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$\mathscr{H}_\infty$ filtering of networked systems with time-varying sampling rates

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Abstract— **In this paper, the problem of robust filter design for networked systems with time-varying sampling rate is investigated. The design conditions are obtained by using the Lyapunov theory and the Finsler's Lemma. A robust filter, that minimizes an upper bound to the $\mathscr{H}_\infty$ performance of the estimation error, is obtained as the solution of an optimization problem. A path-dependent Lyapunov function is used in order to obtain less conservative design conditions. Robust filters based on affine parameter-dependent Lyapunov functions can be obtained as a particular case of the proposed method. Numerical examples illustrate the results.**

I. INTRODUCTION

The use of communication channels in the control of dynamic systems is an important topic which has been much investigated by the control community in the last years. Networked control systems (NCS) have represented a good alternative to implement distributed control and interconnected systems, among others. A better characterization of how real-time networks exchange information between system components is an important step towards a precise description of stability and robustness properties for this class of systems [1, 2]. The study of strategies to deal with packet size constraints, time delays, uncertain sampling rates, and so on, in NCS has received a special attention lately, as can be seen in [3–7].

Over the last decades, the Lyapunov theory has been extensively used for stability analysis, control and filtering of dynamic systems. In many cases, the design conditions can be expressed as a feasibility problem of a set of linear matrix inequalities (LMIs). Concerning the index of performance, there have been two basic criteria, respectively the minimization of $\mathscr{H}_2$ and $\mathscr{H}_\infty$ cost functions. Recent works in the NCS framework include [8] dealing with feedback control of a discrete-time Markovian jump system with random delays, [9] in the context of multipoint-packet system and $\mathscr{H}_2$ optimization, [10] concerned with stabilization of NCSs by means of a packet-loss dependent Lyapunov function, [11] where a Lyapunov-Krasovskii functional is used in the control problem of a time-delay sampled system and [12] that investigates the problem of robust estimation for uncertain polytopic systems subject to limited communication capacity such as measurement quantization, signal transmission delay, and data packet dropout.

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In some particular situations, where either no solution or only sufficient conditions are available in the literature, non-convex approaches have also been applied in the stability analysis of dynamic systems, as for instance bilinear matrix inequality (BMI) tools. More details about optimization problems expressed in terms of BMIs and some specific applications can be seen in the works [13–15] and references therein.

The aim of this paper is to provide robust filters for networked systems subject to time-varying sampling rate. In order to assure the stability of the estimation error dynamic, a path-dependent Lyapunov function [16, 17] is applied. In general, this class of function provides less conservative results when compared with quadratic and affine Lyapunov functions. The robustness of the filter is certified by an $\mathscr{H}_\infty$ guaranteed cost. Using extra variables introduced by the Finsler's Lemma, the design conditions are expressed in terms of BMIs, that can be freely explored in the search for better performance of the networked filtering system. All the sampled system matrices are supposed to be affected by the time-varying parameters, which are modelled inside polytopic domains. The robust filter is then obtained by the solution of an optimization problem that minimizes an upper bound to the $\mathscr{H}_\infty$ index of performance subject to a finite number of BMI constraints formulated only at the vertices of a polytope. Design conditions written in terms of LMIs and based on affine Lyapunov functions can be obtained as a particular case of the main result. Numerical examples illustrate the efficiency of the proposed method.

II. PRELIMINARIES AND PROBLEM STATEMENT

Consider the model described in Figure 1. The stable physical plant is given by the following equations, for $t \geq 0$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t), \ x(0) = 0 \\ z(t) &= C_1x(t) + D_1w(t) \\ y(t) &= C_2x(t) + D_2w(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state space vector, $w(t) \in \mathbb{R}^m$ is the noise input belonging to $L_2[0,\infty)$, $z(t) \in \mathbb{R}^p$ is the signal to be estimated and $y(t) \in \mathbb{R}^q$ is the measured output. All matrices are real, with appropriate dimensions.

System (1) is sampled with a period h, yielding the discrete-time model, for $k \in \mathbb{Z}_+$ and $x(0) = 0$ [18]

$$\begin{aligned} x(kh+h) &= A_s(h)x(kh) + B_s(h)w(kh) \\ z(kh) &= C_{1s}(h)x(kh) + D_{1s}(h)w(kh) \\ y(kh) &= C_{2s}(h)x(kh) + D_{2s}(h)w(kh) \end{aligned} \tag{2}$$

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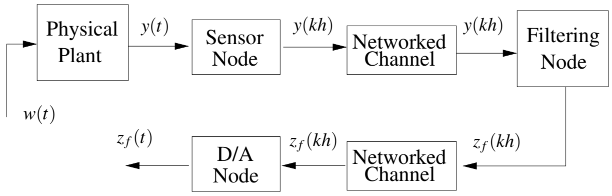

Fig. 1. Networked Filtering Model.

The system matrices $A_s(h)$, $B_s(h)$, $C_{1s}(h)$, $D_{1s}(h)$, $C_{2s}(h)$ and $D_{1s}(h)$ are given by

$$\begin{gathered} A_s(h)=e^{Ah},\ B_s(h)=\int_0^h e^{As}dsB,\ C_{1s}(h)=C_1 \\ D_{1s}(h)=D_1,\ C_{2s}(h)=C_2,\ D_{2s}(h)=D_2 \end{gathered} \tag{3}$$

In order to make model (2) closer to real systems, the sampling period h must be considered a time-varying parameter. As discussed in [5], h may change its value at run-time due to different reasons, as bandwidth allocation and scheduling decisions. Nevertheless, bounds in such variations can be determined, guaranteeing that the actual value of h at each instant k (i.e., h_k) lie inside finite discrete sets as specified below

$$h_k \in \{h_{min},\dots,h_{max}\},\ h_k=\kappa\cdot g,\ \kappa\in\mathbb{N} \tag{4}$$

The parameter g is known as the processor/network clock granularity and is closely related with the number of possible values of (4), [5]. As for instance, the smaller g the bigger the number of elements of (4). As a result, the sampled system becomes an uncertain system with parameters that are time-varying.

