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Finite-Time Control for Uncertain Linear Systems with Disturbance Inputs

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Abstract

In this paper we consider the static output feedback, finite-time disturbance rejection problem for linear systems with time-varying norm-bounded uncertainties. The first result provided in the paper is a sufficient condition for finite-time *state feedback* disturbance rejection in the presence of *constant* disturbances. This condition requires the solution of an LMI. Then we consider the more general output feedback case, which is shown to be reducible to the solution of an optimization problem involving *Bilinear Matrix Inequalities.* Finally we deal with the case in which the disturbance is time-varying and generated by a linear system.

1 Introduction

In recent years many results have been published on the robust stability problem for linear systems (see for example the reprint volumes [5] and [6]). The work of control scientists and engineers has mainly focused on robust *Lyapunov Stability*; as it is well known, Lyapunov stability deals with the steady-state behavior of linear systems, because it looks at the asymptotic pattern of system trajectories.

In many cases, however, we are more interested in what happens over a finite-time interval rather than asymptotically. This is the case, for instance, when we want to control the state trajectory from an initial point to a final point in a prescribed time interval. Two different problems can arise: what happens if we have some initial conditions different from zero? And what happens if, in addition to some initial conditions, we also have some disturbances acting on the system? To address the former problem, in [1] we used the concept of Finite-Time Stability [4]. In this paper we generalize this idea, introducing in Section 2 the concept of Finite-Time Boundedness.

These finite-time properties are distinct and independent from their asymptotic counterparts: for instance a system which is finite-time stable could be Lyapunov unstable, whereas a Lyapunov stable system could be finite-time unstable if its state exceeds the prescribed bounds during the transient period.

The first result provided in this paper (Section 3) is a sufficient condition for *state feedback* finite-time disturbance rejection in presence of *constant* disturbances. This condition reduces to an LMI Problem [7]. This result is generalized in Section 4 to solve the *static output feedback* (SOF) case. However the SOF problem requires the solution of Bilinear Matrix Inequalities (BMIs). Most BMIs algorithms are guaranteed to converge but not necessarily to the global optimum (like LMIs based problems) and the solution is dependent on the initial data; however there exist efficient local minima algorithms which work well in many situations (see [9], [10] and the bibliography therein).

The last result of the paper, provided in Section 5, is a sufficient condition for finite-time disturbance rejection in presence of *time-varying* disturbances generated by a zero-input LTI system.

2 Problem Statement and Preliminaries

In this paper we consider the following linear system subject to time-varying uncertainties and to exogenous disturbances

$$\begin{aligned}\dot{x}(t) &= [A+\Delta A(t)]\,x(t)+[B+\Delta B(t)]\,u(t)+ \\ &\quad\; Gw(t) \qquad (1a)\\ y &= Cx \qquad (1b)\end{aligned}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $G \in \mathbb{R}^{n\times q}$.

We shall assume the following.

A1) The uncertain part in (1a) is in the so-called *structured, one block form*

$$[\,\Delta A(t) \quad \Delta B(t)\,] = F\Delta(t)\,[E_1 \quad E_2]$$

where $F \in \mathbb{R}^{n\times r}$, $E_1 \in \mathbb{R}^{s\times n}$ and $E_2 \in \mathbb{R}^{s\times m}$ and the unknown, real matrix-valued function Δ belongs to the class

$$\mathcal{D} := \{\Delta : [0,+\infty) \mapsto \mathbb{R}^{r\times s} |\ \Delta \text{ is Lebesgue measurable, } \Delta(t)^T\Delta(t) \le I\}.$$

A2) The exogenous disturbance w is constant and satisfies

$$w^T w \le d \tag{2}$$

where $d \ge 0$.

Assumption A2 will be removed in Section 5, where we shall consider the more general case of disturbances generated by a LTI system.

Concerning system (1), we consider the following output feedback controller

$$u = Ky \tag{3}$$

with $K \in \mathbb{R}^{m\times p}$.

The aim of this paper is to find some sufficient conditions which guarantee that the closed loop system given by the interconnection of (1) with (3) exhibits a given level of disturbance rejection over a finite-time interval. The general idea of *finite-time boundedness* concerns the boundedness of the state of a system over a finite time interval given both some initial conditions and an external disturbance acting on the system. This concept can be formalized through the following definition, which is an extension of the one in [1].

