix} -100 & 10 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(k) \end{aligned} \tag{20}$$

where $\underline{\theta} \le \theta(k) \le \overline{\theta}$ and $|\Delta\theta| \le \delta$. A polytopic model (1) with two vertices is obtained evaluating system (20) at $\theta(k) = \underline{\theta}$ and $\theta(k) = \overline{\theta}$. Furthermore, noting that $\forall \alpha(k) \in \mathscr{U}$

$$\sum_{i=1}^{N} \Delta\alpha_i(k) = 0 \Rightarrow \Delta\alpha_1(k) = -\Delta\alpha_2(k)$$

it follows that

$$\begin{aligned} \theta(k) &= \alpha_1(k)\underline{\theta} + \alpha_2(k)\overline{\theta} \\ &= \underline{\theta} + \alpha_2(k)(\overline{\theta} - \underline{\theta}) \\ \Rightarrow \Delta\theta(k) &= \Delta\alpha_2(k)(\overline{\theta} - \underline{\theta}) \\ \Rightarrow |\Delta\theta(k)| &= |\Delta\alpha_2(k)||\overline{\theta} - \underline{\theta}| \le \delta \\ \Rightarrow |\Delta\alpha_2(k)| &= |\Delta\alpha_1(k)| \le \frac{\delta}{|\overline{\theta} - \underline{\theta}|} = b \end{aligned}$$

System (20) is analyzed for the cases where $\overline{\theta} = -\underline{\theta} = 0.3$ (in order to ensure quadratic stability) and $0 \le \delta \le 0.6$, which corresponds to $0 \le b \le 1$. Theorem 1 was solved by using the Alternating Semi-Definite Programming. Each

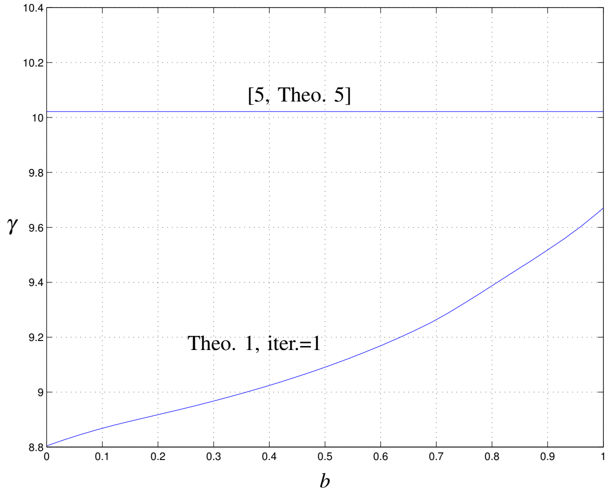

Fig. 1. $\mathscr{H}_\infty$ upper bound attained by using strictly proper fiters.

iteration consists in two steps. First the problem is solved with $G(\cdot) = \mathbf{0}$ and $H(\cdot) = \mathbf{0}$ and second, $G(\cdot)$ and $H(\cdot)$ are explored in the search for a better $\mathscr{H}_\infty$ upper bound γ. Figure 1 shows the minimum γ achieved with strictly proper filters (D_f=$\mathbf{0}$) designed for different cases where $0 \leq b \leq 1$. With only one iteration, Theorem 1 was able to provide approximately the same curve as the one given by the biquadratic method proposed in [9, Theorem 2]. As the number of iterations evolves better results are obtained. Figure 2 shows the improvement after 10 iterations.

It is important to stress that the Lyapunov function used to obtain the conditions of Theorem 1 is linear in the parameter α. Nevertheless, it still provides better results when contrasted with the biquadratic method, obtained by using a Lyapunov function that is quadratic in the parameter θ, making clearer the efficiency of the proposed approach.

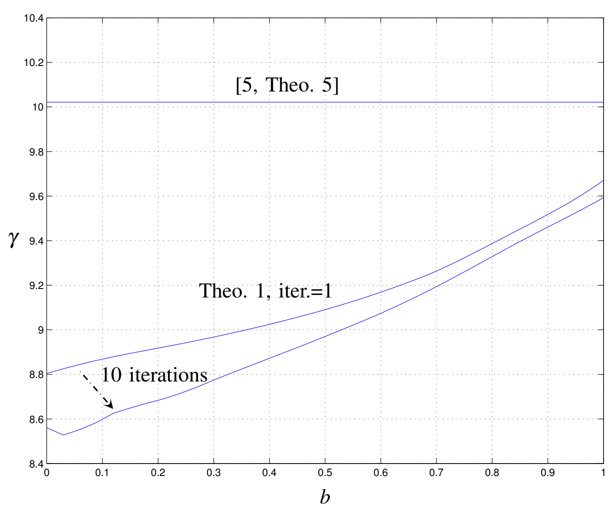

Fig. 2. Improvements in the $\mathscr{H}_\infty$ upper bound due to the BMIs.

