43rd IEEE Conference on Decision and Control
December 14-17, 2004
Atlantis, Paradise Island, Bahamas

Suboptimal Control Techniques for Networked Hybrid Systems

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Abstract— **In this paper we discuss a feasible solution to the hybrid optimal switching problem, where the measurement and feedback signals are communicated via a network and undergo a constant delay. We utilize the concept of model prediction to compensate for the induced delay and periodic update of state measurements. A bound on the resulting estimation error is provided, and an example is provided to illustrate the method.**

I. Introduction

Over the past decade, major advancements in the area of communication and computer networks have led to their inclusion in real time feedback control systems design. A new paradigm in control systems analysis and design resulted, i.e., Networked Control Systems. *Networked Control Systems (NCSs), are control systems whose control loop is closed around a communications network*. Here, the feedback is no longer instantaneous as in classical control systems. Examples of NCSs include: Automotive industry, teleautonomy, teleoperation of robots, and automated manufacturing systems. Including the network into the design of systems has made it possible to increase mobility, reduce the cost of dedicated cabling, ease upgrading of systems, and render maintenance easier and cheaper. The drawback, however, is that the complexity of analysis and design increases manyfold (see [10] for an overview).

In this paper we explore another setting for networked systems, specifically a hybrid system operating in a suboptimal fashion via a network. Our framework is general enough to encompass several applications that have drawn special attention over the past decades teleautonomous robots, hybrid electric vehicles (HEV), and distributed fuel cell systems.

Such scenarios encompass a set of plants that interact and must be individually controlled (decentralized) and coordinated by a supervisory controller through a communication link to achieve some optimal performance. Assuming no disturbances, in the simplest case, sensor measurements to the supervisor are inherently delayed, and the control signals issued by the supervisor undergo a similar delay. Figure 1 illustrates the basic idea. The problem then is to robustly solve a hybrid optimal control problem in the presence of delayed measurements and delayed control inputs. The approach here is to combine recent advances in hybrid optimal control with the emerging work on model predictive control (MPC) to achieve a viable solution to this very important networked control system (NCS) problem.

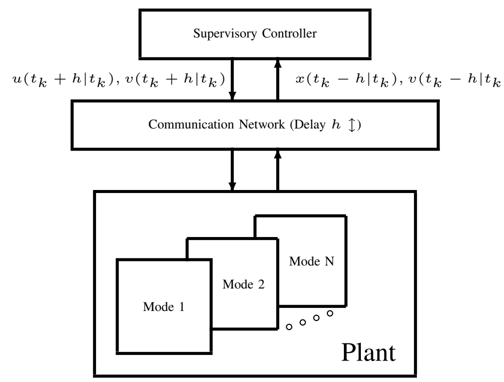

Fig. 1. Supervisory control of a multi-mode plant over a communication network with delays.

II. Problem Statement and Proposed Approach

We consider a plant that operates in one of a set of possible structures at any one time. This necessitates two control inputs: a continuous-valued input and a discrete-valued switching input that selects the proper structure at each instant of time. In the presence of state and input constraints the objective is to minimize a cost functional by appropriately selecting the structure and generating its controls.

In the absence of communication network delays, the problem can be solved by embedding the switched system into a larger family of systems and then solving a more general hybrid optimal control problem [3]. Implementation of a solution requires instantaneous and continuous availability of the state measurements, which is not possible due to the delays inherent in a real communication network. Nevertheless, we assume that the supervisory controller receives periodic updates of the plant state. These periodic updates of the delayed state initialize a possible uncertain prediction of the state trajectory for use in a MPC algorithm that also accounts for the communication delay to the plant.

For the rest of this paper we consider, for notational tractability, a switching plant model that consists of two possible modes of operation. However, the theoretical results can be scaled to encompass n-modes. In our subsequent analysis, we will use the notation pertinent to MPC: (i) for any control signal u, $u(t_1|t_2)$ is the value generated at

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(based on the information up to) time t_2 and that is predicted to be applied to the system at time $t_1(\geq t_2)$; (ii) for any state signal x, $x(t_1|t_2)$ is the value generated at time t_2 and is estimated for time t_1 (in this case t_1 and t_2 can be in any ordering relationship).

A. Problem Formulation

Within the aforementioned framework, we assume that the communication network imposes a constant delay h on the communicated signals as shown in Figure 1. Moreover, for minimizing the communication bandwidth, we assume that the supervisory controller receives the plant state, over the total optimization problem horizon $[t_0, t_f]$, only at times $t_k \in \{t_0+h, t_0+2h, ..., t_f-2h\}$; therefore, the supervisory controller receives an exact state measurement, $x(t_k-h)$. Using $x(t_k-h)$, the supervisor predicts the state trajectory forward in time and simultaneously solves an optimization problem to generate the optimal input $u^\star$ and switching signal $v^\star$ over the interval $[t_k+h, t_k+h+N_f]$, where N_f is a suitable multiple of h and represents a variable horizon. The interval of optimization begins with (t_k+h) to account for the communication delay from the supervisor to the plant. In accordance with MPC strategies [6], [11], only the "optimal" inputs $u^\star$ and $v^\star$ over the interval $[t_k+h, t_k+2h]$ are sent. The optimal control problem is then resolved at (t_k+h) for the interval $[t_k+2h, t_k+2h+N_f']$ where N_f' may be different from N_f. And the process repeats.

