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A BMI approach for $\mathscr{H}_\infty$ gain scheduling of discrete time-varying systems

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SUMMARY

The problem of gain-scheduled state feedback control for discrete-time linear systems with time-varying parameters is considered in this paper. The time-varying parameters are assumed to belong to the unit simplex and to have bounded rates of variation, which depend on the values of the parameters and can vary from slow to arbitrarily fast. An augmented state vector is defined to take into account possible time-delayed inputs, allowing a simplified closed-loop analysis by means of parameter-dependent Lyapunov functions. A gain-scheduled state feedback controller that minimizes an upper bound to the $\mathscr{H}_\infty$ performance of the closed-loop system is proposed. No grids in the parametric space are used. The design conditions are expressed in terms of bilinear matrix inequalities (BMIs) due to the use of extra variables introduced by the Finsler's lemma. By fixing some of the extra variables, the BMIs reduce to a convex optimization problem, providing an alternate semi-definite programming algorithm to solve the problem. Robust controllers for time-invariant uncertain parameters, as well as gain-scheduled controllers for arbitrarily time-varying parameters, can be obtained as particular cases of the proposed conditions. As illustrated by numerical examples, the extra variables in the BMIs can provide better results in terms of the closed-loop $\mathscr{H}_\infty$ performance.

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1. INTRODUCTION

One cannot deny the fact that gain scheduling has become an important topic within control system theory [1, 2]. As shown in [3], this technique can extend the validity of the linearization approach of non-linear systems to a range of operating points. Consequently, gain-scheduled controllers are guaranteed to work in a bigger region instead of only in a certain neighborhood of a single operating point. The main idea is to model the system in such a way that the different operating points are parametrized by one or more variables, commonly called scheduling variables [3]. The stability is then assured by a closed-loop Lyapunov function and a family of linear controllers, whose parameters are changed in accordance with the scheduling rules. Although there are other articles addressing the topic of gain scheduling [4–6] can be considered as pioneering works.

The use of linear parameter varying (LPV) structures to model certain classes of non-linear systems has

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provided an interesting framework for gain-scheduling control by means of convex optimization [2, 7–9]. It is worth mentioning that the state-space dynamic matrices of LPV systems depend on time-varying parameters that are assumed to be measured online. The use of such parameters in defining the scheduling rules brings extra information during the design step what may lead to less conservative results when contrasted to robust control strategies.

The Lyapunov theory has been extensively used as a main tool to deal with synthesis of gain-scheduled controllers. In many cases, it might be possible to express the design conditions as an optimization problem in terms of linear matrix inequalities (LMIs), which can be numerically handled by specific softwares [10–12]. As a way to guarantee robustness against practical disturbances, the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norms have been frequently applied as indexes of performance. Recent works include [13] where the problem of stabilizability and $\mathscr{H}_\infty$ control of discrete-time LPV systems is investigated by means of gain-scheduled state feedback [14] in which gain scheduling for linear fractional transformation (LFT) systems is designed by using parameter-dependent Lyapunov functions [15], where gain scheduled $\mathscr{H}_2$ controllers for affine LPV systems are proposed [16] in which robust and gain-scheduled controllers for LFT parameter-dependent systems are designed by using duality theory [17], where switching $\mathscr{H}_\infty$ controllers for a class of LPV systems scheduled along a measurable parameter trajectory are addressed, among others.

Bilinear matrix inequalities (BMIs) have also been applied in the study of control of LPV systems. It is well known that optimization problems expressed in terms of BMIs are non-convex. Nevertheless, the use of BMIs may represent a good strategy to face problems with either no solution, only sufficient conditions available in the literature or to improve the closed-loop performance. See, for instance, [18–21] and references therein.

Another important aspect observed in a large number of dynamic models, including LPV plants, is the presence of time delays. In many cases, a good characterization of time delays is required since they may represent a source of instability to the system trajectories. When the delay is known, a simple strategy consists of defining an augmented state vector, and then to design a standard controller that takes into account the delayed states (i.e. a memory controller). Other approaches could be used to cope with time delays, as for instance the ones based on the Lypuanov–Krasovskii functionals, resulting, in general, in more complex conditions that demand a higher computational effort.

The aim of this paper is to provide gain-scheduled memory controllers to stabilize discrete time-varying linear systems with bounded rates of variation. A simplified framework for possible time delays is assumed, where the delay is constant and a memory is used to store the delayed information. The use of a memory in the feedback loop allows one to cope with time delays without making use of more complex Lyapunov functionals. All the system matrices are assumed to be affected by the time-varying parameters, which are supposed to lie inside polytopic domains. An $\mathscr{H}_\infty$ guaranteed cost, which reflects the worst-case energy gain of the system, provides robustness with respect to unmodeled uncertainties. A preliminary version of this paper appeared in [22], where the time-varying parameters were allowed to vary arbitrarily fast inside the polytope. Here, a more precise parameter variation modeling is used to take into account the bounds on the rates of variation, providing synthesis procedures to cope with parameters that can be frozen or can vary slowly or arbitrarily fast. The Lyapunov theory is applied to assure the closed-loop stability with $\mathscr{H}_\infty$ disturbance attenuation, with a parameter-dependent Lyapunov function that reduces the conservatism of the proposed method, resulting in a more general approach when compared with methods based on quadratic stability. Extra variables introduced by the Finsler's lemma, that may be freely explored in the search for better performance of the LPV system, lead to design conditions expressed in terms of BMIs. The gain-scheduled memory controller is then obtained by the solution of an optimization problem that minimizes an upper bound to the $\mathscr{H}_\infty$ index of performance subject to a finite number of BMI constraints formulated only in terms of the vertices of the polytopic model. An iterative scheme is proposed, exploiting the fact that the BMIs reduce to LMIs by fixing some variables and also using line searches. Some results

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from the literature concerned with stability without time delays can be obtained as a particular case of the proposed method. Numerical examples illustrate the proposed conditions. The strategy proposed here could also be adapted to cope with the design of gain-scheduling controllers based on other types of storage functions, such as Lyapunov–Krasovskii functionals.

2. PRELIMINARIES AND PROBLEM STATEMENT

Consider the time-varying discrete-time system,

$$\begin{aligned} x(k+1) &= A(\alpha(k))x(k) + B_{du}(\alpha(k))u(k-\tau) \\ &\quad + B_u(\alpha(k))u(k) + B_w(\alpha(k))w(k) \\ x(0) &= 0 \\ y(k) &= C(\alpha(k))x(k) + D_{du}(\alpha(k))u(k-\tau) \\ &\quad + D_u(\alpha(k))u(k) + D_w(\alpha(k))w(k) \end{aligned} \tag{1}$$

where τ represents the discrete-time delay, $x(k)\in\mathbb{R}^n$ is the state-space vector, $u(k)\in\mathbb{R}^m$ is the control signal, $w(k)\in\mathbb{R}^r$ is the $l_2[0,\infty)$ noise and $y(k)\in\mathbb{R}^q$ is the controlled output. The time-varying vector of parameters $\alpha(k)$ belongs to the unit simplex

$$\mathscr{U}_N = \left\{\psi\in\mathbb{R}^N : \sum_{i=1}^{N}\psi_i = 1, \psi_i \geqslant 0, i=1,\ldots,N\right\}$$

for all $k\geqslant 0$ with bounded rates of variation of percentage $b\in[0,1]$. For instance, $b=0.05$ indicates that the parameters are constrained to vary only 5% of their original values between two instants of time. The time-invariant case is modeled by $b=0$ and arbitrarily fast variations by $b=1$. All matrices are real, with appropriate dimensions, belonging to the polytope

$$\hat{\mathscr{P}} \triangleq \left\{ \left[\begin{array}{c|c} A(\alpha(k)) & B_u(\alpha(k)) \\ \hline B_{du}(\alpha(k)) & B_w(\alpha(k)) \\ \hline C(\alpha(k)) & D_u(\alpha(k)) \\ \hline D_{du}(\alpha(k)) & D_w(\alpha(k)) \end{array}\right] = \sum_{i=1}^{N}\alpha_i(k) \left[\begin{array}{c|c} A_i & B_{ui} \\ \hline B_{dui} & B_{wi} \\ \hline C_i & D_{ui} \\ \hline D_{dui} & D_{wi} \end{array}\right],\ \alpha(k)\in\mathscr{U}_N \right\} \tag{2}$$

