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Robust, Non-Fragile and Optimal Controller Design Via Linear Matrix Inequalities

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Abstract

In this article, we introduce a robust non-fragile state feedback controller which is also optimal with respect to a quadratic performance index, using Linear Matrix Inequalities (LMIs). The uncertainties are assumed to be polytopic, both in the controller gains and the system dynamics. A numerical example is presented to demonstrate the efficiency of this method, and the controller turns out to be robust with respect to the uncertainties in the plant and the controller.

Key Words: Non-fragile controller, Linear Matrix Inequalities, robust stability, Polytopic LDIs.

1 Introduction

One of the most active areas of research in linear control systems is robust and optimal controller design. For the past 15 years several researchers have come up with different methods that enable the controller to cope with uncertainties in the plant dynamics. Some of these methods deal with the so-called structured uncertainty, while others deal with unstructured uncertainty. A majority of these methods rely on the Youla-Kučera Q parameterization of all stabilizing controllers. Elegant techniques for minimizing H_2 [1], H_∞ [11, 12] and L_1 [4] norms of different closed-loop transfer functions have been developed using this parameterization. Although these methods cope with uncertainty in the plant dynamics, they all assume that the controller derived is precise, and exactly implemented. Unfortunately, this is not the case in practice. The controller implementation is subject to round-off errors in numerical computations, in addition to the need of providing the practicing engineer with safe-tuning margins. Therefore, the design has to be able to tolerate some uncertainty in the controller as well as the plant dynamics. Recent results in [6] have brought attention to this problem. The authors in [7] have come up with a method to deal with the uncertainty in a fixed-structure dynamical controller, but have not taken into account the uncertainty in the plant dynamics.

The basic premise of our paper is that one can not achieve "resiliency" if robustness is all that is demanded, and as motivated by [6] and discussed in [7], there exists a trade-off between the system's ability to tolerating both. The numerical examples in [6] suggest that if the only uncertainty is in the plant, all of the available margins will be used, making the closed-loop system extremely fragile with respect to the other type of uncertainties. Since designing a dynamical controller as in [7] for the case where both system and controller are uncertain makes the problem very complicated, we consider in this paper the design of robust, yet resilient static state feedback controllers.

The structure of this paper is as follows. First, the uncertain plant is described by a set of Linear Differential Inclusions (LDIs) [3] in section 2. We consider the robust stability and performance of state feedback controllers with polytopic uncertainty in the controller gains in sections 3 and 4 respectively. A numerical example is presented in section 5 and robustness of the controller with respect to parametric uncertainties in plant and controller gains is demonstrated. Further extensions and our conclusions are discussed in section 6

2 Polytopic Linear Differential Inclusions(LDIs)

A linear differential inclusion (LDI) is given by

$$\dot{x} \in \Omega x \tag{1}$$

where Ω is a subset of R^n. The LDI in (1) may for example describe a family of linear time-varying systems. In this case, every trajectory of the LDI satisfies

$$\dot{x} = A(t)\, x(t)$$

When Ω is a polytope, the LDI is called polytopic or PLDI, that is, $A(t) \in Co\{A_1, A_2, \cdots, A_r\}$ which means that we can write $A(t)$ as a convex combination of the vertices of the polytope as follows

$$A(t,x)=\alpha_1(t,x)A_1 + \alpha_2(t,x)A_2 + \cdots + \alpha_r(t,x)A_r \quad (2)$$

where $\{A_1, \cdots, A_r\}$ are known matrices and $\alpha_1, \ldots, \alpha_r$ are positive scalars which satisfy $\sum_{i=1}^{r} \alpha_i(t,x) = 1$. Using a technique known as global linearization [3], we can use PLDIs to study properties of nonlinear time varying systems. In fact, consider the system

$$\dot{x}=f(t,x,u) \quad (3)$$

If the Jacobian of the system matrix $A(t,x) = \frac{\partial f}{\partial x}$ lies in the convex hull defined in (2), then every trajectory of the nonlinear system is also a trajectory of the LDI defined by Ω (See [3] for more details).