Of course, the time interval elapsed between successive plant state transmissions may be different than the communication delay. However, we make this assumption for avoiding excessive notation. Without loss of generality, we also assume that the computational time required to solve the optimization problems at each step is equal to zero (otherwise, the communication delay can be increased to include this time interval). Furthermore, the proposed method can be appropriately modified for handling situations where the communication delays on the receiving and transmitting channels are unequal.

The two-switched system model adopted in this paper has system state $x(t) \in \mathbb{R}^n$ at time t with dynamics

$$\dot{x}(t) = f_{v(t)}(t, x(t), u(t)), \qquad x(t_0) = x_0 \qquad (1)$$

where at each $t \geq t_0$, $v(t) \in \{0,1\}$ is the switching control, $u(t) \in \Omega \subset \mathbb{R}^m$ is the control input constrained to the convex and compact set Ω, and f_0 and f_1 are real vector-valued functions, $f_0, f_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, of class $\mathcal{C}^1$. The control inputs, $v(t)$ and $u(t)$, are both measurable functions. The state of the system described by (1) does not undergo jump discontinuities, an important assumption in this formulation. The optimization functional over the interval $[t_0, t_f]$ is: $J(x,u,v) = \int_{t_0}^{t_f} f^0_{v(t)}(t, x(t), u(t))dt$ where $f_0^0, f_1^0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are of class $\mathcal{C}^1$. The objective is: $\min_{v,u} J(x,u,v)$, subject to the constraints

(a) $x(\cdot)$ satisfies the (uncertain) model equation given by (1), with a given x_0;

(b) for each $t \in [t_0, t_f]$, $v(t) \in \{0,1\}$ and $u(t) \in \Omega$,

in the presence communication delays, and exact knowledge of the plant state x only at discrete instants, as described above. Relaxation of the above exact knowledge assumption is addressed later.

B. Proposed Suboptimal Solution

Our goal is to offer, in a deterministic context, a suboptimal solution to this problem. Admittedly, the optimization problem is not rigorously stated as one has limited information on the model uncertainties, and therefore, for a given control input, one does not exactly know the corresponding trajectory. Assuming certain structure on the model uncertainties, a rigorous formulation is possible in the context of stochastic optimal control. Remaining in a deterministic context, we make use of the concepts of MPC technique, whose advantages in dealing with uncertainties have been underlined in the literature (for example [6]). However, due to the delays in transmitting the control signal, we cannot directly apply this technique. The adapted strategy is described in the following.

As it usually occurs in engineering applications the plant model (1) is not exactly known. The plant model used by the supervisory controller is a nominal/estimation model described by

$$\dot{\hat{x}}(t) \quad = \quad \hat{f}_{v(t)}(t, \hat{x}(t), u(t)), \quad v(t) \in \{0,1\}. \qquad (2)$$

At each time instant $t_k \in \{t_0+h, t_0+2h, ..., t_f-2h\}$:

(i) The supervisory controller receives the exact plant state $x(t_k-h)$ and can update its estimate (of the plant model) made at time t_k for time t_k-h: $\hat{x}(t_k-h|t_k) = x(t_k-h)$;

(ii) Using the update $\hat{x}(t_k-h|t_k)$, and the control inputs $u([t_k-h, t_k]|t_k-2h), v([t_k-h, t_k]|t_k-2h)$ – generated at time t_k-2h for the interval $[t_k-h, t_k]$ – and $u([t_k, t_k+h]|t_k-h)$, $v([t_k, t_k+h]|t_k-h)$ – generated at time t_k-h for the interval $[t_k, t_k+h]$ –, the supervisory controller estimates the plant state up to the time instant t_k+h. Having the initial state estimate at t_k+h, we can initiate the following optimization problem, over the horizon $N_f(>h)$,

$\min_{v,u} \int_{t_k+h}^{t_k+h+N_f} f^0_{v(\tau|t_k)}(\tau, \hat{x}(\tau|t_k), u(\tau|t_k))d\tau$

subject to the constraints

  (a) $\dot{\hat{x}}(\tau|t_k) = \hat{f}v(\tau|t_k)(\tau, \hat{x}(\tau|t_k), u(\tau|t_k))$, for $\tau \in [t_k+h, t_k+h+N_f]$

  (b) for each $\tau \in [t_k+h, t_k+h+N_f]$, $v(\tau|t_k) \in \{0,1\}$ and $u(\tau|t_k) \in \Omega$.