More specifically, the system matrices are given, for any time $k\geqslant 0$, by the convex combination of the well-defined vertices of the polytope $\hat{\mathscr{P}}$. As usual in gain-scheduling control, it is also assumed that the parameters $\alpha(k)$ are measured online.

In order to guarantee the stability of system (1), a memory state feedback controller with a parameter-dependent gain is designed. Using extra state variables $z(k)$ to store the delayed values of the control signal, system (1) can be rewritten as follows [23]:

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}(\alpha(k))\tilde{x}(k) + \tilde{B}_u(\alpha(k))u(k) \\ &\quad + \tilde{B}_w(\alpha(k))w(k) \\ \tilde{x}(0) &= 0 \\ y(k) &= \tilde{C}(\alpha(k))\tilde{x}(k) + \tilde{D}_u(\alpha(k))u(k) \\ &\quad + \tilde{D}_w(\alpha(k))w(k) \end{aligned} \tag{3}$$

where $\tilde{x}(k) = [x(k)' z(k)']' \in \mathbb{R}^{n+\tau m}$ and

$$\tilde{A}(\alpha(k)) = \begin{bmatrix} A(\alpha(k)) & B_{du}(\alpha(k)) & 0 & \ldots & 0 \\ 0 & 0 & \mathbf{I} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mathbf{I} \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

$$\tilde{B}_u(\alpha(k)) = \begin{bmatrix} B_u(\alpha(k)) \\ 0 \\ 0 \\ \vdots \\ \mathbf{I} \end{bmatrix} \tag{4}$$

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$$\tilde{B}_w(\alpha(k)) = \begin{bmatrix} B_w(\alpha(k)) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\tilde{C}(\alpha(k)) = [C(\alpha(k)) \;\; D_{du}(\alpha(k)) \;\; 0 \;\; \ldots \;\; 0]$$

$$\tilde{D}_u(\alpha(k)) = D_u(\alpha(k)), \quad \tilde{D}_w(\alpha(k)) = D_w(\alpha(k))$$

The memory control law is given by

$$u(k)=[K_x(\alpha(k)) \;\; K_d(\alpha(k))]\begin{bmatrix} x(k) \\ z(k) \end{bmatrix}=K(\alpha(k))\tilde{x}(k) \tag{5}$$

where $K(\alpha(k))=[K_x(\alpha(k)) \;\; K_d(\alpha(k))]$, yielding the closed-loop system

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}_{\mathrm{cl}}(\alpha(k))\tilde{x}(k)+\tilde{B}_w(\alpha(k))w(k) \\ \tilde{x}(0) &= 0 \\ y(k) &= \tilde{C}_{\mathrm{cl}}(\alpha(k))\tilde{x}(k)+\tilde{D}_w(\alpha(k))w(k) \end{aligned} \tag{6}$$

with $\tilde{x}(k)\in\mathbb{R}^{n+\tau m}$, $w(k)\in\mathbb{R}^r$, $y(k)\in\mathbb{R}^q$ and

$$\begin{aligned} \tilde{A}_{\mathrm{cl}}(\alpha(k)) &= \tilde{A}(\alpha(k))+\tilde{B}_u(\alpha(k))K(\alpha(k)) \\ \tilde{C}_{\mathrm{cl}}(\alpha(k)) &= \tilde{C}(\alpha(k))+\tilde{D}_u(\alpha(k))K(\alpha(k)) \end{aligned} \tag{7}$$

The control problem to be dealt with can be stated as follows.

Problem 1
Find parameter-dependent matrices $K_x(\alpha(k))\in\mathbb{R}^{m\times n}$ and $K_d(\alpha(k))\in\mathbb{R}^{m\times\tau m}$ of the control law (5), such that the closed-loop system (6) is asymptotically stable, and an upper bound $\gamma>0$ to the $\mathscr{H}_\infty$ performance is minimized, that is

$$\sup_{w\neq 0}\frac{\|y\|_2^2}{\|w\|_2^2}<\gamma^2 \tag{8}$$

with $w\in l_2[0,\infty)$.

Condition (8) for a given closed-loop discrete time-varying linear system can be characterized by the discrete-time version of the *bounded real lemma* in terms of parameter-dependent LMIs, as for instance it has been presented in [24, 25]. The result is extended here in the context of parameter-dependent time-varying systems, as follows.

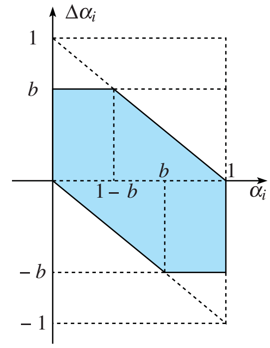

Figure 1. Region on the plane $\Delta\alpha_i\times\alpha_i$ where $\Delta\alpha_i$ can assume values as a function of α_i (dark region).

Lemma 1
For a given γ, if there exists a bounded matrix sequence $P(\alpha(k))'=P(\alpha(k))>0$ such that[‡]

$$\begin{bmatrix} -P(\alpha(k)) & P(\alpha(k))\tilde{A}(\alpha(k))' & P(\alpha(k))\tilde{C}(\alpha(k))' & \mathbf{0} \\ (\star) & -P(\alpha(k+1)) & \mathbf{0} & \tilde{B}(\alpha(k)) \\ (\star) & (\star) & -\gamma\mathbf{I} & \tilde{D}(\alpha(k)) \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} <0 \tag{9}$$

for all $\alpha(k)\in\mathscr{U}_N$, then the closed-loop system (6) is asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.

Note that, since the parameters lie inside a unit simplex, the rates of variation are intrinsically lower bounded by $-b$ and upper bounded by $b, b\in[0,1]$. In order to develop a model[§] for the parameter variation when $-b<\Delta\alpha_i(k)<b, b\neq 0$, note that the feasible values of $\Delta\alpha_i(k)$ depend on the actual values of $\alpha_i(k)$, as shown in Figure 1 (darken area).

[‡]The symbol $(\star)$ indicates symmetric blocks in the LMIs.
[§]For simplicity, the same b is considered for all $\alpha_i, i=1,\ldots,N$.

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Thus, any pair $(\alpha_i, \Delta\alpha_i)$ belongs to the polytope $\Lambda_i, i=1,\ldots,N$ given by

$$\Lambda_i \triangleq \left\{\delta\in\mathbb{R}^2 : \delta=\sum_{j=1}^{6}\lambda_j s_j, \lambda\in\mathscr{U}_6\right\}$$
$$S=[s_1\ldots s_6]=\left[\begin{array}{c|c|c|c|c|c} 0 & 0 & 1-b & 1 & 1 & b \\ 0 & b & b & 0 & -b & -b \end{array}\right] \tag{10}$$

that is, Λ_i represents the convex combination of the extremes (vertices) of the feasible area.

