Linear-Quadratic Simultaneous Performance Design

Robert A. Luke, Peter Dorato, and Chaouki T. Abdallah
Department of Electrical and Computer Engineering
University of New Mexico
Albuquerque, NM 87131-1356 USA
luke@eece.unm.edu
http://www.eece.unm.edu/students/luke/

Abstract

In this paper the problem of designing a fixed state feedback control law which minimizes an upper bound on linear-quadratic performance measures for m distinct plants is reduced to a convex programming problem.

Keywords: simultaneous stabilization, simultaneous control, semidefinite programming, state feedback control, linear matrix inequalities, numerical optimization

1. Introduction

The problem considered here is the design of a fixed state feedback control law $u(t) = -Kx(t)$ which minimizes an upper bound on the performance measures

$$\mathcal{J}_j \triangleq \mathbb{E}\left\{\int_0^\infty \left[x_j^T(t)Q_j x_j(t) \right.\right.$$
$$\left.\left. +u_j^T(t)R_j u_j(t)\right] dt\right\} , \quad (1)$$

each associated with one of the plants described by state space equations

$$\dot{x}_j(t) = A_j x_j(t) + B_j u_j(t) \qquad (2)$$

for all $j \in I_m \triangleq \{1,\ldots,m\}$. In (1) the expectation operator $\mathbb{E}(\cdot)$ is taken over random initial conditions satisfying $\mathbb{E}\{x(0)\} = 0$ and $\mathbb{E}\{x(0)x^T(0)\} = I$. This is referred to here as *simultaneous performance design*. Standard linear-quadratic assumptions are made for each system, namely, all members of collection $\{Q_j\}_{j\in I_m}$ are positive semidefinite and all members of $\{R_j\}_{j\in I_m}$ are strictly positive definite. See a similar treatment in Balakrishnan and Vandenberghe [1]. Further, all systems are (A_j, B_j)-controllable and $(A_j, Q_j^{1/2})$-observable.

It is well known (see, for example, Dorato, et al. [5]) that the performance measures in (1) are given by

$$\mathcal{J}_j = \text{tr}\left\{P_j\right\}$$

where each P_j satisfies, for a given K, the Lyapunov equation

$$P_j\left(A_j - B_j K\right) + \left(A_j - B_j K\right)^T P_j$$
$$+ Q_j + K^T R_j K = 0 . \quad (3)$$

In Paskota, et al. [11], nonlinear programming techniques are used to minimize the combined performance measure

$$S = \sum_{j=1}^m \text{tr}\left\{P_j\right\} \qquad (4)$$

for single-input, single-output systems. Because this optimization problem is not convex, only a local minimum is assured. Thus even though a performance index of the form in (4), that is, a positive linear combination of performance measures $\text{tr}\{P_j\}$, is often useful in generating Pareto optimal [10] solutions, a local minimum guarantees only a locally Pareto optimal solution. See, for example, Vincent and Grantham [14]. Little can be said about bounds on each term $\text{tr}\{P_j\}$.

In this paper a guaranteed-cost approach (see Chang and Peng [4]) is used and a minimization of a bound on all the performance measures, $\text{tr}\{P\}$ is sought, where

$$\text{tr}\left\{P_j\right\} \leqslant \text{tr}\left\{P\right\} \qquad (5)$$

for all system indices j in set I_m. It is well known (see for example, Boyd, et al. [3] or Dorato, et al. [5]) that with a change in matrix variables introduced in Bernussou, et al. [2], this problem can be reduced to one of convex programming with linear matrix inequality ("LMI") constraints, which can be solved numerically with commercially available software. For example, an LMI Control Toolbox (Gahinet, et al. [6]) is available for use with MATLAB. This approach has been suggested in both References [3] and [5] when $Q_j = Q$ and $R_j = R$ for all system indices contained by set I_m. Here, performance function weighting matrices $\{Q_j\}_{j\in I_m}$ and $\{R_j\}_{j\in I_m}$ vary with each system and the details of reducing the simultaneous performance design problem is carried to a linear matrix inequality convex programming problem. Finally, the optimal guaranteed cost solution is compared to the results of the numerical example given in [11].