(iii) Solving the optimization problem in (ii) produces the control signals $u([t_k+h, t_k+h+N_f]|t_k)$ and $v([t_k+h, t_k+h+N_f]|t_k)$ which, due to the delay h are applied to the plant on the desired interval $[t_k+h, t_k+2h]$.

The rigorously stated optimization problem at the second step is solved using the method indicated in Section III.

III. Piecewise Estimated Optimal Solution

In this section, we solve the subproblem of step (ii) in subsection II-B. This is a standard switched optimal control problem. Our approach is to embed the switched (estimation model) system (2) into a larger family of systems and to reformulate the problem for the latter. The viability of the approach is given by a certain relationship between the set of trajectories of the switched and embedded systems.

A. Switched Optimal Control Problem and a Reformulation

In step (ii) of subsection II-B, the controls $u([t_k+h, t_k+h+N_f]|t_k)$, and $v([t_k+h, t_k+h+N_f]|t_k)$ are the solution to a standard switched optimal control problem:

Switched optimal control problem (SOCP): minimize the cost $J_S(u,v) = \int\limits_{t_k+h}^{t_k+h+N_f} f^0_{v(t)}(t, \hat{x}(t), u(t))dt$ over the controls u and v, subject to the constraints

(a) $\hat{x}(\cdot)$ satisfies (2), with the initial condition $\hat{x}(t_k+h|t_k)$ computed as mentioned at step (ii) in subsection II-B;

(b) for each $t \in [t_k+h, t_k+h+N_f]$, $v(t) \in \{0,1\}$ and $u(t) \in \Omega$.

For related problem formulations and approaches see Branicky *et al.* [5], Giua *et al.* [8], Riedinger *et al.* [12], [13], Sussman [15], Xu and Antsaklis [16].

The difficulty of the SOCP stems in the presence of both continuous-time dyanmics–$\hat{f}_0$ and $\hat{f}_1$–and discrete events–the switching instants. To overcome this difficulty we embed the switching system (2) into a larger family of systems as follows: let $v(t)$ now take values in the interval $[0,1]$, i.e. $v(t) \in [0,1]$, and let $u_i(t)$ define the control input corresponding to vector field $\hat{f}_i$, $i = 0,1$; then define the family of systems parametrized by $v(t) \in [0,1]$ as

$$\dot{\hat{x}}(t) = [1-v(t)]\hat{f}_0(t,\hat{x}(t),u_0(t)) + v(t)\hat{f}_1(t,\hat{x}(t),u_1(t)) \quad (3)$$

with associated cost functional

$$J_E(v,u_0,u_1) = \int\limits_{t_k+h}^{t_k+h+N_f} \Big\{[1-v(t)]\cdot f^0_0(t,\hat{x}(t),u_0(t)) + v(t)\cdot f^0_1(t,\hat{x}(t),u_1(t))\Big\} \quad (4)$$

The optimal control problem of interest now becomes

Embedded optimal control problem(EOCP): minimize the functional (4) over all functions v, u_0, and u_1, subject to the following constraints:

(i) $\hat{x}(\cdot)$ satisfies equation (3), with initial condition $\hat{x}(t_k+h|t_k)$;

(ii) for each $t \in [t_k, t_k+h+N_f]$, $v(t) \in [0,1]$ and $u_0(t), u_1(t) \in \Omega$ (convex and compact).

The EOCP is a classical optimization problem for which we can study [2], [3] sufficient and necessary conditions for optimality via the classical tools of optimal control theory. The viability of our approach stems from the fact that the set of trajectories of the system (2) is dense in the set of trajectories of system (3). Each trajectory, $\hat{x}_E$, of (3) can be approximated within any desired accuracy, ϵ, by a trajectory, $\hat{x}_S$, of the (estimation model) switched system (2) corresponding to a proper choice of the switching input $v_\epsilon(\cdot)$, with values in the set $\{0,1\}$, and of the control input $u_\epsilon(\cdot)$.

Theorem 1: Let $u_0(t) \in \Omega$, $u_1(t) \in \Omega$, and $v(t) \in [0,1]$ be a control triplet for the embedded system (3). Let both systems, (2) and (3), have initial point $\hat{x}(t_k+h|t_k)$. Let $\hat{x}_E(\cdot)$ be a trajectory for the system (3) corresponding to $u_0(\cdot), u_1(\cdot), v(\cdot)$ on $[t_k+h, t_k+h+N_f]$. Then, for each $\epsilon > 0$, there are controls $v_\epsilon(t) \in \{0,1\}$ and $u_\epsilon(\cdot)$ defined on $[t_k+h, t_k+h+N_f]$, with the following properties: For almost all $t \in [t_k+h, t_k+h+N_f]$, $u_\epsilon(t) \in \Omega$, and the trajectory $\hat{x}_{S,\epsilon}(\cdot)$ (of the switched system (2) corresponding to the controls $v_\epsilon(\cdot)$ and $u_\epsilon(\cdot)$) satisfies: for all $t \in [t_k+h, t_k+h+N_f]$, $|\hat{x}_{S,\epsilon}(t) - \hat{x}_E(t)| < \epsilon$.