A full order proper robust filter is investigated here, being given by

$$\begin{aligned} \dot{x}_f(kh+h)&=A_f x_f(kh)+B_f y(kh),\ x_f(0)=0 \\ z_f(kh)&=C_f x_f(kh)+D_f y(kh) \end{aligned} \tag{5}$$

where $x_f(t)\in\mathbb{R}^n$ is the filter state space vector and $z_f(t)\in\mathbb{R}^p$ the estimated signal. All filter matrices are real, with appropriate dimensions.

In order to represent the set of all possible matrices in system (2) due to the time-varying uncertainties (4), a polytopic model is considered. More specifically, the system matrices, for any time $kh\geq 0$, are described as a convex combination of well-defined vertices, which are given by the arrangements of the extreme values of (4).

The estimation error dynamics is given by

$$\begin{aligned} \varsigma(kh+h)&=\hat{A}(\alpha(kh))\varsigma(kh)+\hat{B}(\alpha(kh))w(kh),\varsigma(0)=0 \\ e(kh)&=\hat{C}(\alpha(kh))\varsigma(kh)+\hat{D}(\alpha(kh))w(kh) \end{aligned} \tag{6}$$

where $\varsigma(kh)=[x(kh)' \quad x_f(kh)']'$, $e(kh)=z(kh)-z_f(kh)$, $\alpha(kh)$ represents the time-varying uncertainties and[1]

$$\hat{A}(\alpha)=\begin{bmatrix} A_s(\alpha) & \mathbf{0} \\ B_f C_{2s}(\alpha) & A_f \end{bmatrix},\ \hat{B}(\alpha)=\begin{bmatrix} B_s(\alpha) \\ B_f D_{2s}(\alpha) \end{bmatrix}$$

[1]The time dependence of α(kh) will be omitted to lighten the notation.

$$\begin{aligned} \hat{C}(\alpha)&=\begin{bmatrix} C_{1s}(\alpha)-D_f C_{2s}(\alpha) & -C_f \end{bmatrix}, \\ \hat{D}(\alpha)&=\begin{bmatrix} D_{1s}(\alpha)-D_f D_{2s}(\alpha) \end{bmatrix} \end{aligned} \tag{7}$$

The whole of possible outcomes for the set (7) belongs to the polytope

$$\mathscr{P}\triangleq\left\{\left[\begin{array}{c|c} \hat{A}(\alpha) & \hat{B}(\alpha) \\ \hline \hat{C}(\alpha) & \hat{D}(\alpha) \end{array}\right]=\sum_{i=1}^{N}\alpha_i\left[\begin{array}{c|c} \hat{A}_i & \hat{B}_i \\ \hline \hat{C}_i & \hat{D}_i \end{array}\right]\right\} \tag{8}$$

with the time-varying vector α lying inside the unit simplex

$$\mathscr{U}=\left\{\alpha\in\mathbb{R}^N:\sum_{i=1}^{N}\alpha_i=1,\ \alpha_i\geq 0\ ,\ i=1,\dots,N\right\}$$

for all $kh\geq 0$.

The filtering problem to be dealt with can be stated as follows.

Problem 1: Find matrices $A_f\in\mathbb{R}^{n\times n}$, $B_f\in\mathbb{R}^{n\times q}$, $C_f\in\mathbb{R}^{p\times n}$ and $D_f\in\mathbb{R}^{p\times q}$ of the filter (5), such that the estimation error system (6) is asymptotically stable, and an upper bound γ to the $\mathscr{H}_\infty$ estimation error performance is minimized, that is, for all $kh\geq 0$

$$\sup_{w(kh)\neq 0}\frac{\|e(kh)\|_2^2}{\|w(kh)\|_2^2}<\gamma^2 \tag{9}$$

with $w(kh)\in l_2[0,\infty)$.

Before proceeding to the solution of Problem 1, a previous result is needed.

Lemma 1: (*Finsler*) Let $\xi\in\mathbb{R}^a$, $\mathscr{Q}=\mathscr{Q}'\in\mathbb{R}^{a\times a}$, $\mathscr{B}\in\mathbb{R}^{b\times a}$ with $rank(\mathscr{B})<a$, and $\mathscr{B}^\perp$ a basis for the null-space of $\mathscr{B}$ (*i.e.* $\mathscr{B}\mathscr{B}^\perp=0$). The following statements are equivalent.

i) $\xi'\mathscr{Q}\xi<0,\ \forall\mathscr{B}\xi=0,\ \xi\neq 0$;

ii) $\mathscr{B}^{\perp\prime}\mathscr{Q}\mathscr{B}^\perp<0$;

iii) $\exists\ \mu\in\mathbb{R}:\mathscr{Q}-\mu\mathscr{B}'\mathscr{B}<0$;

iv) $\exists\ \mathscr{X}\in\mathbb{R}^{a\times b}:\mathscr{Q}+\mathscr{X}\mathscr{B}+\mathscr{B}'\mathscr{X}'<0$.

Proof: See [19]. ■

By applying the Bounded Real Lemma [20], combined with the Finsler's Lemma (1), the condition (9) can be guaranteed as follows.

