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A Randomized approach to the H_2/H_∞ Control Problem via Q-parameterization

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Abstract

In this paper we show that the mixed H_2/H_∞ control problem can be efficiently solved using randomized algorithms. Q-parameterization provides a mechanism to search over all stabilizing controllers, and thus gives us the ability to search for H_2 minimizing controllers, while still providing stability robustness. Finally, we are able to show that we can get results comparable to a more traditional approach such as gradient search, but in addition, we can solve more complex problems. With very little modification, we are able to deal with multiple objectives, plant uncertainty, and fixed order controllers.

1 Introduction

The need for practical controllers to simultaneously satisfy multiple performance criteria has led to the investigation of multi-objective optimal control, combining two or more standard performance metrics into a single optimal synthesis problem. Examples of multi-objective optimal control problems include the H_2/H_∞ [1, 2, 3, 4], H_2/μ, and H_2/L_1 problems [5]. A common feature of such mixed-norm problems is the fact that the resulting optimization problem is nonconvex, so that the usual Riccati-equation-based solution techniques optimal control problem no longer apply. As a result, the control engineer is left with nonlinear programming techniques such as gradient search [6], homotopy or continuation [7], or bilinear matrix inequalities [8].

Recently, however, it has been shown that many such nonconvex optimal control problems are in fact computationally intractable [9], and that the time required for any algorithmic solution scales in a non-polynomial fashion with the "size" or the problem. It was then proposed that randomized search techniques [10, 11] offer a practical alternative to designers faced with such nonconvex problems. By posing the question in a "soft" manner, i.e. by answering the problem in a probabilistic sense, randomized algorithms have been shown to yield solutions in a time that scales in a polynomial fashion with the problem size. In this approach, a search range for each optimization parameter is selected, and a weighting function for the combination of the multiple objectives is defined. A controller is then selected randomly according to a user-specified probability distribution (which turns out to be non-critical), and the multiobjective performance function is evaluated. The controller that minimizes the multiobjective performance is then declared the "solution" of the optimization process. It is important to note that the questions answered by randomized search techniques are relaxed versions of the more traditional approach, as certain components of the multiobjective criterion that are traditionally considered "hard" constraints (i.e. closed-loop stability) may or may not be satis-

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fied by the "solution". Additionally, the "solution" cannot be claimed to be optimal, but rather that with high confidence, the probability of finding a better controller in the defined search space is less than a user-specified value.

This paper investigates the performance of randomized algortihms on problems with "hard" (e.g. internal stability) constraints, and the importance of the choice of parameter ranges in such cases. To illustrate these points, a randomized search algorithm is applied to the mixed-$\mathcal{H}_2/\mathcal{H}_\infty$ optimal-control problem, with specific emphasis on the importance of the search parameter range specification. This is similar to solutions obtained via numerical techniques such as gradient search over the set of Youla-Kučera (or Q) parameterization of controllers. However, randomized algorithms can easily be extended to handle multiple performance objectives, and plant parameter uncertainty, while traditional methods can not.

The remainder of this paper is divided as follows: Section 2 contains a brief introduction to randomized search techniques, and presents an algorithm suitable for solving the mixed-norm $\mathcal{H}_2/\mathcal{H}_\infty$ optimal control problem. Section 3 shows the difficulty of finding a search parameter range, and justifies the use of Q parameterization. We then briefly explain the Q parameterization as used in this paper. Section 4 presents a numerical example drawn from the literature [5], and provides a comparison of the results of the randomized search algorithm with previous work. Finally section 5 contains our conclusions and suggestions for future research.

2 A Randomized Search Algorithm for Optimization

As noted in the introduction, randomized search algorithms have been proposed as a practical tool for solving computationally difficult controller synthesis problems [12, 10, 13, 14]. The algorithms offer several advantages over traditional numerical algorithms for the synthesis of nonconvex optimal controllers (e.g. homotopy, gradient-search, bilinear LMI's), such as: polynomial scaling of solution time with problem dimension, applicability to nonsmooth/discontinuous objective functions, applicability to nonsmooth/discontinuous controller parameterizations, computational requirements known ***a priori***, Ccmputational burden comprised of independent function evaluations (and thus may be performed completely in parallel), ease of incorporation of (direct) constraints on controller parameter values, ability to solve fixed order controller design problems, ability to solve fixed structure controller design problems, can easily deal with plant uncertainty, and can be made to minimize a function of multiple competing constraints.

This is not to imply that randomized search algorithms are not without their own problems. In particular, the performance on problems with "hard" constraints and the importance of the choice of parameter ranges to the overall performance are issues that have not been properly emphasized in the literature. These are questions that we seek to address.