To construct the $(\alpha, \Delta\alpha)$-space, the Cartesian product of all $\Lambda_i, i=1,\ldots,N$ must be considered, taking into account that the new vertices must satisfy $\alpha_1+\cdots+\alpha_N=1$ and $\Delta\alpha_1+\cdots+\Delta\alpha_N=0$. The resulting polytope, called Λ, is then given by

$$\Lambda\triangleq\left\{\delta\in\mathbb{R}^{2N} : \delta=\sum_{i=1}^{M}\lambda_i q_i,\ \lambda\in\mathscr{U}_M\right\} \tag{11}$$

where $q_i\in\mathbb{R}^{2N}$ are given vectors. Thus, the first step to search for a solution to any LMI/BMI depending on both α and $\Delta\alpha$ is to lift the inequalities to the λ-space, by observing that from (11) one has

$$\begin{bmatrix}\alpha\\ \Delta\alpha\end{bmatrix}=Q\lambda, \quad Q=[q_1\ldots q_M]\in\mathbb{R}^{2N\times M}$$
$$\lambda\in\mathscr{U}_M \tag{12}$$

Therefore, in the case of affine on $\alpha(k)$ parameter-dependent matrices, that is

$$X(\alpha(k))=\sum_{i=1}^{N}\alpha_i(k)X_i, \quad \alpha_i(k)=\sum_{j=1}^{M}\lambda_j Q_{ij} \tag{13}$$

$$X(\alpha(k+1))=\sum_{i=1}^{N}(\alpha_i(k)+\Delta\alpha_i(k))X_i$$
$$\Delta\alpha_i(k)=\sum_{j=1}^{M}\lambda_j Q_{(i+N)j} \tag{14}$$

it follows that

$$\bar{X}(\lambda)=\sum_{i=1}^{N}\sum_{j=1}^{M}\lambda_j Q_{ij}X_i=\sum_{j=1}^{M}\lambda_j\bar{X}_j \tag{15}$$

$$\tilde{X}(\lambda)=\sum_{i=1}^{N}\sum_{j=1}^{M}\lambda_j(Q_{ij}+Q_{(i+N)j})X_i=\sum_{j=1}^{M}\lambda_j\tilde{X}_j \tag{16}$$

where

$$\bar{X}_j=\sum_{i=1}^{N}Q_{ij}X_i \tag{17}$$

$$\tilde{X}_j=\sum_{i=1}^{N}(Q_{ij}+Q_{(i+N)j})X_i \tag{18}$$

Another preliminary result, the Finsler's lemma, is reproduced here for convenience.

Lemma 2
Let $\xi\in\mathbb{R}^a, \mathscr{Q}=\mathscr{Q}'\in\mathbb{R}^{a\times a}, \mathscr{B}\in\mathbb{R}^{b\times a}$ with $\text{rank}(\mathscr{B})<a$, and $\mathscr{B}^\perp$ a basis for the null-space of $\mathscr{B}$ (i.e. $\mathscr{B}\mathscr{B}^\perp=0$). The following statements are equivalent.

(i) $\xi'\mathscr{Q}\xi<0, \forall\mathscr{B}\xi=0, \xi\neq0$;
(ii) $\mathscr{B}^{\perp'}\mathscr{Q}\mathscr{B}^\perp<0$;
(iii) $\exists\mu\in\mathbb{R}:\mathscr{Q}-\mu\mathscr{B}'\mathscr{B}<0$;
(iv) $\exists\mathscr{X}\in\mathbb{R}^{a\times b}:\mathscr{Q}+\mathscr{X}\mathscr{B}+\mathscr{B}'\mathscr{X}'<0$.

Proof
See [26]. □

The variables μ and $\mathscr{X}$ in statements (iii) and (iv) of Lemma 2 allow one to present a more general version of Lemma 1. As pointed out in [26], these variables represent extra degree of freedom that may be used, for instance, for design purposes. By considering the particular structure

$$\mathscr{X}=[F(\alpha(k))'\ F(\alpha(k))'G(\alpha(k+1))'$$
$$F(\alpha(k))'H(\alpha(k+1))']' \tag{19}$$

the following condition is obtained.

Theorem 1
For a given $\gamma>0$, if there exists a bounded matrix sequence $F(\alpha(k)), G(\alpha(k)), P(\alpha(k))'=P(\alpha(k))>0$ and

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$H(\alpha(k))$, such that

$$\begin{bmatrix} P(\alpha(k+1))-F(\alpha(k))-F(\alpha(k))' & \hat{\mathscr{F}}_{12} & \hat{\mathscr{F}}_{13} & \mathbf{0} \\ (\star) & \hat{\mathscr{F}}_{22} & \hat{\mathscr{F}}_{23} & \tilde{B}_{w\mathrm{cl}}(\alpha(k)) \\ (\star) & (\star) & \hat{\mathscr{F}}_{33} & \tilde{D}_{w\mathrm{cl}}(\alpha(k)) \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \quad (20)$$

$$\begin{aligned} \hat{\mathscr{F}}_{12} &= F(\alpha(k))\tilde{A}_{\mathrm{cl}}(\alpha(k))' - F(\alpha(k))'G(\alpha(k+1))' \\ \hat{\mathscr{F}}_{13} &= F(\alpha(k))\tilde{C}_{\mathrm{cl}}(\alpha(k))' - F(\alpha(k))'H(\alpha(k+1))' \\ \hat{\mathscr{F}}_{22} &= G(\alpha(k+1))F(\alpha(k))\tilde{A}_{\mathrm{cl}}(\alpha(k))' \\ &\quad +\tilde{A}_{\mathrm{cl}}(\alpha(k))F(\alpha(k))'G(\alpha(k+1))' - P(\alpha(k)) \\ \hat{\mathscr{F}}_{23} &= G(\alpha(k+1))F(\alpha(k))\tilde{C}_{\mathrm{cl}}(\alpha(k))' \\ &\quad +\tilde{A}_{\mathrm{cl}}(\alpha(k))F(\alpha(k))'H(\alpha(k+1))' \\ \hat{\mathscr{F}}_{33} &= H(\alpha(k+1))F(\alpha(k))\tilde{C}_{\mathrm{cl}}(\alpha(k))' \\ &\quad +\tilde{C}_{\mathrm{cl}}(\alpha(k))F(\alpha(k))'H(\alpha(k+1))' - \gamma\mathbf{I} \end{aligned}$$

for all $(\alpha(k),\Delta\alpha(k))\in\Lambda$, then the closed-loop system (6) is asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.