Lemma 2: For a given γ, if there exists a parameter-dependent matrix $P(\alpha,\alpha_+)'=P(\alpha,\alpha_+)>0$ such that the statements of Lemma 1 are satisfied for

$$\begin{aligned} \mathscr{Q}=&\begin{bmatrix} P(\alpha_+,\alpha_{++}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -P(\alpha,\alpha_+) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ &+\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma^{-1}\hat{B}(\alpha)\hat{B}(\alpha)' & \gamma^{-1}\hat{B}(\alpha)(\alpha)\hat{D}(\alpha)' \\ \mathbf{0} & \gamma^{-1}\hat{D}(\alpha)\hat{B}(\alpha)' & \gamma^{-1}\hat{D}(\alpha)\hat{D}(\alpha)'-\gamma\mathbf{I} \end{bmatrix} \end{aligned}$$

$$\mathscr{B}=\begin{bmatrix} -\mathbf{I} & \hat{A}(\alpha)' & \hat{C}(\alpha)' \end{bmatrix},$$

$$\mathscr{B}^\perp=\begin{bmatrix} \hat{A}(\alpha)' & \hat{C}(\alpha)' \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},\ \xi=\begin{bmatrix} \varsigma(kh+h)' & \varsigma(kh)' & w(kh)' \end{bmatrix}'$$

where $\alpha_+=\alpha(k+1)$ and $\alpha_{++}=\alpha(k+2)$, then the error dynamic (6) is asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.

Proof: Let $v(kh) = \varsigma(kh)'P(\alpha,\alpha_+)\varsigma(kh)$ be a *path-dependent* Lyapunov function given by

$$P(\alpha,\alpha_+) = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_{+_j} P_{ij} \tag{10}$$

Considering the dual system (*i. e.* $\hat{A} = \hat{A}'$, $\hat{B} = \hat{C}'$, $\hat{C} = \hat{B}'$ and $\hat{D} = \hat{D}'$), it is straightforward, in accordance with statement *i)* of Lemma 1, that Lemma 2 ensures $v(kh) > 0$ and

$$\Delta v(kh) < -\gamma^{-1} e(kh)'e(kh) + \gamma w(kh)'w(kh)$$

with the choice $\xi = [\varsigma(kh+h)' \;\; \varsigma(kh)' \;\; w(kh)']'$. Therefore, from the Bounded Real Lemma, system (6) has an upper bound γ to the $\mathscr{H}_\infty$ performance and from the Lyapunov theory [21] is asymptotically stable. ■

The conditions of Lemma 2 appear as nonlinearities that must be tested at all points of the simplex $\mathscr{U}$, *i.e.*, at an infinite number of points. Hence, the main goal hereafter is to obtain finite-dimensional BMI conditions in terms of the vertices of the polytope $\mathscr{P}$ to solve Problem 1. Using Schur complement, change of variables and exploring the extra variables provided by Lemma 1, finite-dimensional BMIs assuring the existence of such filters are given in the next section.

III. Main Results

Theorem 1: ($\mathscr{H}_\infty$ Networked Filtering) Given the sampled system (2), if there exist matrices $Z, Y, R, Q \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times q}$, $J \in \mathbb{R}^{p\times n}$, $\tilde{D}_f \in \mathbb{R}^{p\times q}$, G_{ij}, $M_{ij} = M_{ij}' > 0 \in \mathbb{R}^{2n\times 2n}$, $H_{ij} \in \mathbb{R}^{p\times 2n}$ $i,j = 1,\ldots,N$ and a scalar $\gamma > 0$ such that[2]

$$\Xi_{ijk} \triangleq \begin{bmatrix} \mathscr{F}_{11} & \mathscr{F}_{12} & \hat{F}_{3i} - \hat{F}_1' H_{jk}' & \mathbf{0} \\ (\star) & \mathscr{F}_{22} & G_{jk}\hat{F}_{3i} + \hat{F}_{2i}' H_{jk}' & \hat{F}_{4i} \\ (\star) & (\star) & H_{jk}\hat{F}_{3i} + \hat{F}_{3i}' H_{jk}' - \gamma\mathbf{I} & \mathscr{F}_{34} \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \tag{11}$$

$$i = 1,\ldots,N, \;\; j = 1,\ldots,N, \;\; k = 1,\ldots,N$$
$$\mathscr{F}_{11} = M_{jk} - \hat{F}_1 - \hat{F}_1', \;\; \mathscr{F}_{12} = \hat{F}_{2i} - \hat{F}_1' G_{jk}'$$
$$\mathscr{F}_{22} = G_{jk}\hat{F}_{2i} + \hat{F}_{2i}' G_{jk}' - M_{ij}, \;\; \mathscr{F}_{34} = D_{1si} - \tilde{D}_f D_{2si}$$

$$\hat{F}_1 = \begin{bmatrix} Z & Y' + R' \\ Z & Y' \end{bmatrix}, \;\; \hat{F}_{2i} = \begin{bmatrix} A_{si}'Z & A_{si}'Y' + C_{2si}'L' + Q' \\ A_{si}'Z & A_{si}'Y' + C_{2si}'L' \end{bmatrix}$$
$$\hat{F}_{3i} = \begin{bmatrix} C_{1si}' - C_{2si}'\tilde{D}_f' - J' \\ C_{1si}' - C_{2si}'\tilde{D}_f' \end{bmatrix}, \;\; \hat{F}_{4i} = \begin{bmatrix} Z'B_{si} \\ YB_{si} + LD_{2si} \end{bmatrix}$$

then there exists a proper robust filter in the form of (5), ensuring the asymptotic stability of the estimation error dynamic (6) and an $\mathscr{H}_\infty$ guaranteed cost γ, with matrices given by

$$\begin{matrix} A_f = \hat{V}^{-1}Q(UZ)^{-1}, \;\; B_f = \hat{V}^{-1}L \\ C_f = J(UZ)^{-1}, \;\; D_f = \tilde{D}_f \end{matrix} \tag{12}$$

where $U \in \mathbb{R}^{n\times n}$ and $\hat{V} \in \mathbb{R}^{n\times n}$ are matrices arbitrarily chosen such that $R = \hat{V}UZ$.