Definition 1 Let $\mathcal{W}$ be a class of disturbance signals. The time-varying linear system

$$\dot{x}(t) = A(t)x(t) + G(t)w \qquad t \in [0,T] \tag{4}$$

subject to an exogenous disturbance $w \in \mathcal{W}$, is said to be Finite-Time Bounded (FTB) with respect to the given quadruple (c_1, c_2, d, T), with $c_2 > c_1$ if

$$x^T(0)x(0) \le c_1 \Rightarrow x^T(t)x(t) < c_2 \ \ \forall t \in [0,T], \forall w \in \mathcal{W}$$

■

On the basis of the above considerations the first aim of this paper is the solution of the following problem.

Static Output Feedback Problem (SOFP) Given system (1) and the quadruple (c_1, c_2, d, T), find an output feedback controller in the form (3) such that the closed loop system given by the interconnection of (1) with (3) is FTB with respect to (c_1, c_2, d, T) for all $\Delta \in \mathcal{D}$. ■

A particular case of SOFP is the one in which the whole state of system (1) is available for feedback ($C = I$); we will denote the related problem by SFP.

The following lemma states a sufficient condition for the FTB of a system in the form

$$\dot{x}(t) = (A + F\Delta(t)E_1)\,x(t) + Gw \tag{5}$$

which is fundamental to prove the main results of the following sections.

Lemma 1 *System (5) is FTB with respect to (c_1, c_2, d, T) for all $\Delta \in \mathcal{D}$ if there exist a positive scalar α and two symmetric positive definite matrices $Q_1 \in \mathbb{R}^{n\times n}$ and $Q_2 \in \mathbb{R}^{q\times q}$ such that*

$$\begin{pmatrix} AQ_1 + Q_1A^T + FF^T - \alpha Q_1 + \frac{1}{\alpha}GQ_2G^T & Q_1E_1^T \\ E_1Q_1 & -I \end{pmatrix} < 0 \tag{6a}$$

$$cond \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} < \frac{c_2}{c_1 + d} e^{-\alpha T} \tag{6b}$$

Proof: Let $V(x,w) = x^TQ_1^{-1}x + w^TQ_2^{-1}w$ and denote, as usual, by $\dot{V}$ the derivative of V along the solution of system (5). Suppose that the condition

$$\dot{V}(x(t), w) < \alpha V(x(t), w) \tag{7}$$

holds for all $t \in [0,T]$ and all $w \in \mathcal{W}$. We will first demonstrate that conditions (7) and (6b) imply that system (5) is FTB with respect to (c_1, c_2, d, T). Then, to conclude the proof, we will show that condition (7) is implied by (6a).

Our first claim is that conditions (7) and (6b) imply the FT Boundedness of system (5) with respect to (c_1, c_2, d, T). Introducing the matrix

$$P = \begin{pmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{pmatrix}$$

and the vector

$$z = \begin{pmatrix} x \\ w \end{pmatrix}$$

it is easy to show that from (7) it follows that

$$z^T(t)Pz(t) < z^T(0)Pz(0)e^{\alpha t} \tag{8}$$

Now we have

$$\begin{aligned} z^T(t)Pz(t) &\ge \lambda_{min}(P)z^T(t)z(t) \\ &\ge \lambda_{min}(P)x^T(t)x(t) \qquad (9a) \\ z^T(0)Pz(0)e^{\alpha t} &\le \lambda_{max}(P)z^T(0)z(0)e^{\alpha t} \\ &\le \lambda_{max}(P)(c_1 + d)e^{\alpha T} \qquad (9b) \end{aligned}$$

Putting together (8) and (9) we have

$$x^T(t)x(t) < cond\begin{pmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1}\end{pmatrix}(c_1+d)e^{\alpha T}$$
$$= cond\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2\end{pmatrix}(c_1+d)e^{\alpha T} \quad (10)$$

From (10) it readily follows that (6b) implies, for all $t \in [0,T]$, $x^T(t)x(t) < c_2$; from this last consideration our first claim follows.

Now we need to prove that condition (6a) implies (7).