The performance of a randomized search algorithm can be characterized by the type of solution produced by the algorrithm. This section, which closely follows [12], provides a summary of the characteristics of the algorithm used in later developments. Let Y be a given set, $f : Y \longrightarrow \mathbb{R}$ be a measurable function, and define $f^* \triangleq \inf_{y \in Y} f(y)$ where $\mathbb{R}$ is the set of real numbers, and $\inf = infinum$. The following definition provides a precise statement of the characteristics of the output of a randomized search algorithm.

Definition 1 *Suppose $f : Y \to \mathbb{R}$, W is a given probability measure on Y, and that $\alpha > 0$ is a given number. A number $f_0 \in \mathbb{R}$ is said to be a* **Type 2 near minimum of** $f(\cdot)$ **to level** α, *if $f_0 \geq f^*$, and, in addition, $W\{y \in Y : f(y) < f_0\} \leq \alpha$.*

As noted in [12], this definition does not provide for any bound on $|f_0 - f^*|$.

An efficient randomized algorithm was proposed in [12] for the computation of this type of near minimum, and is restated here for completeness.

Algorithm 1 *Given: A probability W on $\mathcal{Y}$, A measurable function $f : Y \longrightarrow \mathbb{R}$, A level parameter $\alpha \in (0, 1)$, and A confidence parameter $\delta \in (0, 1)$.*

Then, choose an integer m such that

$$m \geq \frac{\log(2/\delta)}{\log[1/(1-\alpha)]}, \tag{1}$$

and generate independent identically distributed (i.i.d.) samples $y_1, y_2, \cdots, y_m \in Y$ distributed ac-

cording to W. Define,

$$\bar{f} \triangleq \min_{1 \leq i \leq m} f(y_i).$$

Then with confidence at least $1 - \delta$, $\bar{f}$ is a probably approximate near minimum of $f(y)$ to level α; independent of W.

3 Finding the Controller

A general observation of the eigenvalue locations for the controller (when found using Riccati-type equations), is that they tend to lay in the same spectral region as those of the plant. Therefore we decided to randomly sample poles and zeros inside the semicircle of the left half plane, with radius equal to the largest magnitude of the open-loop plant G. Not too surprisingly, not one stabilizing controller was found using this method. A few trial and error values for different radius values, also did not yield a single stabilizing controller. Therefore, it it obvious that a more intelligent approach is needed.

It is well known that the set of all stabilizing controllers can be parameterized using Q parameterization (see for example [15]). Furthermore, if the conditions of the follwoing theorem are met, it is guaranteed that Q parameterizes all robustly stabilizing controllers. In the following, we use the notation on pages 288-290 of [15]

Theorem: *Suppose a LTI plant G satisfies the assumptions in* [15],

1. *There exists an admissible controller $K(s)$ such that $\|\mathcal{F}_\ell(G,K)\|_\infty < \gamma$. (i.e., $\|T_{zw}\|_\infty < \gamma$) if and only if*
   - (a) $\gamma > max(\bar{\sigma}[D_{1111}, D_{1112}], \bar{\sigma}[D^*_{1111}, D^*_{1121}])$;
   - (b) *$H_\infty \in dom(Ric)$ with $X_\infty = Ric(H_\infty) \geq 0$;*
   - (c) *$J_\infty \in dom(Ric)$ with $Y_\infty = Ric(J_\infty) \geq 0$;*
   - (d) $\rho(X_\infty Y_\infty) \leq \gamma^2$.
2. *Given that the conditions of* 1. *above are satisfied, then all rational internally stabilizing controllers $K(s)$ satisfying $\|\mathcal{F}_\ell(G,K)\|_\infty \leq \gamma$ are given by*

$$K = \mathcal{F}_\ell(M_\infty, Q) \text{ for arbitrary } Q \in \mathcal{RH}_\infty \text{ such that } \|Q\|_\infty \leq \gamma.$$

4 Numerical Example

As an illustrative example of the utility of randomized search algorithms in solving nonconvex optimal control problems, we select a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ synthesis problem detailed in [5]. The problem represents the (simplified) SISO control of the longitudinal dynamics of an F-16 fighter, with the $\mathcal{H}_2$ portion of the problem providing disturbance rejection and control energy minimization, while the $\mathcal{H}_\infty$ constraint incorporates tracking performance and vector gain and phase margins. The mixed-objective plant is described by a three input-three output system, five-state linear time-invariant system, denoted

$$\begin{bmatrix} e \\ z \\ y \end{bmatrix} = P(s) \begin{bmatrix} d \\ w \\ u \end{bmatrix}$$

where y denotes the measurement signals available for feedback, u represents the actuator command signals, z represents the ($\mathcal{H}_2$) performance signals of interest, w represents the ($\mathcal{H}_2$) exogenous disturbances, and d and e represent the ($\mathcal{H}_\infty$) disturbance and performance, respectively. A realization for $P(s)$ is given in [5] inputs or outputs,

We note that this nonconvex synthesis problem implicitly involves two "hard" constraints: closed-loop internal stability and the satisfaction of a closed-loop $\mathcal{H}_\infty$ constraint. Using traditional optimization algorithms such as gradient-search techniques to solve such problems, the user may start within a feasible set of controllers and constrain any search or continuation to remain within the feasible set. Alternatively, the user could attempt to parameterize the solution in such a manner that the constraints are automatically satisfied for any set of parameter values specified. This latter approach, while obviously preferable, is not always feasible, and therefore the former approach is generally the de-facto standard [5, 16].