Proof
First, using Schur complement, inequality (20) can be rewritten as follows:

$$\begin{bmatrix} P(\alpha(k+1))-F(\alpha(k))-F(\alpha(k))' & \hat{\mathscr{F}}_{12} & F\tilde{C}_{\mathrm{cl}}(\alpha(k))' - F'H(\alpha(k+1))' \\ (\star) & \hat{\mathscr{F}}_{22} & \hat{\mathscr{F}}_{23} \\ (\star) & (\star) & \hat{\mathscr{F}}_{33} \end{bmatrix} + \gamma^{-1}\hat{\mathscr{F}}_4(\alpha(k))\hat{\mathscr{F}}_4(\alpha(k))' < 0 \quad (21)$$

where

$$\hat{\mathscr{F}}_4(\alpha(k)) = [\mathbf{0}\ \ \tilde{B}_{w\mathrm{cl}}(\alpha(k))'\ \ \tilde{D}_{w\mathrm{cl}}(\alpha(k))']'$$

Second, by setting

$$\mathscr{Q} = \begin{bmatrix} P(\alpha(k+1)) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma^{-1}\tilde{B}_{w\mathrm{cl}}(\alpha(k))\tilde{B}_{w\mathrm{cl}}(\alpha(k))' - P(\alpha(k)) & \gamma^{-1}\tilde{B}_{w\mathrm{cl}}(\alpha(k))\tilde{D}_{w\mathrm{cl}}(\alpha(k))' \\ \mathbf{0} & \gamma^{-1}\tilde{D}_{w\mathrm{cl}}(\alpha(k))\tilde{B}_{w\mathrm{cl}}(\alpha(k))' & \gamma^{-1}\tilde{D}_{w\mathrm{cl}}(\alpha(k))\tilde{D}_{w\mathrm{cl}}(\alpha(k))' - \gamma\mathbf{I} \end{bmatrix}$$

$$\mathscr{B} = [-\mathbf{I}\ \tilde{A}_{\mathrm{cl}}(\alpha(k))'\ \tilde{C}_{\mathrm{cl}}(\alpha(k))'], \quad \xi = [\tilde{x}(k+1)'\ \ \tilde{x}(k)'\ \ w(k)']'$$

with $\mathscr{X}$ given by (19), inequality (21) yields statement (iv) of Lemma 2. Finally, if statement (iv) of Lemma 2 holds then statement (ii) also holds and Lemma 1 follows immediately. The fact that (iv) $\Rightarrow$ (ii) can be verified by multiplying (21) on the left by $\mathscr{B}^\perp$ and on the right by $\mathscr{B}^{\perp\prime}$, where

$$\mathscr{B}^\perp = \begin{bmatrix} \tilde{A}_{\mathrm{cl}}(\alpha(k))' & \tilde{C}_{\mathrm{cl}}(\alpha(k))' \\ \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix}$$

□

The conditions of Theorem 1 exhibit non-linearities and must be tested at all points of the simplex $\mathscr{U}_N$, that is, at an infinite number of points. Moreover, the unknown parameter-dependent matrices appear as functions of both $\alpha(k+1)$ and $\alpha(k)$. Hence, the main goal hereafter is to obtain finite-dimensional conditions in terms of the vertices of the polytope $\hat{\mathscr{P}}$ to solve Problem 1, considering the particular structure for the Lyapunov matrix (similar structures for $F(\alpha(k)), G(\alpha(k+1))$ and $H(\alpha(k+1))$ have been used)

$$\begin{aligned} P(\alpha(k)) &= \alpha_1(k)P_1 + \alpha_2(k)P_2 + \cdots + \alpha_N(k)P_N \\ \alpha(k) &\in \mathscr{U}_N \end{aligned} \quad (22)$$

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More complex structures, as for instance with polynomial dependence on $\alpha(k)$, could be used following the lines depicted in [27], yielding BMI conditions that would be more precise at expenses of being much more involved. Now, considering the λ-space presented, using Schur complement, change of variables and exploring the extra variables provided by Lemma 2, BMI conditions assuring the existence of such controllers are given in the next section.

3. MAIN RESULTS

Theorem 2

Given the augmented discrete-time system (3) and matrix Q as in (12), if there exist matrices $L_i, H_i, F_i, G_i, P_i = P_i' > 0$, with appropriate dimensions, for $i = 1, \ldots, N$ and a scalar $\gamma > 0$, the control law (5), with matrices given by

$$K(\alpha(k)) = [K_x(\alpha(k)) \;\; K_d(\alpha(k))] = L(\alpha(k))(F(\alpha(k))')^{-1} \tag{23}$$

where

$$L(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k) L_i, \quad F(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k) F_i$$
$$\alpha(k) \in \mathscr{U}_N \tag{24}$$

assures the asymptotic stability of the closed-loop system (6) and an $\mathscr{H}_\infty$ guaranteed cost γ provided that, for matrices $\bar{L}_i, \bar{H}_i, \bar{F}_i, \bar{G}_i, \bar{P}_i, \hat{A}_i, \hat{B}_{ui}, \hat{B}_{wi}, \hat{C}_i, \hat{D}_{ui}$ and $\hat{D}_{wi}$ given as in (17) and $\tilde{H}_i, \tilde{G}_i, \tilde{P}_i$ as in (18)

$$\Xi_i \triangleq \begin{bmatrix} \tilde{P}_i - \bar{F}_i - \bar{F}_i' & \mathscr{F}_{12} & \bar{F}_i\hat{C}_i' + \bar{L}_i'\hat{D}_{ui}' - \bar{F}_i'\tilde{H}_i' & \mathbf{0} \\ (\star) & \mathscr{F}_{22} & \mathscr{F}_{23} & \hat{B}_{wi} \\ (\star) & (\star) & \mathscr{F}_{33} & \hat{D}_{wi} \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0$$
$$i = 1, \ldots, M \tag{25}$$

$$\mathscr{F}_{12} = \bar{F}_i\hat{A}_i' + \bar{L}_i'\hat{B}_{ui}' - \bar{F}_i'\tilde{G}_i'$$
$$\mathscr{F}_{22} = \tilde{G}_i F_i \hat{A}_i' + \hat{A}_i \bar{F}_i'\tilde{G}_i' + \tilde{G}_i\bar{L}_i'\hat{B}_{ui}' + \hat{B}_{ui}\bar{L}_i\tilde{G}_i' - \bar{P}_i$$
$$\mathscr{F}_{23} = \tilde{G}_i\bar{F}_i\hat{C}_i' + \tilde{G}_i\bar{L}_i'\hat{D}_{ui}' + \hat{A}_i\bar{F}_i'\tilde{H}_i' + \hat{B}_{ui}\bar{L}_i\tilde{H}_i'$$
$$\mathscr{F}_{33} = \tilde{H}_i\bar{F}_i\hat{C}_i' + \hat{C}_i\bar{F}_i'\tilde{H}_i' + \tilde{H}_i\bar{L}_i'\hat{D}_{ui}' + \hat{D}_{ui}\bar{L}_i\tilde{H}_i' - \gamma\mathbf{I}$$

$$\Xi_{ik} \triangleq \begin{bmatrix} \bar{\mathscr{F}}_{11} & \bar{\mathscr{F}}_{12} & \bar{\mathscr{F}}_{13} & \mathbf{0} \\ (\star) & \bar{\mathscr{F}}_{22} + \bar{\mathscr{F}}_{22}' - 2\bar{P}_i - \bar{P}_k & \bar{\mathscr{F}}_{23} & 2\hat{B}_{wi} + \hat{B}_{wk} \\ (\star) & (\star) & \bar{\mathscr{F}}_{33} + \bar{\mathscr{F}}_{33}' - 3\gamma\mathbf{I} & 2\hat{D}_{wi} + \hat{D}_{wk} \\ (\star) & (\star) & (\star) & -3\gamma\mathbf{I} \end{bmatrix} < 0$$
$$i = 1, \ldots, M, \quad k = 1, \ldots, M, \quad i \neq k \tag{26}$$