2. Reduction of the Guaranteed-Cost Problem to a Convex Programming Problem

As suggested in References [3] and [5], each instance of distinct Lyapunov matrices P_j in matrix equation (3) is replaced by a *single* matrix P. Consider the associated set of m Lyapunov matrix *inequalities*

$$P(A_j - B_j K) + (A_j - B_j K)^T P + Q_j + K^T R_j K < 0 . \quad (6)$$

A positive definite matrix solution P which satisfies each of the m matrix inequalities, for a fixed gain K, is a guaranteed upper bound on all performance measures in $\{\mathcal{J}_j\}_{j\in I_m}$ as indicated in expression (5). With the usual Bernussou rational matrix description change of variables

$$P = Y^{-1} , \qquad K = X Y^{-1} ,$$

the matrix inequalities in (6) become

$$A_j Y - B_j X - X^T B_j^T + Y A_j^T + Y Q_j Y + X^T R_j X < 0 \quad (7)$$

for all system indices j in set I_m. The basic "LMI Lemma" (see, for example, Reference [3]) allows the conversion of these *quadratic* (in matrix variables X and Y) inequalities into equivalent *linear* matrix inequalities

$$\left[\begin{array}{c|c} \begin{array}{c} (-A_j Y + B_j X \\ + X^T B_j - Y A_j^T) \end{array} & \left[\begin{array}{c} Q_j^{1/2} Y \\ R_j^{1/2} X \end{array} \right]^T \\ \hline \left[\begin{array}{c} Q_j^{1/2} Y \\ R_j^{1/2} X \end{array} \right] & I \end{array} \right] > 0 \quad (8)$$

where all members of $\{Q_j^{1/2}\}_{j\in I_m}$ and $\{R_j^{1/2}\}_{j\in I_m}$ are symmetric factorizations of weighting matrices in $\{Q_j\}_{j\in I_m}$ and $\{R_j\}_{j\in I_m}$, respectively, satisfying

$$Q_j^{1/2} Q_j^{1/2} = Q_j , \qquad R_j^{1/2} R_j^{1/2} = R_j$$

for all system indices j in I_m.

Note that if all members of $\{Q_j\}_{j\in I_m}$ are also strictly positive definite (a more restrictive condition) then the linear matrix inequality (8) may also be written

$$\left[\begin{array}{c|c|c} \begin{array}{c} (-Y A_j^T - A_j Y \\ + B_j X + X^T B_j^T) \end{array} & Y & X^T \\ \hline Y & Q_j^{-1} & 0 \\ \hline X & 0 & R_j^{-1} \end{array} \right] > 0 . \quad (9)$$

The design objective becomes

$$\min_{K,P} \text{tr}\{P\} = \min_{X,Y} \text{tr}\{Y^{-1}\} . \quad (10)$$

This is a convex programming problem since $\text{tr}\{Y^{-1}\}$ is convex in matrix variable $Y > 0$; and the Lyapunov linear matrix inequality constraints (8) or (9) define convex regions for matrix variables X and Y. However, most of the available software deals only with linear objective functions. In fact, Nesterov and Nemirovskii [9, p. 7] state that " ... to solve a convex problem by an interior point method [as found in the MATLAB LMI Toolbox], we should first reduce the problem to one of minimizing a *linear* [emphasis added] objective over [a] convex domain (which is quite straightforward)." To deal with this limitation the linear matrix inequality

$$\left[\begin{array}{cc} Z & I \\ I & Y \end{array} \right] > 0 \quad (11)$$

is added to the other linear matrix inequality constraints (8) and the objective function

$$\min_{X,Y,Z} \mathcal{J} = \text{tr}\{Z\} .$$

is used instead of the right hand side of (10). Note that the linear matrix inequality (11) implies $Z > Y^{-1}$ via the LMI Lemma, so that effectively a further upper bound is minimized.

3. Example

An example found in Paskota, et al. [11] and a number of other relevant references (Petersen [12]; Wu, et al. [15]; and Howitt and Luus [8]) is now used to demonstrate the usage of the convex problem. A static state feedback gain matrix K is to be found which simultaneously stabilizes four different operating points of an airplane trajectory in the vertical plane and minimizes the upper bound on all linear-quadratic performance objectives (1). The four operating points are given by a set of four state differential equations (2) assuming a scalar input u. The state coefficient matrices are given as

$$A_1 = \left[\begin{array}{ccc} -0.9896 & 17.41 & 96.15 \\ 0.2648 & -0.8512 & -11.89 \\ 0 & 0 & -30 \end{array} \right] \quad (12)$$

$$A_2 = \left[\begin{array}{ccc} -0.6607 & 18.11 & 84.34 \\ 0.08201 & -0.6587 & -10.81 \\ 0 & 0 & -30 \end{array} \right] \quad (13)$$

$$A_3 = \left[\begin{array}{ccc} -1.702 & 50.72 & 263.5 \\ 0.2201 & -1.418 & -31.99 \\ 0 & 0 & -30 \end{array} \right] \quad (14)$$

$$A_4 = \left[\begin{array}{ccc} -0.5162 & 26.96 & 178.9 \\ -0.6896 & -1.225 & -30.38 \\ 0 & 0 & -30 \end{array} \right] . \quad (15)$$