For a proof and a construction of the controls v_ϵ and u_ϵ see [2], [3]. The crucial role in the justification of this result is played by the Chattering Lemma as in [4], [7]. Since the integrands f^0_0 and f^0_1 of the cost functional J_E are of class $\mathcal{C}^1$, the approximating trajectory $\hat{x}_{S,\epsilon}$ (of the switched system (2)) can be constructed such that $|J_S(u_\epsilon, v_\epsilon) - J_E(u_0,u_1,v)| < \epsilon$. Therefore, if $\hat{x}_E$ is an optimal trajectory for the EOCP, then the trajectories $\hat{x}_{S,\epsilon}$ are suboptimal solutions for the SOCP.

B. Necessary Conditions for Optimality

Having motivated, in the previous sections, the study of EOCP, we are interested in sufficient and necessary conditions for optimality for EOCP. It can be shown (see [2] and [3]) that both the assumptions made in Section II-A and the satisfaction of the conditions S1–S4, specified below, guarantee that the EOCP has a solution.

(S1) $f_0(t,x,z_0) = A_0(t,x) + B_0(t,x)\cdot z_0$;

(S2) $f_1(t,x,z_1) = A_1(t,x) + B_1(t,x)\cdot z_1$;

(S3) for each (t,x), the function $f^0_0(t,x,z_0)$ is a convex function of z_0;

(S4) for each (t,x), the function $f^0_1(t,x,z_1)$ is a convex function of z_1.

For the remaining of this section, we presume that EOCP has a solution, $(\hat{x}^\star_E(t), u^\star_0(t), u^\star_1(t), v^\star(t))$. With the system given in (3) and the cost functional given by (4) we associate the Hamiltonian function $H : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\times[0,1]\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ defined as

$$\begin{aligned} &H(t,x,z_0,z_1,\mu,p^0,p) = p^0[(1-\mu)f^0_0(t,x,z_0) + \mu \\ &\cdot f^0_1(t,x,z_1)] + p^T[(1-\mu)\hat{f}_0(t,x,z_0) + \mu\hat{f}_1(t,x,z_1)] \\ &\triangleq \mu E_1(t,x,z_0,z_1,p^0,p) + E_2(t,x,z_0,z_1,p^0,p) \end{aligned} \quad (5)$$

with the obvious definition of the functions E_1 and E_2.

Let $T_k \triangleq \{t \in [t_k+h, t_k+h+N_f]| E_1(t,\hat{x}^\star_E(t),u^\star_0(t),u^\star_1(t),\lambda^0,\lambda(t)) = 0\}$. It can be shown ([3]) that for almost all $t \in [t_k+h, t_k+h+N_f]\setminus T_k$, $v^\star(t) \in \{0,1\}$.

Theorem 2: For almost all $t \in [t_k+h, t_k+h+N_f]\setminus T_k$,

(i) (MODE 0) If $\max\limits_{z_0\in\Omega} H(t,\hat{x}^\star_E(t),z_0,z_1,0,\lambda^0,\lambda(t)) > \max\limits_{z_1\in\Omega} H(t,\hat{x}^\star_E(t),z_0,z_1,1,\lambda^0,\lambda(t))$, then $v^\star(t) = 0$

and

$$\begin{aligned}\dot{\hat{x}}_E^\star(t) &= \hat{f}_0(t, \hat{x}_E^\star(t), u_0^\star(t)) \\ \dot{\lambda}(t) &= -p^0\left[\frac{\partial f_0^0}{\partial x}\right] - \left[\frac{\partial \hat{f}_0}{\partial x}\right]^T \lambda(t)\end{aligned} \quad (6)$$

with the above Jacobians calculated at $(t, \hat{x}_E^\star(t), u_0^\star(t))$. The optimal control $u_1^\star(t)$ is indeterminate in Ω, and $u_0^\star(t) = \arg\max_{z_0\in\Omega}[\lambda^0 f_0^0(t, \hat{x}_E^\star(t), z_0) + \lambda^T(t)\hat{f}_0(t, \hat{x}_E^\star(t), z_0)]$.