$$\begin{aligned} \bar{\mathscr{F}}_{11} &= 2\tilde{P}_i + \tilde{P}_k - 2\bar{F}_i - 2\bar{F}_i' - \bar{F}_k - \bar{F}_k', \bar{\mathscr{F}}_{12} = \bar{F}_i\hat{A}_i' \\ &\quad + \bar{F}_i\hat{A}_k' + \bar{F}_k\hat{A}_i' + \bar{L}_i'\hat{B}_{ui}' + \bar{L}_i'\hat{B}_{uk}' + \bar{L}_k'\hat{B}_{ui}' - \bar{F}_i'\tilde{G}_i' \\ &\quad - \bar{F}_i'\tilde{G}_k' - \bar{F}_k'\tilde{G}_i', \bar{\mathscr{F}}_{13} = \bar{F}_i\hat{C}_i' + \bar{F}_i\hat{C}_k' + \bar{F}_k\hat{C}_i' \\ &\quad + \bar{L}_i'\hat{D}_{ui}' + \bar{L}_i'\hat{D}_{uk}' + \bar{L}_k'\hat{D}_{ui}' - \bar{F}_i'\tilde{H}_i' - \bar{F}_i'\tilde{H}_k' - \bar{F}_k'\tilde{H}_i' \\ \bar{\mathscr{F}}_{22} &= \tilde{G}_i\bar{F}_i\hat{A}_k' + \tilde{G}_k\bar{F}_i\hat{A}_i' + \tilde{G}_i\bar{F}_k\hat{A}_i' + \tilde{G}_i\bar{L}_i'\hat{B}_{uk}' \\ &\quad + \tilde{G}_k\bar{L}_i'\hat{B}_{ui}' + \tilde{G}_i\bar{L}_k'\hat{B}_{ui}'\bar{\mathscr{F}}_{23} = \tilde{G}_i\bar{F}_i\hat{C}_k' + \tilde{G}_k\bar{F}_i\hat{C}_i' \\ &\quad + \tilde{G}_i\bar{F}_k\hat{C}_i' + \tilde{G}_i\bar{L}_i'\hat{D}_{uk}' + \tilde{G}_k\bar{L}_i'\hat{D}_{ui}' + \tilde{G}_i\bar{L}_k'\hat{D}_{ui}' \\ &\quad + \hat{A}_i\bar{F}_i'\tilde{H}_k' + \hat{A}_k\bar{F}_i'\tilde{H}_i' + \hat{A}_i\bar{F}_k'\tilde{H}_i' + \hat{B}_{ui}\bar{L}_i\tilde{H}_k' \\ &\quad + \hat{B}_{uk}\bar{L}_i\tilde{H}_i' + \hat{B}_{ui}\bar{L}_k\tilde{H}_i' \\ \bar{\mathscr{F}}_{33} &= \tilde{H}_i\bar{F}_i\hat{C}_k' + \tilde{H}_k\bar{F}_i\hat{C}_i' + \tilde{H}_i\bar{F}_k\hat{C}_i' + \tilde{H}_i\bar{L}_i'\hat{D}_{uk}' \\ &\quad + \tilde{H}_k\bar{L}_i'\hat{D}_{ui}' + \tilde{H}_i\bar{L}_k'\hat{D}_{ui}' \end{aligned}$$

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$$\Xi_{ik\ell} \triangleq \begin{bmatrix} \tilde{\mathscr{F}}_{11} & \tilde{\mathscr{F}}_{12} & \tilde{\mathscr{F}}_{13} & \mathbf{0} \\ (\star) & \tilde{\mathscr{F}}_{22}+\tilde{\mathscr{F}}_{22}'-2\bar{P}_i-2\bar{P}_k-2\bar{P}_\ell & \tilde{\mathscr{F}}_{23} & 2(\hat{B}_{wi}+\hat{B}_{wk}+\hat{B}_{w\ell}) \\ (\star) & (\star) & \tilde{\mathscr{F}}_{33}+\tilde{\mathscr{F}}_{33}'-6\gamma\mathbf{I} & 2(\hat{D}_{wi}+\hat{D}_{wk}+\hat{D}_{w\ell}) \\ (\star) & (\star) & (\star) & -6\gamma\mathbf{I} \end{bmatrix} < 0$$