[2]The term $(\star)$ indicates symmetric blocks in the LMIs.

Proof: Firstly, applying the following operation

$$\Xi(\alpha) = \sum_{k=1}^{N} \alpha_{++_k} \left\{ \sum_{j=1}^{N} \alpha_{+_j} \left\{ \sum_{i=1}^{N} \alpha_i \Xi_{ijk} \right\} \right\} \tag{13}$$

in the BMI (11) one gets

$$\Xi(\alpha) = \begin{bmatrix} \mathscr{F}_{11}(\alpha) & \mathscr{F}_{12}(\alpha) & \mathscr{F}_{13}(\alpha) & \mathbf{0} \\ (\star) & \mathscr{F}_{22}(\alpha) & \mathscr{F}_{23}(\alpha) & \hat{F}_4(\alpha) \\ (\star) & (\star) & \mathscr{F}_{33}(\alpha) & \mathscr{F}_{34}(\alpha) \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \tag{14}$$

$$\begin{gathered} \mathscr{F}_{11}(\alpha) = M(\alpha_+,\alpha_{++}) - \hat{F}_1 - \hat{F}_1', \\ \mathscr{F}_{12}(\alpha) = \hat{F}_2(\alpha) - \hat{F}_1' G(\alpha_+,\alpha_{++})' \\ \mathscr{F}_{13}(\alpha) = \hat{F}_3(\alpha) - \hat{F}_1' H(\alpha_+,\alpha_{++})' \\ \mathscr{F}_{22}(\alpha) = G(\alpha_+,\alpha_{++})\hat{F}_2(\alpha) \\ +\hat{F}_2(\alpha)' G(\alpha_+,\alpha_{++})' - M(\alpha,\alpha_+), \\ \mathscr{F}_{23}(\alpha) = G(\alpha_+,\alpha_{++})\hat{F}_3(\alpha) + \hat{F}_2(\alpha)' H(\alpha_+,\alpha_{++})' \\ \mathscr{F}_{33}(\alpha) = H(\alpha_+,\alpha_{++})\hat{F}_3(\alpha) + \hat{F}_3(\alpha)' H(\alpha_+,\alpha_{++})' - \gamma\mathbf{I} \\ \mathscr{F}_{34}(\alpha) = D_{1s}(\alpha) - \tilde{D}_f D_{2s}(\alpha) \end{gathered}$$

where

$$\begin{gathered} \hat{F}_2(\alpha) = \begin{bmatrix} A_s(\alpha)'Z & A_s(\alpha)'Y' + C_{2s}(\alpha)'L' + Q' \\ A_s(\alpha)'Z & A_s(\alpha)'Y' + C_{2s}(\alpha)'L' \end{bmatrix} \\ \hat{F}_3(\alpha)' = \begin{bmatrix} C_{1s}(\alpha) - \tilde{D}_f C_{2s}(\alpha) - J & C_{1s}(\alpha) - \tilde{D}_f C_{2s}(\alpha) \end{bmatrix}, \\ \hat{F}_4(\alpha)' = \begin{bmatrix} B_s(\alpha)'Z & B_s(\alpha)'Y' + D_{2s}(\alpha)'L' \end{bmatrix} \end{gathered}$$

Secondly, define the partitioned matrices [22]

$$F = \begin{bmatrix} X' & U' \\ \hat{U}' & \hat{X}' \end{bmatrix}, \;\; F^{-1} = \begin{bmatrix} Y & \hat{V} \\ V & \hat{Y} \end{bmatrix}, \;\; T = \begin{bmatrix} X^{-1} & Y' \\ \mathbf{0} & \hat{V}' \end{bmatrix}$$

together with the following variable transformation

$$\begin{bmatrix} Q & L \\ J & \tilde{D}_f \end{bmatrix} = \begin{bmatrix} \hat{V} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \begin{bmatrix} UZ & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \;\; R = \hat{V}UZ \tag{15}$$

where $Z = X^{-1}$. Then, using the above change of variable, multiply the inequality (14) to the left by S' and to the right by S with

$$S = \begin{bmatrix} \mathscr{S} & \mathbf{0} \\ \star & \mathscr{I} \end{bmatrix}, \;\; \mathscr{S} = \begin{bmatrix} T^{-1} & \mathbf{0} \\ \star & T^{-1} \end{bmatrix}, \;\; \mathscr{I} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \star & \mathbf{I} \end{bmatrix}$$

yielding the following inequality

$$\begin{bmatrix} P(\alpha_+,\alpha_{++}) - F - F' & \mathscr{L}_{12}(\alpha) & \mathscr{L}_{13}(\alpha) & \mathbf{0} \\ (\star) & \mathscr{L}_{22}(\alpha) & \mathscr{L}_{23}(\alpha) & \hat{B}(\alpha) \\ (\star) & (\star) & \mathscr{L}_{33}(\alpha) & \hat{D}(\alpha) \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \tag{16}$$

