

## Extreme-Point Stability Tests for Discrete-Time Polynomials

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### 1 INTRODUCTION

There has recently been an increasing interest in finding extreme-point results for the study of stability of uncertain polynomials. The main purpose of such results is to find a special family of uncertain polynomials, whose stability is equivalent to the stability of a finite number of polynomials. This work finds applications in the robust stabilization of interval plants [1]. The general D-stability of some special regions in the complex plane can also benefit of this technique.

Recently, the authors have extended the class of regions in the coefficients space where Schur stability of the extremes implies the stability of the entire family [2]. Our results allowed a wider class of coupling between the coefficients of the uncertain polynomials. Other extreme-point results are available [3], but require the uncertainty to have a special form, which is difficult to view in the coefficient space. It is well known that a four-corners or even an all-extremes result does not exist for discrete-time polynomials whose coefficients are varying independently. However, Hollot and Bartlett have shown that a variation in the upper-half of the parameters can be allowed with an all-extremes test being valid. [4]. In the present paper we extend the results in [2] and [4] when the degree of the polynomials under consideration is odd.

### 2 PRELIMINARY RESULTS

We will write the polynomials in the complex variable  $s$  with real coefficients in the form

$$P(s) = b_0 s^n + b_1 s^{n-1} + \dots + b_n \quad (1)$$

Throughout this paper, stability for a continuous-time polynomial means stability in the Hurwitz sense. The polynomials in the complex variable  $z$  with real coefficients will be written in the form

$$P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n \quad (2)$$

For these polynomials we will use stability and Schur stability interchangeably. We will first transform a robust stability problem from the discrete-time domain into a continuous-time robust stability problem using the bilinear transformation. To denote the largest integer less than  $x$  and the smallest integer greater than  $x$ , we will use  $[x]$  and  $\lceil x \rceil$ , respectively. Let

$$s = \frac{z+1}{z-1}; \quad z = \frac{s+1}{s-1} \quad (3)$$

Then, the polynomial given in (2) will be stable if and only if the transformed polynomial  $P(s)$  given by

$$P(s) = a_0(s+1)^n + a_1(s+1)^{n-1}(s-1) + \dots + a_n(s-1)^n \quad (4)$$

is stable. Any continuous-time polynomial  $P(s)$  as in (1) can be decomposed into

$$P(s) = P_e(s) + sP_o(s) \quad (5)$$

where  $P_e(s)$  and  $sP_o(s)$  contain, respectively, the even-degree and odd-degree terms of the polynomial  $P(s)$ .

**Definition 1** *The polynomial  $P(s)$  follows an Interlacing Decomposition, if the roots of  $P_e(s)$  and  $P_o(s)$  all lie on the imaginary axis, are different and interlace, and the root closest to the origin is one of  $P_e(s)$ .*

The well-known *Interlacing Property* is easily stated with the previous concept

**Lemma 1** *Let*

$$P(s) = b_0 s^n + b_1 s^{n-1} + \dots + b_n \quad (6)$$

*Then, the polynomial  $P(s)$  is stable if and only if  $b_1/b_0 > 0$  and  $P(s)$  follows an interlacing decomposition.*

We define the difference polynomial  $P_d(\omega)$  in the real variable  $\omega$  as  $P_d(\omega) = P_e(j\omega) - P_o(j\omega)$ . Let  $\omega_{di}$  denote the real positive roots of  $P_d(\omega)$  and form the series

$$0 \leq \omega_{d1} \leq \omega_{d2} \leq \dots \leq \omega_{dl} \quad (7)$$

where  $l$  is the number of roots.

**Corollary 1** *Suppose the polynomial  $P(s)$  of degree  $n$  follows an interlacing decomposition, then the real positive roots  $\omega_{di}$  of the polynomial  $P_d(\omega)$  are such that the series of  $P_e(j\omega_{di})$  takes alternating sign values when  $i$  goes from 1 to  $l$ . Furthermore, either  $l = \lceil (n-1)/2 \rceil$  or  $l = \lceil (n-3)/2 \rceil$ .*

**Proof:** see [5]

With the previous results at hand, we can state the following Lemma, which was first presented in [6].

**Lemma 2** *Let*

$$P(s, \lambda) = Q(s) + \lambda(as + b)P_d(s^2), \quad \lambda \in [0, 1] \quad (8)$$

*be a family of polynomials of constant degree  $n$ , then the stability of the family is equivalent to that of the extreme polynomials,  $P(s, 0)$  and  $P(s, 1)$ .*

**Proof:** See for example [6].

### 3 MAIN RESULTS

We will first show how the results in [4] can be extended for polynomials  $P(z)$  of odd degree. Let  $m = (n+1)/2$  and consider the family of polynomials

$$P(z, a_m) = a_0 z^n + a_1 z^{n-1} + \dots + a_n, \quad a_m \in [a_m^-, a_m^+] \quad (9)$$

where  $n$  is odd. Transforming every polynomial of the family into a continuous-time one by means of the bilinear transformation (4),

$$P(s, a_m) = \sum_{\substack{i=0 \\ i \neq m}}^n a_i (s+1)^{n-i} (s-1)^i + a_m (s+1)^{n-m} (s-1)^m \\ = P^*(s) + a_m (s+1)^{n-m} (s-1)^m \quad (10)$$