$$i=1,\ldots,M-2, \quad k=i+1,\ldots,M-1, \quad \ell=k+1,\ldots,M \qquad (27)$$

$$\begin{aligned}
\tilde{\mathscr{F}}_{11} &= 2\tilde{P}_i+2\tilde{P}_k+2\tilde{P}_\ell-2\bar{F}_i-2\bar{F}_i'-2\bar{F}_k \\
&\quad -2\bar{F}_k'-2\bar{F}_\ell-2\bar{F}_\ell' \\
\tilde{\mathscr{F}}_{12} &= \bar{F}_i\hat{A}_k'+\bar{F}_k\hat{A}_i'+\bar{F}_i\hat{A}_\ell'+\bar{F}_\ell\hat{A}_i'+\bar{F}_\ell\hat{A}_k'+\bar{F}_k\hat{A}_\ell' \\
&\quad +\bar{L}_i'\hat{B}_{uk}'+\bar{L}_k'\hat{B}_{ui}'+\bar{L}_i'\hat{B}_{u\ell}'+\bar{L}_\ell'\hat{B}_{ui}'+\bar{L}_\ell'\hat{B}_{uk}' \\
&\quad +\bar{L}_k'\hat{B}_{u\ell}'-\bar{F}_i'\tilde{G}_k'-\bar{F}_k'\tilde{G}_i'-\bar{F}_i'\tilde{G}_\ell' \\
&\quad -\bar{F}_\ell'\tilde{G}_i'-\bar{F}_\ell'\tilde{G}_k'-\bar{F}_k'\tilde{G}_\ell' \\
\tilde{\mathscr{F}}_{13} &= \bar{F}_i\hat{C}_k'+\bar{F}_k\hat{C}_i'+\bar{F}_i\hat{C}_\ell'+\bar{F}_\ell\hat{C}_i'+\bar{F}_\ell\hat{C}_k' \\
&\quad +\bar{F}_k\hat{C}_\ell'+\bar{L}_i'\hat{D}_{uk}'+\bar{L}_k'\hat{D}_{ui}'+\bar{L}_i'\hat{D}_{u\ell}'+\bar{L}_\ell'\hat{D}_{ui}' \\
&\quad +\bar{L}_\ell'\hat{D}_{uk}'+\bar{L}_k'\hat{D}_{u\ell}'-\bar{F}_i'\tilde{H}_k'-\bar{F}_k'\tilde{H}_i'-\bar{F}_i'\tilde{H}_\ell' \\
&\quad -\bar{F}_\ell'\tilde{H}_i'-\bar{F}_\ell'\tilde{H}_k'-\bar{F}_k'\tilde{H}_\ell' \\
\tilde{\mathscr{F}}_{22} &= \tilde{G}_i\bar{F}_k\hat{A}_\ell'+\tilde{G}_i\bar{F}_\ell\hat{A}_k' \\
&\quad +\tilde{G}_k\bar{F}_i\hat{A}_\ell'+\tilde{G}_k\bar{F}_\ell\hat{A}_i'+\tilde{G}_\ell\bar{F}_i\hat{A}_k'+\tilde{G}_\ell\bar{F}_k\hat{A}_i' \\
&\quad +\tilde{G}_i\bar{L}_k'\hat{B}_{u\ell}'+\tilde{G}_i\bar{L}_\ell'\hat{B}_{uk}'+\tilde{G}_k\bar{L}_i'\hat{B}_{u\ell}'+\tilde{G}_k\bar{L}_\ell'\hat{B}_{ui}' \\
&\quad +\tilde{G}_\ell\bar{L}_i'\hat{B}_{uk}'+\tilde{G}_\ell\bar{L}_k'\hat{B}_{ui}' \\
\tilde{\mathscr{F}}_{23} &= \tilde{G}_i\bar{F}_k\hat{C}_\ell'+\tilde{G}_i\bar{F}_\ell\hat{C}_k'+\tilde{G}_k\bar{F}_i\hat{C}_\ell' \\
&\quad +\tilde{G}_k\bar{F}_\ell\hat{C}_i'+\tilde{G}_\ell\bar{F}_i\hat{C}_k'+\tilde{G}_\ell\bar{F}_k\hat{C}_i' \\
&\quad +G_iL_k'\hat{D}_{u\ell}'+G_iL_\ell'\hat{D}_{uk}'+G_kL_i'\hat{D}_{u\ell}' \\
&\quad +G_kL_\ell'\hat{D}_{ui}'+G_\ell L_i'\hat{D}_{uk}'+G_\ell L_k'\hat{D}_{ui}' \\
&\quad +\hat{A}_i\bar{F}_k'\tilde{H}_\ell'+\hat{A}_i\bar{F}_\ell'\tilde{H}_k'+\hat{A}_k\bar{F}_i'\tilde{H}_\ell'+\hat{A}_k\bar{F}_\ell'\tilde{H}_i' \\
&\quad +\hat{A}_\ell\bar{F}_i'\tilde{H}_k'+\hat{A}_\ell\bar{F}_k'\tilde{H}_i'+\hat{B}_{ui}\bar{L}_k\tilde{H}_\ell'+\hat{B}_{ui}\bar{L}_\ell\tilde{H}_k' \\
&\quad +\hat{B}_{uk}\bar{L}_i\tilde{H}_\ell'+\hat{B}_{uk}\bar{L}_\ell\tilde{H}_i'+\hat{B}_{u\ell}\bar{L}_i\tilde{H}_k'+\hat{B}_{u\ell}\bar{L}_k\tilde{H}_i' \\
\tilde{\mathscr{F}}_{33} &= \tilde{H}_i\bar{F}_k\hat{C}_\ell'+\tilde{H}_i\bar{F}_\ell\hat{C}_k'+\tilde{H}_k\bar{F}_i\hat{C}_\ell'+\tilde{H}_k\bar{F}_\ell\hat{C}_i' \\
&\quad +\tilde{H}_\ell\bar{F}_i\hat{C}_k'+\tilde{H}_\ell\bar{F}_k\hat{C}_i'+\tilde{H}_i\bar{L}_k'\hat{D}_{u\ell}'+\tilde{H}_i\bar{L}_\ell'\hat{D}_{uk}' \\
&\quad +\tilde{H}_k\bar{L}_i'\hat{D}_{u\ell}'+\tilde{H}_k\bar{L}_\ell'\hat{D}_{ui}'+\tilde{H}_\ell\bar{L}_i'\hat{D}_{uk}'+\tilde{H}_\ell\bar{L}_k'\hat{D}_{ui}'
\end{aligned}$$

Proof

Applying the following operation [28]

$$\Xi(\lambda) = \sum_{i=1}^{M}\lambda_i^3\Xi_i + \sum_{i=1}^{M}\sum_{k=1,k\neq i}^{M}\lambda_i^2\lambda_k\Xi_{ik} + \sum_{i=1}^{M-2}\sum_{k=i+1}^{M-1}\sum_{\ell=k+1}^{M}\lambda_i\lambda_k\lambda_\ell\Xi_{ik\ell} \qquad (28)$$

to the BMIs (25), (26) and (27) inequality (20) follows immediately considering the particular structure (22) for the Lyapunov matrix, the change of variables $L(\alpha(k)) = K(\alpha(k))F(\alpha(k))'$ and the lift of the BMI to the λ-space. Note that the choice of $P(\alpha(k))$ given by (22) with $P_i > 0$ assures a lower bound to the sequence. Finally, the parameter-dependent gain $K(\alpha(k))$ is obtained by the change of variables given in (23), what concludes the proof. □

Note that the actual variables are $L_i, H_i, F_i, G_i, P_i = P_i' > 0$, but the BMIs (25)–(27) are written in terms of $\bar{L}_i$, $\bar{H}_i$, $\bar{F}_i$, $\bar{G}_i$, $\bar{P}_i$, $\tilde{H}_i$, $\tilde{G}_i$, and $\tilde{P}_i$.

Corollary 1

The minimum γ attainable by the conditions of Theorem 2 is given by the optimization problem

$$\min \gamma \text{ s.t. } (25), (26), (27) \qquad (29)$$

The use of memory controller brings some advantages when dealing with discrete time-delay systems.

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Using extra variables to store the past values of the control signal, it is possible to cope with Problem 1 without applying more complex Lyapunov functions, (for instance, the Lyapunov–Krasovskii functional). Sophisticated Lyapunov functionals may lead to conditions that require a bigger computational effort to be solved. Whenever possible, the use of memory controller is suggested when dealing with discrete time-delay systems since it simplifies the analysis. Nevertheless, the method could be adapted to cope with other Lyapunov functions, as the Lyapunov–Krasovskii one.

Gain scheduled control of discrete-time systems with time-varying parameters was also addressed by means of affinely parameter-dependent Lyapunov functions in [24, 29] and improved in [13] to cope with systems in which all state-space matrices are supposed to be affected by time-varying parameters. In the above works, the design conditions are given in terms of LMIs. In this paper, however, statement (iv) in Lemma 2 is applied to reach more general BMI conditions with multiplier defined as in (19). The advantages of this approach are due to the extra variables that can be used in the search for better performance of the closed-loop system. For example, lower $\mathscr{H}_\infty$ guaranteed costs may be obtained exploring the new variables $G(\alpha(k+1))$ and $H(\alpha(k+1))$. In this sense, Theorem 1 encompasses the conditions in [29].

The computational time necessary to solve the sufficient BMI conditions presented here can be estimated in terms of the number of scalar variables V and the number of BMIs L. These two parameters can be written as a function of $\tilde{n}$ (number of augmented states) and N (number of vertices) as follows.

$$V = N\left(\frac{\tilde{n}(\tilde{n}+1)}{2} + 2\tilde{n}^2 + \tilde{n}(q+m)\right) + 1$$

$$L = \frac{(M^4+3M^3+2M^2+6M)}{6}, \quad \tilde{n} = n + \tau m$$

When dealing with problems that take into account uncertainties, it is clear that the difficulty in solving the problem increases as the number of uncertain parameters increases. In the framework studied in this paper, this fact can be particularly noted by the number of BMIs in Theorem 2. Considering a system with a large number of uncertainties, the number of vertices used to describe the whole of possible system outcomes will also be large, yielding a large number of inequalities in the conditions of Theorem 2. Naturally, the computation time will also increase since for the LMI/BMI solvers available nowadays the computational time depends on to the number of LMIs/BMIs, on the number of variables of the problem to be solved and, of course, on the computer hardware used.