(ii) (MODE 1) If $\max_{z_1\in\Omega} H(t, \hat{x}_E^\star(t), z_0, z_1, 1, \lambda^0, \lambda(t)) > \max_{z_0\in\Omega} H(t, \hat{x}_E^\star(t), z_0, z_1, 0, \lambda^0, \lambda(t))$, then $v^\star(t) = 1$ and

$$\begin{aligned}\dot{\hat{x}}_E^\star(t) &= \hat{f}_1(t, \hat{x}_E^\star(t), u_1^\star(t)) \\ \dot{\lambda}(t) &= -p^0\left[\frac{\partial f_1^0}{\partial x}\right] - \left[\frac{\partial \hat{f}_1}{\partial x}\right]^T \lambda(t)\end{aligned} \quad (7)$$

with the above Jacobians calculated at $(t, \hat{x}_E^\star(t), u_1^\star(t))$. The optimal control $u_0^\star(t)$ is indeterminate in Ω, and $u_1^\star(t) = \arg\max_{z_1\in\Omega}[\lambda^0 f_1^0(t, \hat{x}_E^\star(t), z_1) + \lambda^T(t)\hat{f}_1(t, \hat{x}_E^\star(t), z_1)]$.

(iii) If $\max_{z_1\in\Omega} H(t, \hat{x}_E^\star(t), z_0, z_1, 1, \lambda^0, \lambda(t)) = \max_{z_0\in\Omega} H(t, \hat{x}_E^\star(t), z_0, z_1, 0, \lambda^0, \lambda(t))$, then either $v^\star(t) = 0$ or $v^\star(t) = 1$, and the above corresponding equations hold.

Proof: The above theorem is proved in [3].

In the singular cases the optimal switching control $v^\star(t)$ (with $t \in T_k$), may take values in the interval $(0, 1)$. However, from the constraints on the system – input constraints, state and costate equations (6),7) – it still may be possible to take $v^\star(t) \in \{0, 1\}$ for almost all $t \in T_k$, and one again can obtain a solution to the SOCP. Recall that, in the singular cases, if none of the solutions of the EOCP is of bang-bang type, then the SOCP may or may not have a solution. In such cases, suboptimal solutions of the SOCP may be constructed as in [3].

IV. Error Analysis

In this section we present some bounds pertaining to the performance of the system, when there is a discrepancy between the actual plant model and the estimation model used by the supervisor to obtain estimates of the state of the plant. Proved in [1], the following lemma is a slight generalization of the Gronwall's Inequality and is useful in our development on the error bounds.

Lemma 1: Let ρ, α, and β be non-negative real valued functions continuous on $[0, \infty]$, such that $\rho(t) \le \alpha(t) + \int_{t_0}^{t} \beta(s)\rho(s)ds \qquad \alpha \ge 0$ for all t_0, t in $[0, \infty)$. Let α, be monotonically increasing. Then $\rho(t) \le \alpha(t) \cdot e^{\int_{t_0}^{t} \beta(s)ds}$.

Consider the plant given in (1) and the estimation model (2). We assume that the difference between the two models is upper bounded for a given state, i.e. for each $t \in [t_0, t_f]$

$$\left| f(t, x, u) - \hat{f}(t, x, u) \right| \le g(t), \quad (8)$$

with g integrable. Note that the input in both equations is exactly the same since the supervisor knows the exact input that it generated and transmitted to the plant. We make the assumption that for each control input $u(t) \in \Omega$ the solution of equation (2) is bounded. Let $\mathcal{X}$ be a compact subset of $\mathbb{R}^n$ such that $x(t) \in \mathcal{X}$ for all $t \in [t_0, t_f]$. Without loss of generality, we assume that the set $\mathcal{X}$ is convex (otherwise we can consider it as a subset of a convex (compact) set). Since, for each $t \in [t_0, t_f]$ the function $\hat{f}_0(t, \cdot, \cdot)$ is $\mathcal{C}^1$ on the compact and convex set $\mathcal{X} \times \Omega$, it follows that the norm of its Jacobian is bounded, and, upon application of Theorem 9.19 in [14], it follows that there is a function $m(t)$ such that

$$|\hat{f}_0(t, x_1, u_1) - \hat{f}_0(t, x_2, u_2)| \le m(t) \left\| \begin{bmatrix} x_1 \\ u_1 \end{bmatrix} - \begin{bmatrix} x_2 \\ u_2 \end{bmatrix} \right\| \quad (9)$$

for all $x_1, x_2 \in \mathcal{X}$, and all $u_1, u_2 \in \Omega$. We make the assumption that the function $m(\cdot)$ is integrable. Without loss of generality, we also assume that the function $\hat{f}_1$ satisfies equation (9).

There are two scenarios that arise in such a setting. The perfect scenario where $f_i(., ., .) \equiv \hat{f}_i(., ., .)$ for $i = 0, 1$, which results in a perfect reconstruction of the state vector and hence both functions will have the same value in a time period of $2h$ after the system is turned on.