Assume that there exist $\alpha > 0$ and two symmetric matrices $Q_1 > 0$ and $Q_2 > 0$ such that inequality (6a) is satisfied. Using the following inequality

$$FF^T + Q_1E_1^TE_1Q_1 \geq FF^T + Q_1E_1^T\Delta^T\Delta E_1Q_1 \geq F\Delta E_1Q_1 + Q_1E_1^T\Delta^TF^T \quad (11)$$

and letting $\hat{A} := (A + F\Delta E_1)$, from (6a) we have

$$\hat{A}Q_1 + Q_1\hat{A}^T - \alpha Q_1 + \frac{1}{\alpha}GQ_2G^T < 0 \quad (12)$$

Pre and post-multiplying (12) by Q_1^{-1} we obtain the equivalent condition

$$Q_1^{-1}\hat{A} + \hat{A}^TQ_1^{-1} - \alpha Q_1^{-1} + \frac{1}{\alpha}Q_1^{-1}GQ_2G^TQ_1^{-1} < 0 \quad (13)$$

Condition (13) is equivalent to

$$\begin{pmatrix} Q_1^{-1}\hat{A} + \hat{A}^TQ_1^{-1} - \alpha Q_1^{-1} & Q_1^{-1}G \\ G^TQ_1^{-1} & -\alpha Q_2^{-1}\end{pmatrix} < 0$$

which, in turn is equivalent to (7). Therefore the proof follows. ∎

Remark 1 Note that there exists a trade-off between satisfying (6a) and (6b), since increasing α will guarantee the negative definiteness of the LMI (6a) but will tighten the bound in (6b). ∎

3 The State Feedback Case

In this section we consider the case in which the whole state of system (1) is available for feedback, namely

$$\dot{x}(t) = [A + F\Delta(t)E_1]\,x(t) + [B + F\Delta(t)E_2]\,u(t) + Gw \quad (14a)$$
$$y = x \quad (14b)$$

For this system we shall provide a sufficient condition for the solution of the SFP by means of a controller in the form

$$u = Kx \quad (15)$$

with $K \in \mathbb{R}^{n\times m}$. These conditions are then turned into an optimization problem involving Linear Matrix Inequalities (LMIs) [3].

The closed loop system given by the connection of (14) with (15) has the expression

$$\dot{x}(t) = [A + BK + F\Delta(t)(E_1 + E_2K)]\,x(t) + Gw \quad (16)$$

Theorem 1 *The SFP admits a solution if there exist a positive scalar α, two symmetric positive definite matrices $Q_1 \in \mathbb{R}^{n\times n}$ and $Q_2 \in \mathbb{R}^{q\times q}$ and a matrix $L \in \mathbb{R}^{m\times n}$ such that*

$$\begin{pmatrix} AQ_1 + Q_1A^T + BL + L^TB^T + FF^T - \alpha Q_1 + \frac{1}{\alpha}GQ_2G^T & (E_1Q_1 + E_2L)^T \\ E_1Q_1 + E_2L & -I\end{pmatrix} < 0$$

$$cond\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2\end{pmatrix} < \frac{c_2}{c_1+d}e^{-\alpha T}$$

In this case a controller which solves the SFP is given by $K = LQ_1^{-1}$.

Proof: The proof easily follows from Lemma 1, replacing A with $(A + BK)$ and E_1 with $(E_1 + E_2K)$ (compare (5) with (16)), and letting, according to [8], $L = KQ_1$. ∎

For a given α, the feasibility of the conditions stated in Theorem 1 can be turned into an LMIs based optimization problem. Indeed these conditions can be equivalently restated as the following *Eigenvalue Problem* (EVP) [3].

Eigenvalue Problem
Given system (14) and the quadruple (c_1, c_2, d, T), fix $\alpha > 0$ and solve

$$\min_{L,Q_1,Q_2} \gamma$$
$$s.t.$$

$$\begin{pmatrix} AQ_1 + Q_1A^T + BL + L^TB^T + FF^T - \alpha Q_1 + \frac{1}{\alpha}GQ_2G^T & (E_1Q_1 + E_2L)^T \\ E_1Q_1 + E_2L & -I\end{pmatrix} < 0$$

$$I < \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2\end{pmatrix} < \gamma I$$

If $\gamma < \frac{c_2}{c_1+d}e^{-\alpha T}$, letting $K = LQ_1^{-1}$, system (16) is FTB with respect to (c_1, c_2, d, T). ∎