Although other methods could be applied to solve problem (29), the following algorithm is proposed.

Algorithm 1
Let $G_i = \mathbf{0}$ and $H_i = \mathbf{0}, i = 1, \ldots, N$. Let ε be given. Set $k = 1$ and iterate:

1. Fix the variables H_i and G_i, minimize w.r.t. γ_k and determine F_i, L_i and P_i.
2. Fix the variables F_i and L_i, minimize w.r.t. γ_k and obtain H_i, G_i and P_i.
3. If $|\gamma_k - \gamma_{k-1}| < \varepsilon$, then stop (no significant changes).
4. Set $k = k+1$ and go to step 1.

This approach is sometimes called an Alternating Semi-Definite Programming method [18]. At each step a convex optimization problem in terms of LMI conditions is solved. It is worth stressing that the aim here is not to develop new strategies to solve BMIs. Whenever feasible, other methods from the literature could be applied to solve Corollary 1, as the ones appeared in [18–21]. Concerning the convergence aspect, the proposed algorithm is a heuristic approach and consequently there is no guaranteed convergence to the local optimum. However, since steps 1 and 2 are *convex* optimization problems, the resulting $\mathscr{H}_\infty$ cost is nonincreasing with the iterations.

An important aspect of Algorithm 1 is the choice of the initial values of the variables G_i and H_i. Initializing them as null matrices produces LMI conditions in step 1 of the first iteration similar to the ones presented in [13, 24] in terms of stabilization, since the only extra variables in the LMIs are F_i and, in this case, the extra degree of freedom provided by G_i and H_i cannot be explored. As a remedy, an alternative structure to

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matrices G_i and H_i is proposed:

$$G_i = \varsigma \mathbf{I}, \quad H_i = [h_{rs}]_i, h_{rs} = \varsigma, \quad i = 1, \dots, N \qquad (30)$$

where ς is a real number. In this case, the conditions of Theorem 2 can be tested as LMIs through line searches.

Corollary 2
Given the augmented discrete-time system (3), matrix Q as in (12) and a scalar $\varsigma \in \mathbb{R}$, if there exist matrices $L_i, F_i, P_i = P_i' > 0$, with appropriate dimensions, $i = 1, \dots, N$ and a scalar $\gamma > 0$ such that (25), (26) and (27) hold with G_i and H_i given by (30), then there exists a memory control law (5), ensuring the asymptotic stability of the closed-loop system (6) and an $\mathscr{H}_\infty$ guaranteed cost γ, with $K(\alpha)$ given as in (23) and (24).

Through a line search on ς, the conditions of Corollary 2 can be used to search for stabilizing controllers even when the conditions [13, 24] and the first iteration of Algorithm 1 fail. Moreover, if Corollary 2 provides a feasible solution, the respective ς can be used to initialize G_i and H_i as in (30), assuring that the first iteration of Algorithm 1 will provide a feasible solution, probably with a less conservative $\mathscr{H}_\infty$ guaranteed cost.

By fixing the variable matrices $F_i = F$ and $L_i = L$ (not depending on $\alpha(k)$), $\mathscr{H}_\infty$ robust memory controllers can be obtained using the conditions of Theorem 2, as stated in the next corollary.

Corollary 3
Given the augmented discrete-time system (3), if BMI (25), for $i = 1, \dots, M$, and BMI (26), for $i = 1, \dots, M-1, j = i+1, \dots, M$, of Theorem 2 are feasible with fixed variable matrices L and F then the closed-loop system (6) is asymptotically stable with a robust memory controller $K = L(F')^{-1}$ and an $\mathscr{H}_\infty$ guaranteed cost γ.

Note that BMI (27) is not necessary in this case, since it would produce redundant conditions. The line search strategy could also be applied in this context, similarly to Corollary 2.

It is worth stressing that for time-varying discrete-time systems, robust stabilizability implies gain-scheduling stabilizability, but the converse is not true [30]. In other words, there may exist systems for which Theorem 2 and Corollary 2 provide feasible solutions but Corollary 3 is unfeasible. This fact points out the importance of studying and improving gain-scheduling strategies for control systems, specially in the discrete-time domain.

Finally, the novelties presented here consist especially in the use of the parameter variation modeling in the λ-space within the gain-scheduling framework and in the use of BMIs as a tool in the search of better $\mathscr{H}_\infty$ performance. To the best of the authors' knowledge, the use of Lemma 2 with the particular structure (19) (that results in Theorem 1) has never been seen in the literature in the context of gain-scheduled control. Consequently, Theorem 2, obtained through Theorem 1 and expressions (17), (18) and (22) represent a novel strategy to face the problem of feedback control for discrete time-varying systems. The conditions provide good results when compared with other methods appeared recently in the literature, as shown in the numerical experiments, and represent a flexible strategy in the sense that it can be used in four different contexts, namely, gain scheduling or robust control of time-varying systems with bounded or unbounded rates of variation.

4. NUMERICAL EXPERIMENTS

All the experiments have been performed in a PC equipped with: Athlon 64X2 6000+ (3.0 GHz), 2GB RAM (800 MHz), using Linux (Ubuntu), Matlab (7.0.1) and the SDP solver SeDuMi [11] interfaced by the parser YALMIP [12]. The numerical complexity associated with the proposed conditions and the ones from the literature used for comparison purposes are estimated in terms of the computational times given in seconds. Only the time required to solve the LMIs is considered, since the time necessary to build the set of LMIs is highly dependent on the LMI parser interface. Particularly with respect to the iterative procedure given in Algorithm 1, the time of the ith iteration is the cumulated total time.

Example I
This example is concerned with the fourth-order two-mass-spring system presented in [31] that is reproduced here in Figure 2. The same transfer function is considered.

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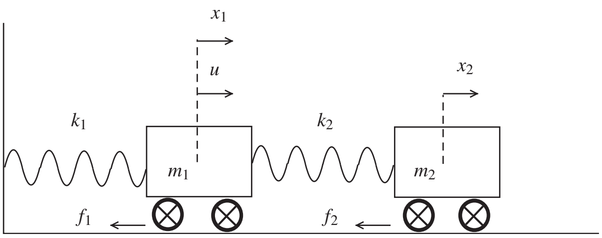

Figure 2. Mass-spring system.

The masses and the stiffness of the second spring are assumed constant as $m_1=2, m_2=1, k_2=0.5$. The friction forces f_1 and f_2 are associated with the viscous friction coefficient c_0. The stiffness of the first spring and the viscous friction coefficient are assumed to be time-varying in the ranges

$$1\leqslant k_1(k)\leqslant 13, \quad 1\leqslant c_0(k)\leqslant 13$$

resulting in a polytope of $N=4$ vertices, obtained by evaluating the following discrete-time equation at the extreme values of the parameters.

$$x(k+1)=\begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ \dfrac{-0.1(k_1+k_2)}{m_1} & \dfrac{0.1k_2}{m_1} & 1-\dfrac{0.1c_0}{m_1} & 0 \\ \dfrac{0.1k_2}{m_2} & \dfrac{-0.1k_2}{m_2} & 0 & 1-\dfrac{0.1c_0}{m_2} \end{bmatrix} x(k)+\begin{bmatrix} 0 \\ 0 \\ \dfrac{0.1}{m_1} \\ 0 \end{bmatrix} u(k) \quad (31)$$

The sampled version (31) of the two-mass-spring system was obtained by using the Euler's first-order approximation for the derivative with a sampling time of 0.1 s. The other system matrices are $C_i=[0\ \ 1\ \ 0\ \ 0]$, $B_{wi}=[0\ \ 0.1\ \ 0.1\ \ 0]'$, $D_{wi}=0.01$, $D_{ui}=0, i=1,\ldots,4$. Additionally, it is also investigated the situation where the model is affected by a one step delayed input, considering $B_{dui}=[0\ 0\ 1\ 0]'$ and $D_{dui}=0, i=1,\ldots,4$. The results obtained by the methods [[24] Theorem 4] (gain-scheduling control), [[24] Theorem 5] (robust control), Theorem 2 and Corollary 3 are shown in Table I for the case of arbitrarily fast variations of the parameters ($b=1$) and for slow variations ($b=0.05$), that is the value of the parameters are constrained to vary only 5% from the instant k to the instant $k+1$.