The more challenging scenario is when the system equations do not match exactly, which results in a discrepancy between the state of the plant and that of the estimation model. In this case, we need to obtain a bound on the estimation error. Since the supervisory controller receives updates on the plant states every h seconds, it is sufficient to study the estimation error on an interval $[t, t+h]$, assuming that, at time t, the supervisory controller receives the plant state at time $t-h$; i.e. $\hat{x}(t-h|t) = x(t-h)$. Utilizing the control inputs on the interval $[t-h, t]$, $u([t-h, t]|t-2h)$ and $v([t-h, t]|t-2h)$, and those on the interval $[t, t+h]$, $u([t, t+h]|t-h)$ and $v([t, t+h]|t-h)$, along with the triangle inequality and the bounds given by (8) and (9), we obtain that, for each $\tau \in [t, t+h]$,

$$|x(\tau) - \hat{x}(\tau|t)| \le \int_{t-h}^{\tau} g(s)ds + \int_{t-h}^{\tau} m(s)|x(s) - \hat{x}(s|t)|ds \quad (10)$$

Observe that the above inequality holds in fact for every $\tau \in [t-h, t+h]$ (although we are interested what happens only on the interval $[t, t+h]$). The function $\int_{t-h}^{\tau} g(s)ds$ is continuous, non-negative and monotonically increasing (because g is non-negative). Using Lemma 1, it follows that

$$|x(\tau) - \hat{x}(\tau|t)| \le \left(\int_{t-h}^{\tau} g(s)ds\right) \cdot \left(e^{\int_{t-h}^{\tau} m(s)ds}\right) \quad (11)$$

for all $\tau \in [t, t+h]$. Notice that in (11), if $g = 0$ then the error is zero; expectedly, since in this case the estimator is an exact replica of the plant model (see (8)). If the delay h equals zero, the error at each time $\tau \in [t, t+h]$ is bounded by $\left(\int_{t}^{\tau} g(s)ds\right) \cdot \left(e^{\int_{t}^{\tau} m(s)ds}\right)$. This estimation error occurs because for $\tau \in [t, t+h]$ the latest available information on the plant state is $x(t)$.

V. Example

In this section, we use the technique developed in this work to solve a version of the optimal control problem considered in [9]. The switched system model considered there is a crude two-dimensional model of a car with two gears. For the problem considered here, the state, x is the velocity of the car (according to some reference coordinate system), with some gears efficiencies, $g_0(\zeta)$ and $g_1(\zeta)$, as plotted in Figure 2.

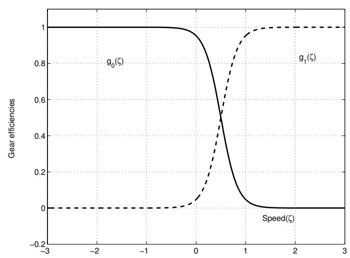

Fig. 2. Gear efficiencies

The dynamics corresponding to each mode are

$$\dot{x}(t) = f_{v(t)}(t, x(t), u(t)) = g_{v(t)}(x(t)) \cdot u(t) + r(t) \cdot x^2 \tag{12}$$

where $v(t) \in \{0, 1\}$, and $r(t)$ is a random function with values in the interval $[-0.06, +0.03]$, which roughly models the road imperfections, drag forces, etc., and the input-throttle command, u, is constrained to the set $\Omega = [-1, 1]$. Initially, the car is at rest, i.e. $x(0) = 0$, and is being controlled through a communication network with delay $h = 0.1$. The goal is to track with minimum control energy the following velocity profile: $x^{des}(t) = 0.5t$ on $[0, 2]$, $x^{des}(t) = 1$ on $[2, 4]$, and $x^{des}(t) = 0.5(4 - t) + 1$ on $[4, 6]$. Accordingly, the performance measure is chosen to be $J_S(v, u) = \int_0^6 \left\{10 \cdot [x(t) - x^{des}(t)]^2 + u^2(t)\right\} dt$.

On the other side of the network, the supervisor controlling the car has only access to the actual state value periodically and uses for the optimization problem the following dynamics of the estimate model:

$$\dot{\hat{x}}(t) = \hat{f}_{v(t)}(t, \hat{x}, u) = g_{v(t)}(\hat{x}) \cdot u, \quad v(t) \in \{0, 1\} \tag{13}$$

Applying the embedding technique discussed earlier to the estimation model results with the following state equation:

$$\dot{\hat{x}}(t) = [1 - v(t)] \cdot g_0(\hat{x}(t)) \cdot u_0(t) + v(t) \cdot g_1(\hat{x}(t)) \cdot u_1(t) \tag{14}$$

where $v(t) \in [0, 1]$ and the controls $u_0(t)$ and $u_1(t)$ are constrained to $[-1, 1]$. The functional cost associated with the embedded system is $J_E(v, u) = \int_0^6 \Big\{10 \cdot [\hat{x}(t) - \hat{x}^{des}(t)]^2 + [1 - v(t)]u_0^2(t) + v(t)u_1^2(t)\Big\} dt$.

For illustrating the suboptimal control technique, we consider that the supervisory controller solves the MPC problem over the following intervals (of type $[t_k + h, t_k + h + N_f]$, as mentioned in Section II-A): $[0.1, 2.1]$, $[2.1, 4.1]$, and $[4.1, 6]$.