Randomized search techniques provide an alternative avenue for attacking such problems, in that the constraints are assigned a weighting function and incorporated into the multiobjective cost function. The tendency of the algorithm to provide solutions that do or do not satisfy the constraints will then be proportional to 1) the weighting assigned to constraints relative to other costs in the multiobjective function, and 2) the relative size of the set of parameter values that satisfy the constraints, to the size of the overall search volume. The first point implies that the algorithm can be directed to provide solutions that satisfy the constraints *at the expense of performing optimization over other factors in the multiobjective cost*, while 2) indicates that if the set of parameter values satisfying the constraint is relatively small compared with the overall search range, then Algorithm 1 may require a pathologically small value of α in order to provide at least one sample value within the set.

As noted above, the elimination of constraints from nonconvex optimal synthesis problems is, in general, not feasible. But for the case of the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, one may utilize the Youla-Kučera or Q-parameterization to provide a set of controllers where both internal stability and closed-loop $\mathcal{H}_\infty$-norm constraints are satisfied for any parameter values in a specified range. The randomized search algorithms may now be applied utilizing the space of stable transfer functions (Q-space) as the search volume, with the guarantee that all sample values satisfy the "hard" constraints; all that remains is to evaluate the remaining objective criteria (i.e. the $\mathcal{H}_2$ cost) to determine the probabilistic minimizer.

Following standard techniques [15], the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimal controllers for the above example were computed via standard Riccati solution techniques, and the respective values of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ costs achieved by each are provided in Table 1. The multiobjective problem is then to determine

$$f(Q) = \arg \min_{K(s,Q)\in\mathcal{S}} ||G_{zw}(s)||_2 \quad (2)$$
$$\text{subject to } ||G_{ed}(s)||_\infty < \gamma,$$

where $K(s)$ represents the controller, $\mathcal{S}$ is the set of all internally stabilizing controllers, and γ represents the desired level of $\mathcal{H}_\infty$ performance. As noted in [5], the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem yields trivial solutions in the case of either a requested $\mathcal{H}_\infty$ constraint lower than that achieved by the $\mathcal{H}_\infty$-optimal controller (in which case there is no solution) or a requested $\mathcal{H}_\infty$ constraint higher than that provided by the $\mathcal{H}_2$-optimal controller (in which case the $\mathcal{H}_2$-optimal controller is also the mixed $\mathcal{H}_2/\mathcal{H}_\infty$-optimal controller). To avoid this degenerate situation, a value of $\gamma = 1.3$ was chosen as the desired level of $\mathcal{H}_\infty$ performance.

Using this value of γ, a controller providing this level of $\mathcal{H}_\infty$ performance was synthesized using the standard Riccati solution techniques, and this controller was utilized to provide a basis for a Q-parameter representation of all internally stabilizing controllers satisfying $||G_{ed}(s)||_\infty < \gamma$. In order to avoid feedthrough terms in the resulting controller $K(s)$ (and therefore a singular $\mathcal{H}_2$ cost), it is necessary that Q-parameters used for the randomized search algorithm be strictly proper. The specific parameterization utilized in this paper is given by

Controller	$\mathcal{H}_2$ Cost	$\mathcal{H}_\infty$ Cost
Optimal H_2 Controller	0.3116	178778
Optimal H^∞ Controller	165.55	1.284
Random Search (6^{th} order)	3.288	1.284
[5] (4^{th}-order)	0.4088	1.490
[5] (8^{th}-order)	0.4088	1.281

Table 1: Summary of Results

$$Q(s) = \frac{a}{s+b}, \quad (3)$$

subject to

$$||Q(s)||_\infty < \gamma \quad (4)$$

where a,b are uniform in $(0, 1]$. Note that a $K(s)$ resulting from (3) will be 6^{th}-order, in general.