$$\begin{gathered} \mathscr{L}_{12}(\alpha) = F\hat{A}(\alpha)' - F'TG(\alpha_+,\alpha_{++})'T^{-1} \\ \mathscr{L}_{13}(\alpha) = F\hat{C}(\alpha)' - F'TH(\alpha_+,\alpha_{++})' \\ \mathscr{L}_{22}(\alpha) = (T')^{-1}G(\alpha_+,\alpha_{++})T'F\hat{A}(\alpha)' \\ +\hat{A}(\alpha)F'TG(\alpha_+,\alpha_{++})'T^{-1} - P(\alpha,\alpha_+), \\ \mathscr{L}_{23}(\alpha) = (T')^{-1}G(\alpha_+,\alpha_{++})T'F\hat{C}(\alpha)' \\ +\hat{A}(\alpha)F'TH(\alpha_+,\alpha_{++})' \\ \mathscr{L}_{33}(\alpha) = H(\alpha_+,\alpha_{++})T'F\hat{C}(\alpha)' \\ +\hat{C}(\alpha)F'TH(\alpha_+,\alpha_{++})' - \gamma\mathbf{I} \end{gathered}$$

where $P(\alpha,\alpha_+) = (T')^{-1}M(\alpha,\alpha_+)T^{-1}$. Using Schur complement, inequality (16) can be rewritten as follows

$$\begin{bmatrix} P(\alpha_+,\alpha_{++}) - F - F' & \mathscr{L}_{12}(\alpha) & \mathscr{L}_{13}(\alpha) \\ (\star) & \mathscr{L}_{22}(\alpha) & \mathscr{L}_{23}(\alpha) \\ (\star) & (\star) & \mathscr{L}_{33}(\alpha) \end{bmatrix} + \Upsilon'\gamma^{-1}\Upsilon < 0 \tag{17}$$

where

$$\Upsilon = \begin{bmatrix} \mathbf{0} & \hat{B}(\alpha)' & \hat{D}(\alpha)' \end{bmatrix}.$$

Defining

$$\mathscr{X} = \begin{bmatrix} F' & F'TG(\alpha_+,\alpha_{++})'T^{-1} & F'TH(\alpha_+,\alpha_{++})' \end{bmatrix}' \tag{18}$$

inequality (17) yields statement *iv)* of Lemma 1 with $\mathscr{Q}$, $\mathscr{B}$ and ξ given as in Lemma 2. Lastly, the filter matrices are obtained by the change of variables (15), what concludes the proof. ■

Corollary 1: The minimum γ attainable by the conditions of Theorem 1 is given by the optimization problem

$$\min \gamma \;\; \text{s.t. } (11) \tag{19}$$

From this point, some remarks are in order.

Remark 1: Theorem 1 provides a robust filter in a single step taking into consideration all possible outcomes of h_k in (4), meanwhile minimizes an upper bound to the $\mathscr{H}_\infty$ performance of the estimation error with respect to $w(kh)$. By choosing $G(\alpha_+,\alpha_{++}) = \mathbf{0}$ and $H(\alpha_+,\alpha_{++}) = \mathbf{0}$ the conditions of Theorem 1 reduce to LMIs. As a consequence, Corollary 1 becomes a convex optimization problem that can be handled by Semi-Definite Programming (SDP) algorithms, as for instance SeDuMi [23] and YALMIP [24] within the Matlab environment. In this paper, statement *iv)* in Lemma 1 was applied to reach BMI conditions with multipliers defined as in Lemma 2 and $\mathscr{X} = [F' \;\; F'TG(\alpha_+,\alpha_{++})'T^{-1} \;\; F'TH(\alpha_+,\alpha_{++})']'$. The advantages of this approach is due to the extra variables that can be used in the search for better performance of the closed-loop system. For example, a lower $\mathscr{H}_\infty$ guaranteed cost may be obtained exploring the new variables $G(\alpha_+,\alpha_{++})$ and $H(\alpha_+,\alpha_{++})$.

Remark 2: The conditions of Theorem 1 were obtained by using *path-dependent* Lyapunov functions. As shown in [16, 17], whenever robust stability analysis is at issue, this class of functions yields necessary and sufficient conditions for arbitrarily time-varying discrete-time systems. Furthermore, in the BMI framework presented here a similar strategy was used in order to write the extra variables $G(\cdot)$ and $H(\cdot)$ as a function of two different instant of time, *i.e.*, $G(\alpha,\alpha_+)$ and $H(\alpha,\alpha_+)$. As a consequence, the results of Theorem 1 may be improved in two different ways. Firstly, a larger *path* of the Lyapunov function (10) can be explored in the search for a tighter upper bound γ, or secondly by using the *path-dependent* variables $G(\cdot)$ and $H(\cdot)$ within the BMI framework. The design conditions when the Lyapunov function is given by $v(kh) = \varsigma(kh)'P(\alpha)\varsigma(kh)$ can be obtained as a particular case of Theorem 1 as stated in Corollary 2.

Corollary 2: A sufficient condition for $\mathscr{H}_\infty$ robust filter design in terms of an affine Lyapunov function, *i. e.* $v(kh) = \varsigma(kh)'P(\alpha)\varsigma(kh)$, is obtained by solving Theorem 1 with matrices $G_{jk} = G_j$, $H_{jk} = H_j$, $M_{ij} = M_i$ and $M_{jk} = M_j$. Note that inequality (11) does not depend on the instant k anymore.