The control coefficient vectors b_j are given as

$$b_1 = \begin{bmatrix} -97.78 \\ 0 \\ 30 \end{bmatrix} , \quad b_2 = \begin{bmatrix} -272.2 \\ 0 \\ 30 \end{bmatrix} ,$$

$$b_3 = \begin{bmatrix} -85.09 \\ 0 \\ 30 \end{bmatrix} , \quad b_4 = \begin{bmatrix} -175.6 \\ 0 \\ 30 \end{bmatrix} .$$

In [11] all members of $\{Q_j\}_{j \in I_m}$ and $\{R_j\}_{j \in I_m}$ are set equal to appropriately sized identity matrices. To be able to compare the solution obtained in [11], where $\sum_{j \in I_m} \text{tr}\{P_j\}$ is minimized, with the results obtained by minimization of an upper bound on all terms $\text{tr}\{P_j\}$, the same identity matrix assumptions are made herein.

The convex optimization problem for this example is to minimize $\text{tr}\{Z\}$ with respect to the matrix variables X, Y, and Z subject to the Lyapunov linear matrix inequality constraints (9).

Using LMITOOL documented by El Ghaoui, et al. [7] and the semidefinite programming algorithm SD documented by Vandenberghe and Boyd [13], the optimal points

$$X^* = \begin{bmatrix} -0.2593 & 0.0061 & 0.0560 \end{bmatrix}$$

$$Y^* = \begin{bmatrix} 3.3514 & -0.3781 & -0.1683 \\ -0.3781 & 0.0569 & 0.0208 \\ -0.1685 & 0.0208 & 0.0387 \end{bmatrix}$$

$$Z^* = \begin{bmatrix} 1.2380 & 7.7939 & 1.2020 \\ 7.7939 & 70.9536 & -4.1979 \\ 1.2020 & -4.1979 & 33.3622 \end{bmatrix}$$

are obtained, resulting in an optimal performance bound of $\text{tr}\{Z^*\} = 105.5538$. The optimal Lyapunov matrix is

$$P^* = (Y^*)^{-1} = \begin{bmatrix} 1.2380 & 7.7939 & 1.2020 \\ 7.7939 & 70.9536 & -4.1979 \\ 1.2020 & -4.1979 & 33.3622 \end{bmatrix} ,$$

which leads to

$$\text{tr}\{P^*\} = 105.5538 .$$

Since $K^* = X^* (Y^*)^{-1}$, the optimal single input static gain K is

$$K^* = \begin{bmatrix} -0.2063 & -1.8247 & 1.5305 \end{bmatrix} .$$

For the purposes of comparison, the "scalarized" cost control results of Paskota, et al. [11] were confirmed using their nonlinear programming scheme. This resulted in a locally optimal gain vector of

$$K^* = \begin{bmatrix} -1.0964 & -8.3140 & 4.2964 \end{bmatrix}$$

Table 1 shows the values of each $\text{tr}\{P_j\}$ when the respective optimal gains K^* are used for control and solving the resulting Lyapunov equation (3).

The following points are worth noting:

Guaranteed-Cost	Scalarization
$\text{tr}\{P_1\} = 2.41$	$\text{tr}\{P_1\} = 2.51$
$\text{tr}\{P_2\} = 10.4$	$\text{tr}\{P_2\} = 9.62$
$\text{tr}\{P_3\} = 15.3$	$\text{tr}\{P_3\} = 14.1$
$\text{tr}\{P_4\} = 5.22$	$\text{tr}\{P_4\} = 5.29$

Table 1: Comparison of Results from Convex Guaranteed-Cost Design and Scalarization Design