3 Robust Stability

In this section, we discuss the robust stability problem. Using the discussion in the previous section, let the dynamics of the uncertain system be defined as follows

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(t,x)(A_i x + B_i u) \quad (4)$$

where, $x(t) \in R^n$, $u(t) \in R^m$, $A_i \in R^{n\times n}$, $B_i \in R^{n\times n}$ $\sum_{i=1}^{r} \alpha_i(t,x) = 1$, and $\alpha_i(t,x) > 0$, $\forall i \in \{1, \ldots, r\}$

For simplicity, we assume that the state is available for measurement and feedback. Using a similar form of polytopic uncertainty for the controller, the control input can be written as the following,

$$u = -\sum_{j=1}^{r} \beta_j(t,x)K_j x \quad (5)$$

$\beta_j(t,x) > 0 \; \forall j \in \{1, \ldots, r\}$, and $\sum_{j=1}^{r} \beta_j(t,x) = 1$. Replacing u in (4) with (5), and keeping in mind that $\sum_{i=1}^{r} \alpha_i(t,x) = 1$, the closed-loop system can be written as

$$\dot{x} = \sum_{j=1}^{r}\sum_{i=1}^{r} \alpha_i(t,x)\beta_j(t,x)(A_i - B_i K_j)x \quad (6)$$

The following theorem then provides sufficient conditions for the stability of the closed-loop system.

Theorem 1 *: The closed loop system (6) is globally asymptotically stable if there exists a common positive definite matrix P that satisfies the following Lyapunov inequalities :*

$$(A_i - B_i K_j)^T P + P(A_i - B_i K_j) < 0 \quad i,j = 1, \ldots, r \quad (7)$$

■

The proof is easily obtained by multiplying inequalities in (7) by $\alpha_i \beta_j$. Pre-multiplying and post-multiplying the inequalities in (7) by $Y = P^{-1}$, and introducing $X_i = K_i Y$, we can write the inequalities (7) as the following LMIs

$$YA_i^T + A_i Y - M_{ij} - M_{ij}^T < 0 \quad i,j = 1, \ldots, r \quad (8)$$

where M_{ij} can be defined as follows:

$$M_{ij} = B_i X_j \quad (9)$$

For later reference, we define

$$N_{ij}=YA_i^T + A_i Y - M_{ij} - M_{ij}^T \quad (10)$$

If the LMI's (10) are feasible, we can obtain the values for K_i and P as,

$$P=Y^{-1}$$
$$K_i=X_i Y^{-1} \quad i = 1, \cdots, r \quad (11)$$

4 Robust Performance

In this section we try to achieve a certain level of performance for the uncertain system (6) using a guaranteed-cost approach [5]. It is a well known result from LQR theory that the problem of minimizing the cost function

$$J = \int_0^\infty (x^T Q x + u^T R u) dt \quad (12)$$

reduces to finding a positive definite solution $P > 0$ of the following Lyapunov equation:

$$(A - BK)^T P + P(A - BK) + Q + K^T R K = 0 \quad (13)$$

where $Q \geq 0$ and $R > 0$. We can write the minimum cost of J as [5]:

$$min\{J\} = x(0)^T P x(0)$$

If we write (13) as a matrix inequality instead of an equality, the solution of the inequality will be an upper bound on the performance measure J, and we can reach $min\{J\}$ by minimizing that upper bound. While this result holds for a single LTI system, we can extend it to the case of equation (6). To avoid the dependency of the cost function of the system on initial conditions of (6), we assume the initial conditions randomized with zero mean and identity covariance, i.e.,

$$\mathbb{E}\{x(0)\} = 0$$
$$\mathbb{E}\{x(0)x(0)^T\} = I \quad (14)$$

Our objective is to minimize the expected value of the performance index J with respect to all possible initial conditions with zero mean and covariance equal to the identity. It can be shown [5, 8] that we can write the upper bound on the performance objective as: $J = tr(P)$ where tr denotes the trace, and where P satisfies the following Lyapunov inequalities:

$$0>(A_i - B_i K_j)^T P + P(A_i - B_i K_j) + Q + \sum_{i=1}^{r} K_i^T R K_i \quad i,j = 1, \ldots, r \quad (15)$$

This follows from the fact that [8]

$$(\sum_{i=1}^{r}\alpha_i K_i)R(\sum_{i=1}^{r}\alpha_i K_i^T) < \sum_{i=1}^{r} K_i^T R K_i \tag{16}$$