With the parameterization given above, it remains to determine the confidence parameter δ, the level parameter α, and the probability distribution W in order to apply Algorithm 1. For the results of this paper, a uniform distribution W was chosen, with confidence parameter $\delta = 0.001$ (yielding confidence 0.999), and level parameter $\alpha = 0.001$. Thus, the required number of sample parameter values in (1) is 7597. The central $\mathcal{H}_\infty$ controller ($Q = 0$) used as the basis for the Q-parameterization yields an $\mathcal{H}_2$ cost of 165.6 with an $\mathcal{H}_\infty$ cost of 1.28. As shown in Table 1, the randomized search algorithm utilizing 7597 samples yielded a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ optimal controller with an $\mathcal{H}_2$ cost of 3.29 an $\mathcal{H}_\infty$ cost of 1.28.

5 Conclusions

This paper has shown that by use of a suitable parameterization, the problem of synthesizing controllers that minimize a closed-loop $\mathcal{H}_2$-norm subject to satisfying a closed-loop $\mathcal{H}_\infty$-norm constraint can be solved using randomized search algorithms. A numerical example drawn from the literature was presented, and the results of the approach compared with those from previous research utilizing more traditional gradient-search techniques. The results obtained compare favorably with those from the literature; however, randomized algorithms have a decidable advantage over other methods in the presence of added constraints. More specifically, the objective function can find a minimum of multiobjective constraints (more than two), and we can include plant uncertainty. Currently, there are no other methods for dealing with either case.

Ongoing research efforts in this area include the investigation of controller parameterizations for non-traditional constraints such as saturation limitations, linear and nonlinear plants with parametric uncertainty, settling time requirements, and the use of randomized search techniques for fixed-structure multiobjective (i.e. mixed $\mathcal{H}_2/\mathcal{H}_\infty$) problems.

References

[1] D. S. Bernstein and W. M. Haddad. LQG control with an $\mathcal{H}_\infty$ performance bound: A Riccati equation approach. *IEEE Trans. Autom. Contr.*, 34:293–305, 1989.

[2] W. M. Haddad and D. S. Bernstein. Generalized Riccati equations for the full- and reduced-order mixed-norm $\mathcal{H}_2/\mathcal{H}_\infty$ standard problem. *Sys. Contr. Lett.*, 14:185–197, 1990.

[3] P. P. Khargonekar and M. A. Rotea. Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control: A convex optimization approach. *IEEE Trans. Autom. Contr.*, 36:824–837, 1991.

[4] M. Sznaier. An exact solution to SISO mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems via convex optimization. *IEEE Trans. Autom. Contr.*, 39:2511–2517, 1994.

[5] D. E. Walker. *$\mathcal{H}_2$-Optimal Control with $\mathcal{H}_\infty$, μ, and $\mathcal{L}_1$ Constraints.* PhD thesis, Air Force Institute of Technology, 1994.

[6] P. M. Mäkilä and H. T. Toivonen. Computational methods for parametric LQ problems—a survey. *IEEE Trans. Autom. Contr*, 32:658–671, 1987.

[7] S. L. Richter and R. A. DeCarlo. Continuation methods: Theory and applications. *IEEE Trans. Cir. Sys.*, 30(6):347–352, 1983.

[8] M. G. Safonov, K. C. Goh, and J. H. Ly. Control system synthesis via bilinear matrix inequalities. In *Proc. Amer. Contr. Conf.*, pages 45–49, Baltimore, MD, June 1994.

[9] V. Blondel and J. N. Tsitsiklis. NP-hardness of some linear control design problems. *SIAM J. Contr. Opt.*, 35(6):2118–2127, Nov. 1997.

[10] M. Vidyasagar. Statistical learning theory and randomized algorithms for control. *Control Systems Magazine*, 18(6):69–85, 1998.

[11] M. Vidasagar. *A Theory of Learning and Generalization.* Springer-Verlag, Berlin–Hiedelberg New York, 1st edition, 1997.

[12] M. Vidyasagar. Statistical learning theory and its applications to randomized algorithms for robust controller synthesis. In *European Control Conference, Plenary Lectures and Mini-Courses*, pages 162–190, Brussels, Belgium, 1997.

[13] V. Koltchinskii, C.T. Abdallah, M. Ariola, P. Dorato, and D. Panchenko. Statistical learning control of uncertain systems: It is better than it seems. Accepted, *IEEE Transactions on Automatic Control*, 1999

[14] V. Koltchinskii, M. Ariola, and C.T. Abdallah. Statistical controller design for the linear benchmark problem. In *Proceedings IEEE Conference on Decision and Control*, Phoenix, AZ, pp. 507–509, 1999.

[15] K. Zhou and J.C. Doyle. *Essentials of Robust Control.* Prentice Hall, Upper Saddle River, New Jersey 07458, 1st edition, 1998.

[16] R. S. Erwin and D. S. Bernstein. Fixed-structure discrete-time $\mathcal{H}_2$-optimal controller synthesis using the delta-operator. In *Proc. Amer. Contr. Conf.*, pages 3185–3189, Albuquerque, NM, June 1997.