1. The guaranteed-cost bound $\text{tr}\{P\} = 105.5064$ is conservative with respect to the actual performance levels achieved for each system

$$\max_{j \in I_m} \text{tr}\{P_j\} = 15.3 .$$

2. The scalarization approach of [11] may yield a locally Pareto optimal solution (for the vector optimization problem with vector components $\text{tr}\{P_j\}$), but not necessarily a globally Pareto optimal point. The guaranteed-cost vector performance measure is not inferior to the scalarization approach since at least one component, that is,

$$\text{tr}\{P_1\} = 2.41$$

is less than the scalarization component, namely,

$$\text{tr}\{P_1\} = 2.51 .$$

3. If the Euclidean norms of the gain matrices K are compared, a scalarization method gain

$$\|K_s\| = 9.4$$

is obtained, versus that of the convex guaranteed-cost method

$$\|K\| = 2.4 .$$

The guaranteed-cost design yields lower feedback gains. However this may be true only for this example.

4. Summary

In this paper a guaranteed-cost approach is taken for the performance design of multiple model systems. The problem is reduced to a convex linear matrix inequality problem which can be solved with commercially available software. A numerical example taken from [11] is used to compare the guaranteed-cost results obtained here with the scalarization results obtained in [11]. From the vector optimization point of view (where one attempts to make each component measure $\text{tr}\{P_j\}$ as small as possible) the two results are not comparable. Neither solution is "superior," in the sense of Pareto, to the other. However in

the particular example considered in [11], the norm of the feedback gain matrix does turn out to be smaller for the guaranteed-cost design than for the scalarization design. It would be of interest to explore the simultaneous performance problem considered in the full context of vector optimization.

References

[1]   V. Balakrishnan and L. Vandenberghe. Connections Between Duality in Control Theory and Convex Optimization. In *Proceedings of the American Control Conference*, pages 4030–4034, Seattle, Washington, June 1995.

[2]   J. Bernussou, P.L.D. Peres, and J.C. Geromel. A Linear Programming Oriented Procedure for Quadratic Stabilization of Uncertain Systems. *Systems & Control Letters*, 13(1):65–72, July 1989.

[3]   S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1994.

[4]   S.S.L. Chang and T.K.C. Peng. Adaptive Guaranteed Cost Control of Systems with Uncertain Parameters. *IEEE Transactions on Automatic Control*, AC-17(4):474–483, August 1972.

[5]   P. Dorato, C.T. Abdallah, and V. Cerone. *Linear Quadratic Control: An Introduction*. Prentice-Hall, Englewood Cliffs, New Jersey, 1995.

[6]   P. Gahinet, A. Nemirovskii, A.J. Laub, and M. Chilali. *LMI Control Toolbox*. The Mathworks, Inc., Natick, Massachusetts, 1995.

[7]   L. El Ghaoui, F. Delebeque, and R. Nikoukhah. LMITOOL: A User-Friendly Interface for LMI Optimization. Preprint downloaded from WWW server at École Nationale Supérieure de Techniques Avançées (ENSTA), France, February 1995.

[8]   G.D. Howitt and R. Luus. Simultaneous Stabilization of Linear Single-Input Systems by Linear State Feedback Control. *International Journal of Control*, 54(4):1015–1030, 1991.

[9]   Y. Nesterov and A. Nemirovskii. *Interior-Point Polynomial Algorithms in Convex Programming*, volume 13 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[10]   V. Pareto. *Cours d'Économie Politique*. Rouge, Lausanne, Switzerland, 1896.

[11]   M. Paskota, V. Sreeram, K.L. Teo, and A.I. Mees. Optimal Simultaneous Stabilization of Linear Single-Input Systems Via Linear State Feedback Control. *International Journal of Control*, 60(4):483–498, 1994.

[12]   I.R. Petersen. A Procedure for Simultaneously Stabilizing a Collection of Single Input Linear Systems Using Non-linear State Feedback Control. *Automatica*, 23(1):33–40, 1987.

[13]   L. Vandenberghe and S. Boyd. *SP: Software for Semidefinite Programming*. Stanford University Information Sciences Laboratory, Stanford, California, beta version edition, November 1994. User's Guide downloaded from anonymous ftp server at isl.stanford.edu in directory /pub/boyd/semidef_prog/.

[14]   T.L. Vincent and W.J. Grantham. *Optimality in Parametric Systems*. John Wiley & Sons, New York, 1981.

[15]   D.-N. Wu, W.-B. Gao, and M. Chen. Algorithm for Simultaneous Stabilization of Single-Input Systems Via Dynamic Feedback. *International Journal of Control*, 51(3):631–642, 1990.