Example 1 Let us consider system (14) with

$$A = \begin{pmatrix} 0 & 20 \\ -1 & 0\end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ -1\end{pmatrix},$$
$$F = \begin{pmatrix} -1 & 0 \\ 1 & -2\end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}$$
$$E_2 = \begin{pmatrix} \frac{1}{2} \\ 0\end{pmatrix}, \quad G = \begin{pmatrix} 3 \\ \frac{3}{2}\end{pmatrix}$$

Moreover let $c_1 = 1$, $c_2 = 100$ and $d = 1$. For the given triplet (c_1, c_2, d), we found the state feedback controller which maximizes the value of T for which the closed loop system is FTB with respect to (c_1, c_2, d, T). This problem has been solved with the aid of the LMI Toolbox [7], finding $T_{max} = 0.32$ for $\alpha = 4.8$; the optimal controller is

$$K = (\,0.32 \quad 2.95\,)\ .$$

■

4 The Output Feedback Case

In this section we go back to the original SOFP concerning system (1).

Theorem 2 *The SOFP admits a solution if there exist a positive scalar α, two symmetric positive definite matrices $Q_1 \in \mathbb{R}^{n\times n}$ and $Q_2 \in \mathbb{R}^{q\times q}$ and a matrix $K \in \mathbb{R}^{m\times p}$ such that*

$$\begin{pmatrix} A_{CL}Q_1 + Q_1A_{CL}^T + FF^T - \alpha Q_1 + \frac{1}{\alpha}GQ_2G^T & Q_1\hat{E}^T \\ \hat{E}Q_1 & -I \end{pmatrix} < 0 \quad (17a)$$

$$cond\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} < \frac{c_2}{c_1 + d}e^{-\alpha T} \quad (17b)$$

where

$$\begin{aligned} A_{CL} &= (A + BKC) \\ \hat{E} &= (E_1 + E_2KC) \end{aligned}$$

Proof: The proof immediately follows from the expression of the closed loop system and from Lemma 1. ■

Unfortunately, the feasibility of the conditions stated in Theorem 2 cannot be directly reduced to LMIs based problems because of the product between the optimization variables; such a problem cannot be overcome via a suitable change of variables, as in the state feedback case. Indeed condition (17a) is a Bilinear Matrix Inequality [10]. To solve this problem we propose a heuristic procedure which consists in alternating the optimization over K, with fixed Q_1 and Q_2, with the optimization over Q_1 and Q_2, with fixed K (see also [9]). In this way each optimization becomes an LMI problem; this procedure is guaranteed to converge but not necessarily to the global minimum and the solution is dependent on the initial data. With this procedure we solved the following example.

Example 2 Let us reconsider the example we introduced in Section 4, this time with the following output matrix C

$$C = (\,-1 \quad 1\,)$$

We found the output feedback controller maximizing the value of T for which the closed loop system is FTB with respect to (c_1, c_2, d, T). We obtained $T_{max} = 0.29$ for $\alpha = 5.2$; the optimal controller is

$$K = -0.88\ .$$

As expected, the value of T_{max} is smaller than in the case of the state feedback. ■

5 Disturbance Rejection with Time-Varying Disturbances

In this section we consider the finite-time disturbance rejection problem in the presence of time-varying disturbances modelled as the output of a zero-input LTI system; to this end let us consider the system

$$\begin{aligned} \dot{x}(t) &= (A + F\Delta(t)E)x(t) + Gw(t) && (18a) \\ \dot{w}(t) &= A_w w(t) \qquad w^T(0)w(0) \le d && (18b) \end{aligned}$$

where A_w may be unstable.

Lemma 2 *System (18) is FTB with respect to (c_1, c_2, d, T) for all $\Delta \in \mathcal{D}$ if there exist a nonnegative scalar α and two symmetric positive definite matrices $Q_1 \in \mathbb{R}^{n\times n}$ and $Q_2 \in \mathbb{R}^{q\times q}$ such that*

$$\begin{pmatrix} AQ_1 + Q_1A^T + FF^T - \alpha Q_1 & GQ_2 & Q_1E^T \\ Q_2G^T & Q_2A_w^T + A_wQ_2 - \alpha Q_2 & 0 \\ EQ_1 & 0 & -I \end{pmatrix} < 0 \quad (19a)$$

$$cond\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} < \frac{c_2}{c_1 + d}e^{-\alpha T} \quad (19b)$$

Proof: As usual we pick the Lyapunov functions $V(x, w) = x^TQ_1^{-1}x + w^TQ_2^{-1}w$ and assume that the condition

$$\dot{V}(x(t), w(t)) < \alpha V(x(t), w(t)) \quad (20)$$

is satisfied for all $t \in [0, T]$.