Proof: The proof is similar to the proof of Theorem 1, except that operation (13) becomes

$$\Xi(\alpha) = \sum_{j=1}^{N} \alpha_{+j} \left\{ \sum_{i=1}^{N} \alpha_i \Xi_{ij} \right\}. \tag{20}$$

■

Remark 3: Concerning the BMI approach, many methods appeared so far in the literature could be applied in the solution of problem (19). Nevertheless, the following algorithms are suggested. The first one is sometimes called an Alternating Semi-Definite Programming (or Gauss-Seidel) method [13] and consists of fixing some variables and solving for others in such a way that at each step a convex optimization problem is solved. The second one is called path-following method [25] and consists of linearizing the BMIs and then compute an increment that slightly improves the controller performance by solving an SDP problem. Although in both case there is no guarantee of convergence to local minimum, these methods are easy to implement and provide good results in many cases.

Remark 4: By setting the variables $G(\alpha_+,\alpha_{++})$ and $H(\alpha_+,\alpha_{++})$ at time $kh+h$ and $kh+2h$ ($\alpha_+ = \alpha(kh+h)$, $\alpha_{++} = \alpha(kh+2h)$) in (18) all products between parameter-dependent matrices appeared in (17) are done in different instants of time. As a consequence, the number of BMIs and the computational time required to solve the optimization problem (19) are reduced. If Theorem 1 was written with $G(\cdot)$ and $H(\cdot)$ at time kh and $kh+h$ a more sophisticated procedure, as the one proposed in [26], should be applied in order to get the BMI conditions expressed just in terms of the vertices of the polytope, resulting in a larger number of BMIs.

Remark 5: Lastly, the use of time-varying uncertainties in polytopic domains to model uncertain delays brings some advantages to face Problem 1. First of all, it does not require the knowledge of the processor/network clock granularity g. Secondly, the time-varying uncertainties, introduced during the sampling stage, can be completely modeled by a polytope of the form (8). The conditions of Theorem 1 are directly applicable to networked systems whose matrices depend affinely on the vector of time-varying parameters, since this class of systems has a polytopic representation whenever the parameters are bounded [27]. Furthermore, and the most interesting one, the number of values in set (4) does not influence the computational burden, in other words, a larger number of h_k does not require a bigger computational effort, what allows the clock granularity to be as small as possible.

IV. Numerical Experiments

Example 1: This example, borrowed from [28], consists of a simplified model of an armature voltage-controlled DC servo motor, consisting of a stationary field and a rotating armature and load. All effects of the field are neglected. The aim is to design an $\mathscr{H}_\infty$ robust filter to estimate the armature

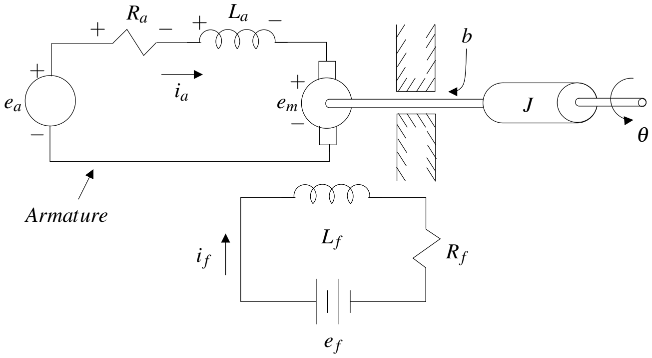

Fig. 2. DC Servo motor as presented in [28].

current given the speed of the shaft. All information is sent through a communication network. The behavior of the DC servo motor shown in Figure 2 can be described by the differential equations

$$\begin{bmatrix}\ddot{\theta}\\ \dot{\rho}_a\end{bmatrix}=\begin{bmatrix}-\frac{b}{J} & \frac{K_T}{J}\\ -\frac{K_\theta}{L_a} & -\frac{R_a}{L_a}\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \rho\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}\omega \qquad (21)$$
$$y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \rho\end{bmatrix}$$

where e_a is the externally applied armature voltage, ρ_a is the armature current, R_a the resistance of the armature winding, L_a the armature winding inductance, e_m the back-emf voltage induced by the rotating armature winding ($e_m = K_\theta\dot{\theta}, K_\theta > 0$), b the viscous damping due to bearing friction, J the moment of inertia of the armature and load and θ the shaft position. Further, the torque generated by the motor is given by $T = K_T i_a$. For $J = 0.01kgm^2/s^2$, $b = 0.1Nms$, $K_T = K_\theta = 0.01Nm/Amp$, $R_a = 1\Omega$ and $L_a = 0.5H$, system (21) can be rewritten in the form (2) with the following system matrices

$$A_s=\begin{bmatrix}e^{-10h_k}-0.0003e^{-2h_k} & 0.125(e^{-2h_k}-e^{-10h_k})\\ 0.002(e^{-10h_k}-e^{-2h_k}) & -0.0003e^{-10h_k}+e^{-2h_k}\end{bmatrix},$$
$$B_s=\begin{bmatrix}0.025e^{-10h_k}-0.125e^{-2h_k}+0.099\\ 0.0000626e^{-10h_k}-0.99e^{-2h_k}+0.99\end{bmatrix}, \qquad (22)$$
$$C_{1s}=\begin{bmatrix}0 & 1\end{bmatrix},\ C_{2s}=\begin{bmatrix}1 & 0\end{bmatrix}\ D_{1s}=0,\ D_{2s}=0.$$