With the change of variable  $\lambda = (a_m - a_m^-)/(a_m^+ - a_m^-)$ , the family can be expressed as

$$P(s, \lambda) = P_0(s) + \lambda(a_m^+ - a_m^-)(s+1)^{n-m}(s-1)^m, \quad (11)$$

where  $\lambda \in [0, 1]$  and  $P_0(s) = P^*(s) + a_m^-(s+1)^{n-m}(s-1)^m$ . Clearly, the polynomial for  $\lambda = 0$  corresponds to  $a_m = a_m^-$ , and the polynomial for  $\lambda = 1$  to  $a_m = a_m^+$ . To use Lemma 2, we only need to show that the term  $(s+1)^{n-m}(s-1)^m$  can be written as  $(as+b)P_\Delta(s^2)$  which is obvious since

$$(s+1)^{n-\frac{n-1}{2}}(s-1)^{\frac{n-1}{2}} = (s+1)(s^2-1)^{\frac{n-1}{2}} \quad (12)$$

The family  $P(z, a_m)$  has no degree dropping, then the family  $P(s, \lambda)$  will not have degree dropping, and Lemma 2 can be combined with Theorem 2 in [4] to state the following theorem.

**Theorem 1** *The family of uncertain polynomials  $\mathcal{F}$  defined as*

$$\mathcal{F} = \{P(z) = \sum_{i=0}^n a_i z^{n-i} : a_j \in [a_j^-, a_j^+], j = \lceil \frac{n-1}{2} \rceil, \dots, n\} \quad (13)$$

*is stable if and only if all the polynomials in  $\mathcal{F}$  for which  $a_j$  is either  $a_j^-$  or  $a_j^+$ , with  $j = \lceil \frac{n-1}{2} \rceil, \dots, n$ , are stable.*

Note that the result in [4] only allows independent variations in coefficients  $a_j, j = \lceil \frac{n-1}{2} \rceil, \dots, n$ . In fact, for monic polynomials of degree 3, Theorem 1 proves that it is necessary and sufficient to check the eight extremes, as was shown in [7]. In order to allow coupling between the uncertain coefficients, we consider the family of polynomials

$$P(z, a_i, a_{n-1-i}) = a_0 z^n + a_1 z^{n-1} + \dots + a_n, \\ a_i \in [a_i^-, a_i^+], a_{n-1-i} \in [a_{n-1-i}^-, a_{n-1-i}^+] \quad (14)$$

where  $n$  is odd,  $i < (n+1)/2$  and where

$$a_i - a_i^- = a_{n-1-i} - a_{n-1-i}^- \quad (15)$$

If we transform the family into continuous-time using the bilinear transformation, we can write

$$P(s, a_i, a_{n-1-i}) = P_0(s) + \lambda(a_i^+ - a_i^-) \cdot \\ \{(s+1)^{n-i}(s-1)^i + (s+1)^{i+1}(s-1)^{n-i-1}\} \quad (16)$$

where  $\lambda = (a_i - a_i^-)/(a_i^+ - a_i^-)$ . Taking  $(s+1)$  as common factor in the second term, we can write

$$P(s, a_i, a_{n-1-i}) = P_0(s) + \lambda(a_i^+ - a_i^-)(s+1) \cdot \\ \{(s+1)^{n-1-i}(s-1)^i + (s+1)^i(s-1)^{n-i-1}\} \quad (17)$$

This latter factor can be shown [8] to be an even polynomial in  $s$ . Therefore, the conditions for Lemma 2 hold, so that the stability of the extremes is necessary and sufficient for the stability of the family of polynomials described by (14), and (15). This result can be combined with Theorem 4.3 in [2] to formulate the following general test.

**Theorem 2** *Consider a polytope of polynomials in the coefficients space where each pair  $(a_i, a_k), 0 \leq i \leq n, 2\lfloor n/2 \rfloor - i \leq k \leq n$  is varying inside a polygon with edges sloped in the interval  $[\pi/4, 3\pi/4]$  and where the pairwise variations  $(a_i, a_k)$  and  $(a_i, a_l), k \neq l$  are not allowed simultaneously. Then, every polynomial in the polytope will be stable if and only if all the polynomials obtained at all the polygon corners are stable.*

## 4 CONCLUSIONS

We have provided an extension to some robust stability results for discrete-time polynomials expressed directly in the coefficient domain by making use of extreme-point results for continuous-time polynomials. Specifically, we can allow independent variations, not only in the upper-half of the coefficients but also on the coefficient prior to the center one in the case where  $n$  is odd. The case of dependent coefficients was also extended from our previous results.

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