In the following, on each of the aforementioned subintervals, an optimization is solved as indicated in step (ii) of Section II-B. Note that on each of these subintervals an optimal solution exists, which is denoted by $(\hat{x}^\star(t), u_0^\star(t), u_1^\star(t), v^\star(t))$. The Hamiltonian of system (14) is $H : \mathbb{R}^7 \to \mathbb{R}$

$$\begin{aligned} H(t, x, z_0, z_1, \mu, p^0, p) = \\ \mu \left\{ p^0[z_1^2 - z_0^2] + p[g_1(x)z_1 - g_0(x)z_0] \right\} \\ + p^0 \left\{ [(x - x^{des}(t)]^2 + z_0^2 \right\} + pg_0(x)z_0 \end{aligned} \tag{15}$$

Except on the last subinterval, the final state $\hat{x}(t_k + h + N_f)$ is free. Applying the transversality conditions ([4], page 190) it follows that $\lambda(t_f + h + N_f) = 0$, which in turns implies, from the Maximum Principle, that $\lambda^0 = -1$. On each subinterval, the solution $(\hat{x}^\star(t), u_0^\star(t), u_1^\star(t), v^\star(t))$ of the EOCP is not guaranteed to be a solution of the SOCP. However, that is guaranteed by the following proposition.

Proposition 1: There is an optimal switching control $v^\star(t)$ with values in the set $\{0, 1\}$. (See [1] for the proof)

A particular aspect of this example is that the optimal controls, $u_0^\star$ and $u_1^\star$, can be explicitly computed, as functions of the co-state, $\lambda(t)$. Therefore, if the initial value of lambda, $\lambda(t_k + h)$, and the initial state, $\hat{x}^\star(t_k + h)$, are known, the optimal controls and estimated optimal trajectories can be determined using the above equations for the two modes of operation. But, due to model uncertainties, it is not possible to exactly know the state of the system and therefore it is not possible to know beforehand the values $\lambda(t_k + h)$. However, for this example, the on-line computational burden can be reduced as described below.

The following computations are performed off-line.

(a) Since the delay $h = 0.1$, on the interval $[0, 0.1]$ the control is zero; hence the velocity at $t = 0.1$ is zero.

(b) Using $\hat{x}^\star(0.1) = 0$, the costate, $\lambda(0.1)$ can be determined numerically by comparing the costs associated with the trajectories and controls for values of λ in an appropriately chosen interval. This is achieved by generating the trajectories and controls over the interval $[0.1, 2.1]$ for each λ in an appropriately chosen (via simulations in this case) set.

(c) Using $\lambda(0.1)$, the corresponding estimated optimal trajectory, $\hat{x}^\star$, on $[0.1, 2.1]$, and the bounds on the uncertainties, an interval is determined for the state $x(2.1)$ of system (12). For each value of x in this interval, the optimal value of λ is generated as in described at (b), but now the problem is considered on the interval $[2.1, 4.1]$. Therefore, we numerically obtain a correspondence between the possible values of $x(2.1)$ and the corresponding $\lambda(2.1)$ which generate the optimal controls on the interval $[2.1, 4.1]$

(d) This part is similar to part (c), but it is performed for the interval $[4.1, 6]$. Another difference is that the correspondence between x and λ, at $t = 4.1$, is determined such that $x(6) = 0$.

The on-line procedure for the suboptimal control of this networked system is described in the following. On the interval $[0, 2.1]$, the optimal controls are computed off-line, since the initial state of (12) is known exactly. At $t = 2$, the supervisory controller receives the system state $x(1.9)$. Therefore, as explained at step (ii) in Section II-B, the state of the system (12) can be estimated. At $t = 2$, using the estimated state $\hat{x}(2.1)$, the value of $\lambda(2.1)$ is determined based on the off-line correspondence calculated as aforementioned at step (b). Using this value of $\lambda(2.1)$ the optimal controls on the interval $[2.1, 4.1]$ are generated. These computations require a small amount of time and, hence, are suitable for the on-line control scheme. At $t = 4$, the supervisory controller receives the state $x(3.9)$ (of the system (12)), and the procedure for generating the optimal controls on $[4.1, 6.1]$ follows the same steps as for the the interval $[2.1, 4.1]$.

The estimated trajectory (of the system (13)) and the trajectory of the system (12) are plotted in Figure 3. The discontinuities present at $t = 2$ and $t = 4$ in the estimated velocity are due to the updates based on the information received at these instances. The controls are illustrated in Figure 4. As expected, the gear switchings occur when the *estimated* velocity equals 0.5, where the two gear efficiencies are equal. Observe that the car velocity at $t = 6$ is slightly smaller than zero. This is due to the disturbances (uniformly distributed in $[-0.06, 0.03]$). For meeting the final constraint, the control $u(t)$ can be set to zero as soon as the velocity equals zero. Finally, for this problem the overall estimated cost is 2.9702 and the overall cost associated with the trajectory of system (12) is 3.0790.