The sampling rate is allowed to vary within the interval $h_k \in [0.001\ 0.099]$. The estimation error is then expressed by polytope (8) with two vertices ($N = 2$), where the parameters α_i are related to h_k. Theorem 1 was solved by using the Alternating Semi-Definite Programming. Each iteration consists of two steps. First the problem is solved with $G(\cdot) = \mathbf{0}$ and $H(\cdot) = \mathbf{0}$ and second, $G(\cdot)$ and $H(\cdot)$ are explored in the search for a better $\mathscr{H}_\infty$ upper bound γ. With one iteration, Theorem 1 provided a robust filter with $\mathscr{H}_\infty$ upper bound $\gamma = 1.8174$ and matrices given by

$$A_f=\begin{bmatrix}-0.3912 & -2.4208\\ 0.2457 & 1.3843\end{bmatrix},\ B_f=\begin{bmatrix}101.7747\\ -13.2249\end{bmatrix},$$
$$C_f=\begin{bmatrix}0.0042 & 0.0227\end{bmatrix},\ D_f=\begin{bmatrix}1.4816\end{bmatrix}.$$

Example II: Consider a sampled system (2) described by a polytope with two vertices given by

$$A_{s1}=\begin{bmatrix}0.2463 & -0.9935 & 0.3908\\ 0.8745 & -0.8092 & 0.7600\\ 0.1245 & -0.1056 & 1.0363\end{bmatrix},\ B_{ws1}=\begin{bmatrix}0\\1\\0\end{bmatrix}$$
$$A_{s2}=\begin{bmatrix}-0.3492 & -0.1346 & 0.5364\\ -0.1717 & 0.4711 & 0.5756\\ 0.5425 & 0.1483 & -0.4035\end{bmatrix},B_{ws2}=\begin{bmatrix}1\\1\\1\end{bmatrix}$$
$$C_{1s1}=C_{1s2}=\begin{bmatrix}1 & 0 & 1\end{bmatrix},C_{2s1}=C_{2s2}=\begin{bmatrix}-1 & 0 & 1\end{bmatrix}$$
$$D_{1s1}=D_{1s2}=\begin{bmatrix}0\end{bmatrix},D_{2s1}=D_{2s2}=\begin{bmatrix}1\end{bmatrix}.$$

In order to illustrate the efficiency of the proposed method, Theorem 1 was contrasted with Corollary 2. With one iteration, the $\mathscr{H}_\infty$ upper bound γ was reduced in approximately 24.68%. This improvement is concerned with the use of a path-dependent Lyapunov function in Theorem 1. It is important to mention that robust filtering methods based on quadratic Lyapunov functions, as the one proposed in [29, Theo. 5], can also be applied in the framework studied here, although it will be in general more conservative. In this example, [29, Theo. 5] was not able to find a feasible solution. The results are summarized in Table I.

TABLE I
RESULTS AND IMPROVEMENT ASSOCIATED TO THE PROPOSED CONDITIONS AND [29, THEO. 5] IN THE ROBUST FILTERING DESIGN GIVEN IN EXAMPLE II.

Method	γ	Improvement	No. iteration
[29, *Theo.*5]	infeasible	–	–
Corollary 2	10.3652	–	1
Theorem 1	7.8068	24.68 %	1

As the number of iterations increases, better bounds on the $\mathscr{H}_\infty$ cost can be obtained. The behavior of γ within 10 iterations is shown in Figure 3.

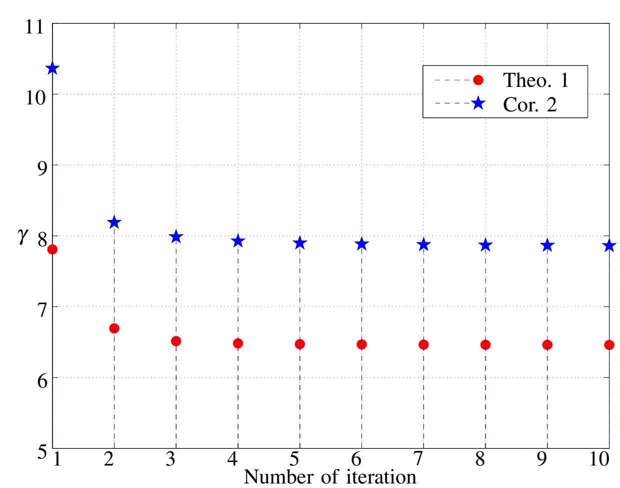

Fig. 3. $\mathscr{H}_\infty$ upper bounds for Example II.

V. Conclusion

The $\mathscr{H}_\infty$ robust filtering for networked systems with time-varying sampling rate modeled as a time-varying parameter belonging to a polytope has been addressed in this paper. A sufficient condition has been stated by using a path-dependent Lyapunov function. Extra variables provided by the Finsler's Lemma were used to derive BMI conditions that may be explored in the search for a better $\mathscr{H}_\infty$ performance. The filter design is accomplished by means of an optimization problem described only at the vertices of the polytope. $\mathscr{H}_\infty$ robust filter design conditions based on affine parameter-dependent Lyapunov functions can be obtained as a particular case of the proposed method. The strategy presented also provides some improvements when compared with other methods from the literature in the context of robust filter design for discrete-time systems with time-varying uncertainties, what increases its reliability when applied in networked systems.

References

[1] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Control Syst. Mag.*, vol. 21, no. 1, pp. 84–99, February 2001.

[2] G. C. Walsh, H. Ye, and L. G. Bushnell, "Stability analysis of networked control systems," *IEEE Trans. Contr. Syst. Technol.*, vol. 10, no. 3, pp. 438–446, May 2002.

[3] D. Nešić and A. R. Teel, "$\mathscr{L}_p$ stability of networked control systems," in *Proc. 42nd IEEE Conf. Decision Contr.*, Maui, HI, USA, December 2003, pp. 1188–1193.