In the case $b=1$, the conditions of [24] and the ones proposed in this paper produce practically the same $\mathscr{H}_\infty$ guaranteed costs. On the other hand, for $b=0.05$ (slow parameter variation), the method proposed yields significative less conservative results, illustrating that the proposed approach can take advantage when bounds on the parameters variation are considered. In general, this is the case for mechanical systems, as in this example, where the parameters c_0 and k_2 are assumed to vary slowly. The improvements in the $\mathscr{H}_\infty$ guaranteed costs, when considering bounds on the rates of variation, obtained by Theorem 2 and Corollary 3 were 47 and 30%, respectively, for the free delay case. For the delayed input case, the improvements are larger, that is 54 and 40%, respectively, for Theorem 2 and Corollary 3. Concerning the computational complexity, the time demanded by the proposed approach is higher due to the conversion of the parameters to the λ-space domain. In this example, the four parameters in the original polytopic domain yield 28 vertices in the λ-space domain. This is the price to be paid in order to take into account limited rates of variation.

Finally, a time simulation has been performed for the delayed input case with the gain-scheduled controllers obtained through the proposed conditions. The parameters $k_2(k)$ and $c_o(k)$ vary $\approx 4\%$ per instant of time, starting from their minimum values until their maximum. The input noise was generated using the Matlab command $w(k)=0.2*\texttt{randn}$ for $0\leqslant k\leqslant 100$. The noise and the outputs (considering $D_w=0$ and $x_0=0$) of the system, using the synthesized gain-scheduling controllers for $b=1$ and $b=0.05$, are depicted in Figure 3. Clearly, the case $b=0.05$ presents a better disturbance rejection. In fact, the total error $e=\sum_{i=1}^{100}|y(k)|$ is $e=7.91$ and $e=4.29$ for the cases $b=1$ and $b=0.05$, respectively, yielding an improvement of 45%.

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Table I. Results and numerical complexity associated with the methods of [24] and the conditions of Theorem 2 and Corollary 3 for the control design problem in Example I.

Method	[[24], T4]	[[24], T5]	$T2_{it=1}$	$C3_{it=2}$	$T2_{it=1}$	$C3_{it=2}$
b	1	1	1	1	0.05	0.05
γ ($\tau=0$)	0.80	1.50	0.82	1.48	0.43	1.03
Time (s)	0.5	0.4	460.8	69.7	810.5	63.2
γ ($\tau=1$)	1.40	2.52	1.39	2.49	0.63	1.55
Time (s)	0.6	0.4	1415.5	92.7	1351.0	116.3

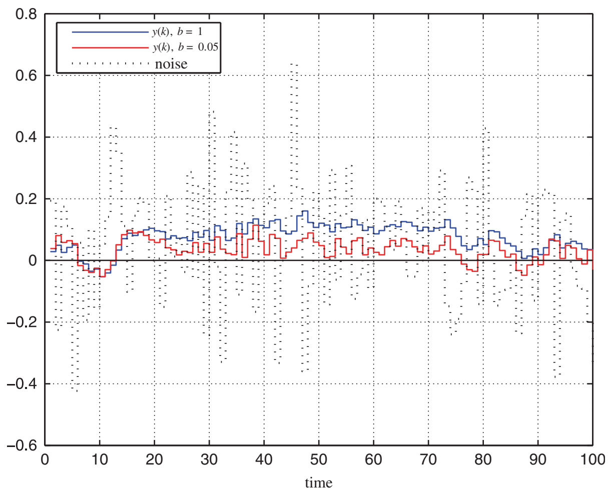

Figure 3. Time simulation of the mass-spring system, with a one-step delay, stabilized through the conditions of Theorem 2 for the cases $b=1$ and $b=0.05$.

Table II. Results and numerical complexity associated with the methods [13] and the conditions of Theorem 2 in the gain-scheduling control design given in Example II.

Method	[13]	$T2_{it=1}$	$T2_{it=2}$	$T2_{it=3}$	$T2_{it=4}$	$T2_{it=5}$	$T2_{it=6}$
γ	20.09	14.39	9.63	8.60	8.27	8.14	8.06
Improvement (%)	—	28.33	52.04	57.15	58.82	59.46	59.87
Time (s)	0.12	1.12	2.07	2.98	3.91	4.80	5.70

Example II

Consider system (3) with vertices (borrowed from [[13] Example II]) given by

$$\tilde{A}_1=\begin{bmatrix}0.28 & -0.315\\ 0.63 & -0.84\end{bmatrix},\quad \tilde{A}_2=\begin{bmatrix}0.52 & 0.77\\ -0.7 & -0.07\end{bmatrix}$$

$$\tilde{B}_{u1}=\begin{bmatrix}1\\ 0\end{bmatrix},\quad \tilde{B}_{u2}=\begin{bmatrix}0\\ 1\end{bmatrix} \tag{32}$$

$\tilde{B}_{wi}=[1\ \ 0]', \tilde{C}_i=[1\ \ 0]$ and $\tilde{D}_{wi}=\tilde{D}_{ui}=0, i=1,2$. This system with arbitrarily fast parameters was also studied in [24], but in a simpler case where matrix $\tilde{B}_u$ was fixed and time-invariant (i.e. $\tilde{B}_{u1}=\tilde{B}_{u2}$). The aim here is to compare the gain-scheduling design conditions from [13] (capable to cope with time-varying $B_u(\alpha(k))$) with the BMI approach proposed in Theorem 2. Additionally, it is considered that the system is affected by one single delay (one step) with

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$B_{dui} = [0\ 1]'$ and $D_{dui} = 0, i = 1, 2$. Table II shows the improvements due to the BMI approach over [13] as the number of iterations evolve. As can be seen in Table II, the $\mathscr{H}_\infty$ upper bound γ was reduced in approximately 59.87% with six iterations, providing better rejection of disturbances.

5. CONCLUSION

The $\mathscr{H}_\infty$ gain-scheduled memory controller for LPV discrete-time systems, with bounded rates of variation, belonging to a polytope has been addressed in this paper. The memory of the controller, used to store the previous values of the control signal, was modeled as a new state-space variable leading to an augmented system representation. A sufficient condition has been proposed in terms of BMIs described only at the vertices of the polytope. Extra variables provided by the Finsler's lemma were used to derive the BMI conditions. The controller design is accomplished by means of an optimization problem that combines convex optimization and line searches. An extension to deal with the design of $\mathscr{H}_\infty$ robust memory controllers has also been given. The conditions provide good results when compared with other methods appeared recently in the literature, as shown in the numerical experiments.

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