Example II: Consider a time-varying system with state-space matrices given by

$$A = \begin{bmatrix} 0.9 & 0.1+0.06\xi(k) \\ 0.01+0.05\eta(k) & 0.9 \end{bmatrix}, B_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, D_{w1} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, D_{w2} = \begin{bmatrix} 0 & 0 & \sqrt{2} \end{bmatrix},$$

where $|\xi(k)| \leq \rho$, $|\Delta\xi(k)| \leq \varsigma$, $|\eta(k)| \leq \kappa$ and $|\Delta\eta(k)| \leq \nu$. This system was analyzed in [5] for arbitrarily time-varying parameters (*i.e.*, $b = 1$).

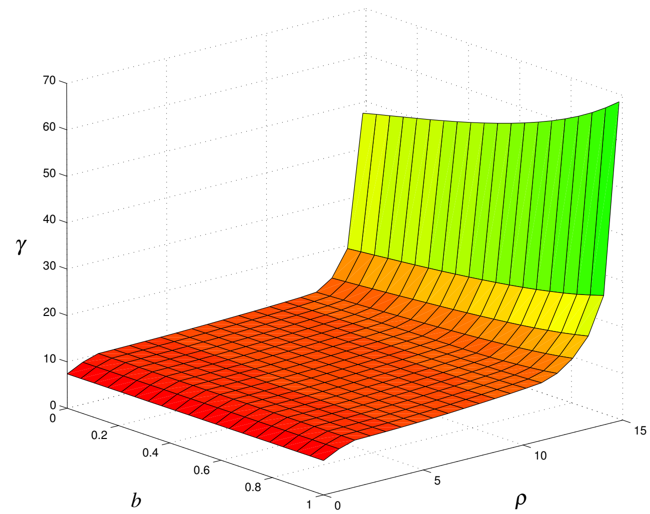

Fig. 3. $\mathscr{H}_\infty$ upper bound as a function of ρ and b.

For simplicity let $\kappa = 0$ and $\nu = 0$. Following the same steps of the previous example, it can be shown that $b = \varsigma/(2\rho)$. The aim was to define the maximum ρ such that Theorem 1 still provides a solution to Problem 1. For different values of $b \in [0,1]$, strictly proper filters were designed in order to attain the minimum $\mathscr{H}_\infty$ guaranteed cost, that can be seen in Figure 3 as a function of b and ρ. Note that as ρ approaches 15 the value of γ increases considerably, what is reasonable since this system becomes unstable for $\rho \geq 15$. For illustration purposes, the case $b = 0.3$ was analyzed and the results compared with [5, Theo. 5]. The values of γ for different values of ρ can be seen in Figure 4. The filter matrices for the point shown in Figure 4 are given by

$$A_f = \begin{bmatrix} -0.5236 & -0.1034 \\ 3.3463 & 1.1768 \end{bmatrix}, B_f = \begin{bmatrix} 0.0100 \\ -0.0199 \end{bmatrix},$$
$$C_f = \begin{bmatrix} 163.4488 & 12.5693 \end{bmatrix}.$$

An improvement of approximately 24.19% was provided by Theorem 1 when contrasted with [5, Theo. 5] ($\gamma = 18.4960$). Better results can still be achieved as the number of iterations increases, at the price of a bigger computational time.

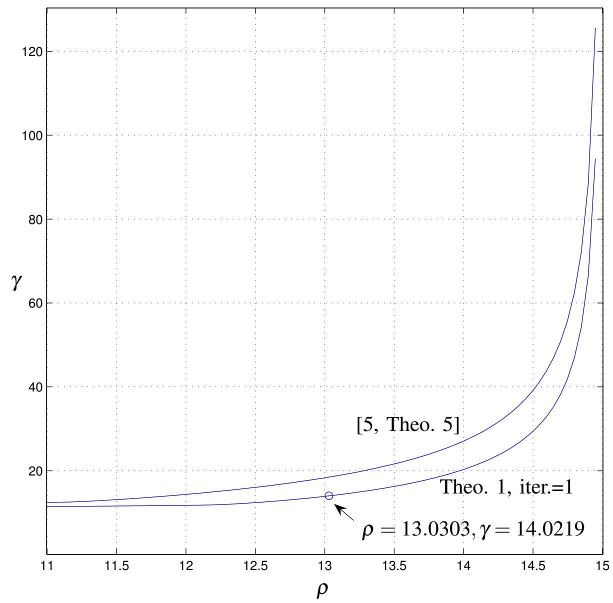

Fig. 4. Minimum γ achieved with strictly proper filters for $b = 0.3$.

V. CONCLUSION

The $\mathscr{H}_\infty$ robust filtering for uncertain discrete-time systems with bounded time-varying parameters has been addressed in this paper. With a more precise description of the parameter time variation, a less conservative design condition was obtained. Extra variables provided by the Finsler's Lemma were used to derive BMI conditions that may be explored in the search for a better $\mathscr{H}_\infty$ performance. The filter design is accomplished by means of an optimization problem, formulated only in terms of the vertices of the polytope, where all system matrices are considered to be affected by time-varying parameters. The proposed approach also provides some improvements when compared to other methods from the literature, as illustrated by examples.

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