As in the proof of Lemma 1, conditions (19b) and (20) imply the statement of the Lemma. We have only to prove that condition (19a) implies (20).

Using the LMI Lemma (see [3]) it is simple to recognize that inequality (19a) is equivalent to the following

$$\begin{pmatrix} AQ_1 + Q_1A^T + FF^T - \alpha Q_1 + Q_1E^TEQ_1 & GQ_2 \\ Q_2G^T & Q_2A_w^T + A_wQ_2 - \alpha Q_2 \end{pmatrix} < 0 \quad (21)$$

Now using inequality (11) and letting $\hat{A} := (A + F\Delta E)$ we have that (21) implies

$$\begin{pmatrix} \hat{A}Q_1 + Q_1\hat{A}^T - \alpha Q_1 & GQ_2 \\ Q_2G^T & Q_2A_w^T + A_wQ_2 - \alpha Q_2 \end{pmatrix} < 0 \tag{22}$$

Pre and post-multiplying (22) by

$$\begin{pmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{pmatrix}$$

we obtain the equivalent condition

$$\begin{pmatrix} Q_1^{-1}\hat{A} + \hat{A}^TQ_1^{-1} - \alpha Q_1^{-1} & Q_1^{-1}G \\ G^TQ_1^{-1} & A_w^TQ_2^{-1} + Q_2^{-1}A_w - \alpha Q_2^{-1} \end{pmatrix} < 0$$

Finally we have that the last inequality is equivalent to (20). ■

Remark 2 If $\alpha = 0$, inequality (19b) is independent of T. In this case it is easy to show that FT Boundedness *implies* Quadratic Stability which in turn implies Lyapunov Asymptotic Stability for all $\Delta \in \mathcal{D}$ [2]. It is also easy to prove that a necessary condition to find a solution to (19) for $\alpha = 0$ is that A_w be a Hurwitz matrix. ■

Now consider the forced system

$$\begin{aligned} \dot{x}(t) &= (A + F\Delta(t)E_1)x(t) + (B + F\Delta(t)E_2)u(t) + \\ &\quad Gw(t) && \text{(23a)} \\ \dot{w}(t) &= A_w w(t) \qquad w^T(0)w(0) \le d\,. && \text{(23b)} \end{aligned}$$

The following theorem immediately follows from Lemma 2; this theorem states a sufficient condition for FTB in presence of time-varying disturbances.

Theorem 3 *The SOFP (stated for system (23a)) admits a solution if there exist a nonnegative scalar α, two symmetric positive definite matrices $Q_1 \in \mathbb{R}^{n\times n}$ and $Q_2 \in \mathbb{R}^{q\times q}$ and a matrix $K \in \mathbb{R}^{m\times p}$ such that*

$$\begin{pmatrix} A_{CL}Q_1 + Q_1A_{CL}^T + FF^T - \alpha Q_1 & GQ_2 & Q_1\hat{E}^T \\ Q_2G^T & Q_2A_w^T + A_wQ_2 - \alpha Q_2 & 0 \\ \hat{E}Q_1 & 0 & -I \end{pmatrix} < 0$$

$$cond\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} < \frac{c_2}{c_1 + d}e^{-\alpha T}$$

where

$$\begin{aligned} A_{CL} &= (A + BKC) \\ \hat{E} &= (E_1 + E_2KC) \end{aligned}$$

■

Regarding the solution of the SOFP for system (23a), considerations similar to those in Section 4 can be repeated.

6 Concluding Remarks

In this paper we have considered the finite-time stabilization problem for a linear system subject to norm bounded uncertainties and to unknown disturbances. We have provided a sufficient condition for finite-time stabilization via state feedback in the presence of constant disturbances which can be turned into an optimization problem involving LMIs; then the more general static output feedback case has been considered. However, in this last case, the solution of the optimization problem is constrained by BMIs. Finally the situation in which the disturbance is time-varying and generated by a zero-input system has been considered.

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