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Statistical Controller Design for the Linear Benchmark Problem

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Abstract

In this paper some fixed-order controllers are designed via statistical methods for the Benchmark Problem originally presented at the 1990 American Control Conference. Based on some recent results by the authors, it is shown that the statistical approach is a valid method to design robust controllers. Two different controllers are proposed and their performance are compared with controllers with the same structure, designed using different techniques.

1 Introduction

It has recently become clear that many control problems are too difficult to admit analytic solutions. New results have also emerged to show that the computational complexity of some "solved" control problems is prohibitive. In order to get around such problems, many authors have recently advanced the notion of probabilistic methods in control analysis and design. In control theory, some of the original (Monte Carlo) ideas have already been used, among the others, by Ray and Stengel [1], Tempo et al. [2], and by Khargonakar and Tikku [3], to solve *robust analysis* problems while Vidyasagar used learning theory to solve *robust control* problems [4], [5].

Unfortunately, and as acknowledged by the various authors, probabilistic methods, while more efficient than gridding techniques (which suffer from the curse of dimensionality), still require a large number of samples in order to guarantee accurate designs. In [6], we proposed a new control design algorithm which greatly reduces the required number of plants sampled in order to achieve a certain performance level. In the current paper, we illustrate the use of our algorithm in designing fixed-order robust controllers for the linear benchmark problem (see also [7]).

2 Problem Formulation

The benchmark problem was originally proposed in [8]. The plant consists of a two-mass/spring system with non-collocated sensor and actuator. The system can be represented in dimensionless state-space form as (see [8], [9])

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{u} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \frac{-k}{m_1} & \frac{k}{m_1} & \frac{-c}{m_1} & \frac{c}{m_1} & \frac{f}{m_1} \\ \frac{k}{m_2} & \frac{-k}{m_2} & \frac{c}{m_2} & \frac{-c}{m_2} & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{\tau} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix} u_c + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \\ 0 \end{bmatrix} w$$

$$y = x_2 + v$$

$$z = x_2$$

where x_1 and x_2 are the positions of the masses, c is the internal damping between the masses, f is a loop-gain uncertainty, u is the control input force, τ is a time constant for a first-order delay between the controller command u_c and the actuator response u, w is the plant disturbance, y is the sensor measurement corrupted by the noise v, and z is the output to be controlled.

Three design problems were proposed in [8]. The most demanding one requires: *i*) Closed-loop stability when the parameters m_1, m_2 and k are uncertain with mean value 1; *ii*) A 15 s settling time for unit disturbance impulse for the nominal plant $m_1 = m_2 = k = 1$; *iii*) The minimization of the control effort and of the controller complexity.

Many controllers were proposed for this problem. They are collected and analyzed in [9], where the authors, after evaluating the nominal performance, carry out a Stochastic Robustness Analysis in order to analyze the behavior when the plant parameters change. The six uncertain parameters are assumed to be uniform independent random variables in the following intervals: $0.5 < k < 2$, $0.5 < m_1 < 1.5$, $0.5 < m_2 < 1.5$, $0 < c < 0.1$, $0.9 < f < 1.1$ and $0.001 < \tau < 0.4$. We shall denote by $X \in \mathcal{X} \subseteq \mathbb{R}^6$ the vector of these uncertain parameters $X = \begin{bmatrix} k & m_1 & m_2 & c & f & \tau \end{bmatrix}^T$.

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Based on the specifications, our target is to design a fixed-structure controller such that: *i)* The nominal plant is stabilized; *ii)* The 15 s settling time specification is satisfied for the nominal plant; *iii)* The control effort does not exceed a one unit saturation limit in response to a unit w disturbance, for the nominal plant; *iv)* A certain cost function is minimized. This cost function accounts for the closed-loop stability and the performance in the presence of parameter variations.

3 Statistical Design of Fixed-Order Controllers

In this section, using a randomized algorithm which is described in detail and proven elsewhere [6], we shall design two fixed-order controllers. One of the characteristics of the algorithm is that the number of plants tested depends only on the accuracy and the confidence chosen but not on the complexity of the plant nor of the controller.

Denoting by $Y \in \mathcal{Y} \subseteq \mathbb{R}^m$ the vector of controller coefficients, the two chosen controllers have the following structures

$$K_1(s,Y) = \frac{a_0 s^2 + a_1 s + a_2}{s^2 + b_1 s + b_2} \qquad (1)$$

$$K_2(s,Y) = \frac{a_1 s^2 + a_2 s + a_3}{s^3 + b_1 s^2 + b_2 s + b_3} \qquad (2)$$

The coefficients of the controllers are chosen to have uniform distributions. For the controller (1) these coefficients take values in the intervals $a_0 \in [0.5, 10]$, $a_1 \in [-2, -0.5]$, $a_2 \in [-0.3, -0.1]$, $b_1 \in [1, 5]$, $b_2 \in [1, 6]$, whereas for the controller (2) they take values in $a_1 \in [-50, 50]$, $a_2 \in [-120, -40]$, $a_3 \in [-40, -10]$, $b_1 \in [70, 170]$, $b_2 \in [80, 160]$, $b_3 \in [100, 140]$.