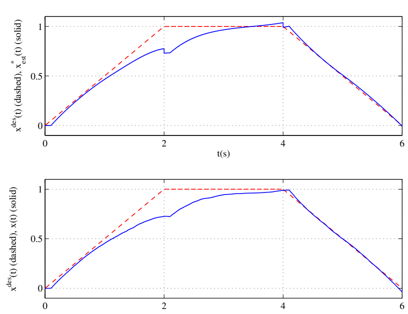

Fig. 3. Desired (dashed), Estimated (solid, upper), and Real (solid, bottom) Trajectories

VI. CONCLUSIONS

In this paper we solve the hybrid optimal switching control problem with delays introduced by a communication network that resides between the multi-mode plant and the supervisor. Due to the presence of delays, only a suboptimal solution can be obtained via utilizing a model-predictive approach and solving the optimal switching problem over subintervals. We illustrated our approach through a gear-switching example with input constraints. For future work we propose expanding our approach to include information losses between the plant and the supervisor, and studying the robustness of our method in that context.

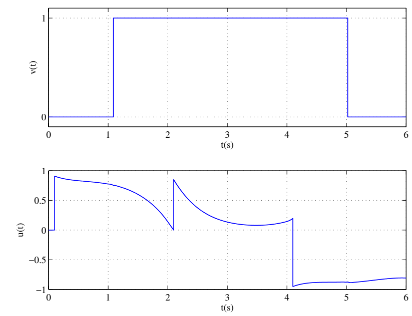

Fig. 4. Suboptimal Controls

REFERENCES

[1] Bengea, S.C., Hokayem P.F., DeCarlo R.A. and Abdallah C.T., "Suboptimal Control Techniques for Networked Hybrid Systems" in *Advances in Communication Control Networks*, S. Tarbouriech, C.T. Abdallah and J. Chiasson eds., Lecture Notes in Control and Information Sciences , Vol. 308, Springer-Verlag, NY, to appear.

[2] Bengea, S. C., DeCarlo, R. A., "Optimal and Suboptimal Control of Switching Systems," In *Proceedings of the IEEE Conference on Control and Decision*, Maui, Hawaii, Dec. 9-12, 2003, pp. 5295-5300.

[3] Bengea, S. C., DeCarlo, R. A., "Optimal Control of Switching Systems," To appear in *Automatica*, Vol. 41, No.1, Jan. 2005.

[4] Berkovitz, L. D., *Optimal Control Theory*, Springer-Verlag, New York, 1974.

[5] Branicky, M.S., Borkar, V.S., Mitter, S.K., "A Unified Framework for Hybrid Control: Model and Optimal Control Theory," in *IEEE Transactions on Automatic Control*, Vol. 43, No. 1, January 1998, pp. 31-45.

[6] Camacho, E.F. and C. Bordons, "Model Predictive Control," Springer-Verlag, London, 1998.

[7] Gamkrelidze, R. V., *Principles of optimal control theory*, Plenum Press, New York, 1978.

[8] Giua, A., Seatzu, C., Van Der Mee, C., "Optimal control of switched autonomous linear systems," in *Proceedings of the* 40^{th} *IEEE Conference on Decision and Control*, Orlando, Florida, USA, December 2001, pp. 2472-2477.

[9] Hedlund, S., Rantzer, A., "Optimal Control for Hybrid Systems," in *Proceedings of the* 38^{th} *IEEE Conference on Decision and Control*, Phoenix, Arizona, USA, December 1999, pp. 3972-3977.

[10] Hokayem, P.F. and Abdallah, C.T, "Inherent Issues in Networked Control Systems: A Survey," in *Proceedings of the American Control Conference*, Boston, MA, June 30-July 2, 2004.

[11] Maciejowski, J.M., "Predictive Control with Constraints," Prentice Hall, 2002.

[12] Riedinger, P., Zanne, C., Kratz, F., "Time Optimal Control of Hybrid Systems," in *Proceedings of the American Control Conference*, San Diego, California, June 1999, pp. 2466-2470.

[13] Riedinger, P., Kratz, F., Iung, C., Zanne, C., "Linear Quadratic Optimization for Hybrid Systems," in *Proceedings of the* 38^{th} *Conference on Decision and Control*, Pheonix, Arizona, USA, December 1999, pp. 3059-3064.

[14] Rudin, W., "Principles of Mathematical Analysis," McGraw-Hill, New York, 1976.

[15] Sussmann, H., "A maximum principle for hybrid optimal control problems, " in *Proceedings of the* 38^{th} *IEEE Conference on Decision and Control*, Pheonix, Arizona, USA, December 1999, pp. 425-430.

[16] Xu, X., Antsaklis, P.J., "Optimal Control of Switched Systems: New Results and Open Problems," in *Proceedings of the American Control Conference*, Chicago, Illinois, USA, June 2000, pp. 2683-2687.