Using the same change of variables as in (11), and by using the LMI lemma [5], we can write the Lyapunov inequalities (15) as the following LMI's:

$$\begin{bmatrix} N_{ij} & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ Q^{1/2}Y & -I_n & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & -I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & \cdots & -I_m \end{bmatrix} < 0$$

$$i, j = 1, \cdots, r \tag{17}$$

where N_{ij} is defined in (10). To obtain the least possible upper bound provable by a quadratic Lyapunov function, we have the following optimization problem

Min $tr(Y^{-1})$
Subject To: LMIs in (17)

This is a convex optimization problem which can be solved in polynomial time [10] using one of the available LMI toolboxes. To make it possible to use MATLAB® LMI Toolbox, we introduce an artificial variable Z as an upper bound on Y^{-1}, and minimize $tr(Z)$ instead, i.e, we recast the problem in the following form:

Min $tr(Z)$
Subject To LMIs in(17), and

$$\begin{bmatrix} Z & I_n \\ I_n & Y \end{bmatrix} > 0 \tag{18}$$

If the above LMIs are feasible, we can calculate the controller gains as

$$K_i = X_i Y^{-1}$$

and u as

$$u = -\sum_{i=1}^{r} \beta_j(t,x) K_j x$$

i.e., we can write u as any convex combination of controller gains K_is. We demonstrate a numerical example in the next section.

5 Numerical Examples

To illustrate this design approach, consider the problem of balancing an inverted pendulum on a cart. The equations of motion for the pendulum are [13]

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g\sin(x_1) - amlx_2^2\sin(2x_1)/2 - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)}$$

where x_1 denotes the angle of the pendulum (in radians) from the vertical axis, x_2 is the angular velocity of the pendulum, $g = 9.8\ m/s^2$ is the gravity constant, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum, and u is the force applied to the cart. The following numerical values were used in the simulation: $a = \frac{1}{m+M}, m = 2kg, M = 8.0kg, 2l = 1.0m$. We approximate the nonlinear equation by a Polytopic LDI (PLDI) defined by the following two vertices

$$A(x) \in Co\{A_1, A_2\}; \quad B(x) \in Co\{B_1, B_2\}$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ \frac{g}{4l/3 - aml} & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -\frac{a}{4l/3 - aml} \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ \frac{9g}{4\pi(4l/3 - aml\beta^2)} & 0 \end{bmatrix}$$
$$B_2 = \begin{bmatrix} 0 \\ -\frac{a\beta}{4l/3 - aml\beta^2} \end{bmatrix}$$
$$\beta = \cos(80^\circ)$$
$$C = [1 \ \ 0] \tag{19}$$

we choose the following values for Q and R

$$Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$R = 0.1 \tag{20}$$

Note that (A_1, B_1, C) and (A_2, B_2, C) are the linearizations of the system equation around the points 0 and 80°, i.e., $\cos(x_1)$ is approximated by β in A_2 and B_2, $\sin(x_1)$ is approximated by x_1 and $\frac{9}{4\pi}x_1$ at 0 and 80° in A_1 and A_2 respectively. The system is not controllable at $\pi/2$. As it was mentioned before, we can let u be any convex combination of controller gains, as long as the closed-loop LDI approximates the closed loop nonlinear system. One such choice can be

$$u = -(0.4K_1 + .6K_2)x \tag{21}$$

Simulations indicate that the above control law can balance the pendulum for initial conditions between [−80°, 80°]. The initial conditions response for an angle of 80° are plotted in figures 1 and 2. In figure 3, we plot the control action u.To illustrate the robustness of the controller, we increased the cart mass by 30%, and the pole length by 40%. Results are plotted in figures 4 and 5 . It is worthwhile to note that we can design nonlinear controllers for the plant based on extended linearization techniques, but these controllers are usually very complicated. One example of such controllers is as follows [2]

$$u = k(x_1, x_2)$$
$$= -\frac{g}{a}\tan(x_1) - \frac{4le_1e_2}{3a}\ln[\sec(x_1) + \tan(x_1)]$$
$$+ e_1e_2ml\sin(x_1)$$
$$+ \frac{(e_1+e_2)x_2}{a}[\frac{4l}{3}\sec(x_1) - aml\cos(x_1)]$$

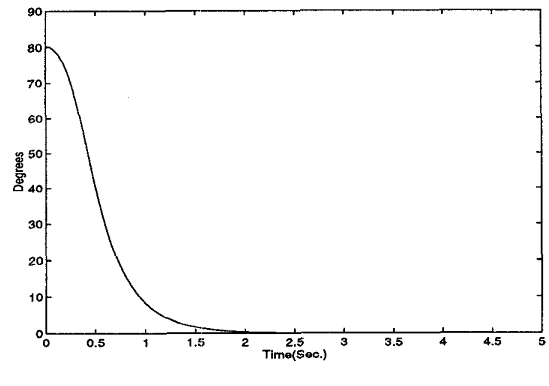

Figure 1: Initial condition Response of the Angle.