In order to use the randomized algorithm methodology, this problem has been reformulated in the following way (see also [6], [5]). Let us define a cost function

$$\Psi(Y) = \max\{\psi_1(Y), \psi_2(Y)\} \qquad (3)$$

where

$$\psi_1(Y) = \begin{cases} 0 & \text{if all requirements on the nominal plant are met} \\ 1 & \text{otherwise} \end{cases}$$

and

$$\psi_2(Y) = E\left(\zeta(X, Y)\right), \qquad (4)$$

where E indicates the *expected value* with respect to X, and

$$\zeta(X,Y) = \begin{cases} 1 & \text{if the random plant is not stabilized} \\ 2/3 & \text{if both the control limit and the settling time specifications are not satisfied} \\ 1/3 & \text{if either the control limit or the settling time specification is not satisfied} \\ 0 & \text{otherwise} \end{cases}$$

Our aim is to minimize the cost function (3) over $\mathcal{Y}$. What we shall find is a *suboptimal* solution, a probably approximate near minimum of $\Psi(Y)$ with confidence $1-\delta$, level α and accuracy ϵ (see [10]), using the following Procedure, which was derived in [6].

Procedure

1. Let $k = 0$
2. Choose n controllers with random uniformly distributed coefficients $Y_1, \dots, Y_n \in \mathcal{Y}$, where (we indicate by $\lfloor \cdot \rfloor$ the floor operator)
$$n = \left\lfloor \frac{\log(2/\delta)}{\log[1/(1-\alpha)]} \right\rfloor$$
Evaluate for these controllers the function ψ_1 (4) and discard those controllers for which $\psi_1 = 1$. Let $\hat{n}$ be the number of the remaining controllers.
3. Choose m plants generating random parameters $X_1, \dots, X_m \in \mathcal{X}$ with uniform distribution, where
$$m = 2^k \left\{ \left\lfloor \frac{100}{\epsilon^2} \log\left(\frac{8}{\delta}\right) \right\rfloor + 1 \right\}$$
4. Evaluate the stopping variable
$$\gamma = \max_{1 \le j \le \hat{n}} \left| \frac{1}{m} \sum_{i=1}^{m} r_i \zeta(X_i, Y_j) \right|$$
where r_i are *Rademacher* random variables, i.e. independent identically distributed random variables taking values $+1$ and -1 with probability 1/2 each. If $\gamma \le \epsilon/5$, stop. The value of m is large enough to guarantee the required probability levels. If $\gamma > \epsilon/5$, let $k = k+1$ and go back to step 3
5. Choose the controller which minimizes the function
$$\frac{1}{m} \sum_{i=1}^{m} \zeta(X_i, \cdot)$$
This is the *suboptimal* controller in the sense defined above.

Remark 1 The proposed algorithm consists of two distinct parts: the estimate of the expected value in (4), which is given with an accuracy ϵ and a confidence $1 - \delta/2$, and the minimization procedure which is carried out with a confidence $1 - \delta/2$ and introduces the level α. As it can be seen from the Procedure, the number m of samples in $\mathcal{X}$ which are needed to achieve the estimate of the expected value (4), known as the *sample complexity*, is not known *a priori* but is itself a random variable. The upper bounds for this random sample complexity however, are of the same order of those that can be found in [4].

In both cases of our controllers, the procedure needed just one iteration to converge, i.e. $k = 0$. Therefore, for $\delta = 0.05$, $\alpha = 0.005$ and $\epsilon = 0.1$, n evaluated to 736 controllers and m evaluated to 50,753 plants. The *suboptimal* controllers are

$$K_1(s) = \frac{1.1110s^2 - 1.7393s - 0.2615}{s^2 + 3.6814s + 2.9353} \tag{5}$$

$$K_2(s) = \frac{31.9432s^2 - 76.6527s - 12.7876}{s^3 + 92.1586s^2 + 123.3358s + 131.8229} \tag{6}$$

and the corresponding values of the cost function are $\Psi_0 = 0.2683$ for (5) and $\Psi_0 = 0.2062$ for (6). As expected, with a more complex controller we get a better result.

4 Analysis of the Performance

In this section we shall compare the performance of the two controllers (5)–(6) with those of 4 other controllers analyzed in [9] with the same structure as the ones proposed here. The transfer functions of these 4 controllers can be found in [9].

The performance are quantified using a Monte Carlo evaluation. According to the distributions of the parameters, 20,000 plants (see [9] for the choice of this number) are randomly generated and estimates of the following three metrics are calculated: 1. P_I: Probability of instability. This is the probability that a randomly generated plant is not stabilized. 2. P_{T_S}: Probability of exceeding the settling time. This is the probability that the 15 s settling time specification is not satisfied. 3. P_u: Probability of exceeding the control limit. This is the probability that the actuator displacement will exceed a one unit saturation limit in response to a unit w disturbance.

Table 1: Robust Performance

Controller	P_I	P_{T_S}	P_u
K_1	0.002	0.803	0.002
C	0.041	0.874	0.041
E	0.125	0.999	0.409
K_2	0.033	0.547	0.033
A	0.165	0.793	0.165
B	0.039	0.963	0.047

The data for the controllers A, B, C and E are taken from [9]

The settling time and the control limit specifications were considered to be violated also when the controller fails to stabilize a plant. As shown in Table 1, the two controllers designed with the statistical approach exhibit a better behavior in all the three cases.

5 Conclusions

In this paper, we have illustrated the use of sequential learning algorithms in designing robust controllers for linear systems. This approach has been developed in [6] and presents a significant extension of the results of [5]. The approach is not limited to linear problems, nor to the finite-dimensional case as one can apply similar design steps to nonlinear systems and to delay-differential systems. In addition, discrete-time systems may be dealt with in exactly the same fashion.

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