[4] ——, "Input-output stability properties of networked control systems," *IEEE Trans. Automat. Contr.*, vol. 49, no. 10, pp. 1650–1667, October 2004.

[5] M. Velasco, P. Marti, R. Villa, and J. M. Fuertes, "Stability of networked control systems with bounded sampling rates and time delays," in *31st Annual Conference of IEEE Industrial Electronics Society*, Raleigh, NC, USA, November 2005, pp. 2417–2422.

[6] F. Yang, Z. Wang, Y. S. Hung, and M. Gani, "$\mathscr{H}_\infty$ control for networked systems with random communication delays," *IEEE Trans. Automat. Contr.*, vol. 51, no. 3, pp. 511–518, March 2006.

[7] C. H. Chen, C. L. Lin, and T. S. Hwang, "Stability of networked control systems with time-varying delays," *IEEE Comm. Letts.*, vol. 11, no. 3, pp. 270–272, March 2007.

[8] Y. Wang, Z. Q. Sun, and F. C. Sun, "Modeling and control of networked control systems with random delays," in *Proceedings of the 8th International Workshop on Hybrid Systems: Computation and Control*, Zurich, Switzerland, March 2005, pp. 655–666.

[9] D. Georgiev and D. M. Tilbury, "Packet-based control: the $\mathscr{H}_2$-optimal solution," *Automatica*, vol. 42, no. 1, pp. 137–144, January 2006.

[10] J. Xiong and J. Lam, "Stabilization of linear systems over networks with bounded packet loss," *Automatica*, vol. 43, no. 1, pp. 80–87, January 2007.

[11] L. S. Hu, T. Bai, P. Shi, and Z. Wu, "Sampled-data control of networked linear control systems," *Automatica*, vol. 43, no. 5, pp. 903–911, May 2007.

[12] H. Gao and T. Chen, "$\mathscr{H}_\infty$ estimation for uncertain systems with limited communication capacity," *IEEE Trans. Automat. Contr.*, vol. 52, no. 11, pp. 2070–2084, November 2007.

[13] M. Fukuda and M. Kojima, "Branch-and-cut algorithms for the bilinear matrix inequality eigenvalue problem," *Computat. Optim. Appl.*, vol. 19, no. 1, pp. 79–105, 2001.

[14] F. Zheng, Q. G. Wang, and T. H. Lee, "A heuristic approach to solving a class of bilinear matrix inequality problems," *Syst. Contr. Letts.*, vol. 47, no. 2, pp. 111–119, October 2002.

[15] H. D. Tuan and P. Apkarian, "Low nonconvexity-rank bilinear matrix inequalities: algorithms and applications in robust controller and structure designs," *IEEE Trans. Automat. Contr.*, vol. 45, no. 11, pp. 2111–2117, November 2002.

[16] J.-W. Lee and G. E. Dullerud, "Uniform stabilization of discrete-time switched and Markovian jump linear systems," *Automatica*, vol. 42, no. 2, pp. 205–218, February 2006.

[17] J.-W. Lee, "On uniform stabilization of discrete-time linear parameter-varying control systems," *IEEE Trans. Automat. Contr.*, vol. 51, no. 10, pp. 1714–1721, October 2006.

[18] K. J. Åström and B. Wittenmark, *Computer Controlled Systems: Theory and Design*. Englewood Cliffs, NJ: Prentice Hall Inc., 1984.

[19] M. C. de Oliveira and R. E. Skelton, "Stability tests for constrained linear systems," in *Perspectives in Robust Control*, ser. Lecture Notes in Control and Information Science, S. O. Reza Moheimani, Ed. New York: Springer-Verlag, 2001, vol. 268, pp. 241–257.

[20] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM Studies in Applied Mathematics, 1994.

[21] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2002.

[22] M. Chilali and P. Gahinet, "$\mathscr{H}_\infty$ design with pole placement constraints: an LMI approach," *IEEE Trans. Automat. Contr.*, vol. 41, no. 3, pp. 358–367, March 1996.

[23] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optim. Method Softw.*, vol. 11–12, pp. 625–653, 1999, http://sedumi.mcmaster.ca/.

[24] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proc. 2004 IEEE Int. Symp. on Comput. Aided Control Syst. Des.*, Taipei, Taiwan, September 2004, pp. 284–289, http://control.ee.ethz.ch/~joloef/yalmip.php.

[25] A. Hassibi, J. How, and S. Boyd, "A path-following method for solving BMI problems in control," in *Proc. 1999 Amer. Control Conf.*, San Diego, CA, USA, June 1999, pp. 1385–1389.

[26] D. C. W. Ramos and P. L. D. Peres, "An LMI condition for the robust stability of uncertain continuous-time linear systems," *IEEE Trans. Automat. Contr.*, vol. 47, no. 4, pp. 675–678, April 2002.

[27] W. Xie, "$\mathscr{H}_2$ gain scheduled state feedback for LPV system with new LMI formulation," *IEE Proc. — Contr. Theory and Appl.*, vol. 152, no. 6, pp. 693–697, November 2005.

[28] P. J. Antsaklis and A. N. Michel, *Linear Systems*. Boston: Birkhäuser, 2006.

[29] J. C. Geromel, J. Bernussou, G. Garcia, and M. C. de Oliveira, "$\mathscr{H}_2$ and $\mathscr{H}_\infty$ robust filtering for discrete-time linear systems," *SIAM J. Control Optim.*, vol. 38, no. 5, pp. 1353–1368, May 2000.