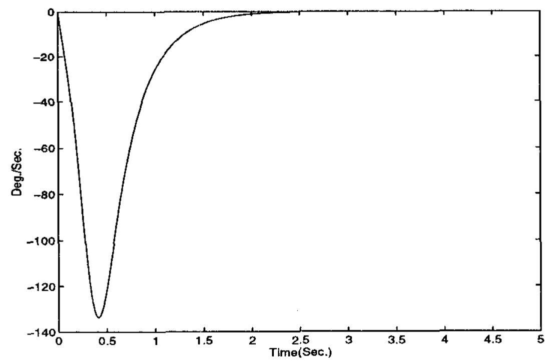

Figure 2: Initial condition Response of Angular Velocity.

where e_1, e_2 are desired closed loop eigenvalues. Note that here we don't have any measure for optimality. Instead, By linearizing the dynamics of the system for the angles greater than $\pi/2$ and also close to π, we can balance the pendulum at any initial condition while feedback linearization works only in the $[-\pi/2, \pi/2]$ interval [13].

6 Conclusion

The purpose of this note was to present a simple solution to the problem of non-fragile controller design, without losing the robustness with respect to uncertainties in the plant model. The next step would be to assume a dynamic observer/controller system and to assume the same form of uncertainty for the observer gains. Using the separation principle proven in [9], we can design the observer and controller separately, and we still end up with LMIs.

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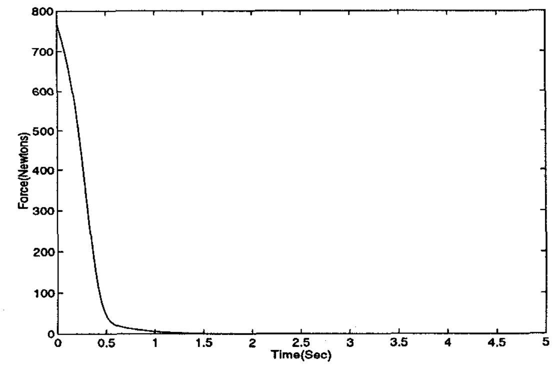

Figure 3: Control Action.

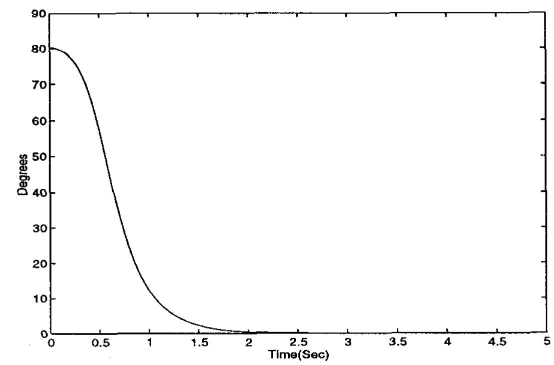

Figure 4: Angle Response with 30% change in cart mass and 40% in pole length.

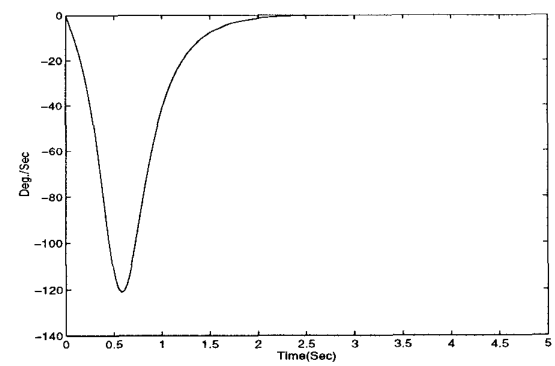

Figure 5: Angular velocity with 30% change in cart mass and 40% in pole length.

Technology.

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