Proceedings of the 32nd Conference
on Decision and Control
San Antonio, Texas • December 1993

ALGEBRAIC TESTS FOR OUTPUT STABILIZABILITY

F. Pérez *, C. Abdallah †, P. Dorato †, and D. Docampo *

* ETSI Telecomunicacion, Universidad de Vigo, 36200-VIGO, SPAIN

† EECE Dept., University of New Mexico, Albuquerque, NM 87131, USA.

Abstract

In this paper, we provide algebraic tests to determine whether a linear Single-Input-Single-Output (SISO) system, is stabilisable with a constant output feedback.

1 Introduction

The problem of output stabilisability of linear systems remains one of the most challenging problems in systems theory. While it is true that many techniques exist to stabilize systems using only output measurements, the fundamental question of the existence of such controllers is still open. In other words, given a linear, time-invariant system (LTI), the existence of a constant output feedback that will stabilise the system can not in general be answered, short of using a root-locus or Nyquist approach that will actually answer the existence question by finding such a stabilising controller. One might argue that with the advent of graphing software, the question is moot since one can answer the question graphically for almost any LTI, SISO system. It is however important to obtain an algebraic answer to the stabilizability question for many reasons. First, a constant output feedback is the simplest member of the hierarchy of fixed-structure controllers, and an answer to the constant output feedback stabilizability might provide an answer to the more general fixed-structure controllers, where a graphical approach is not available. Second, the algebraic conditions may provide the designer with a negative answer to the stabilizability question without actually solving the problem. Finally, these conditions will provide an alternate view to the root-locus and Nyquist methods of analysis which may be extended to the robust stabilizability problem. We call the attention of the reader to the paper [1] for a related approach.

This paper is organized as follows: The problem is stated in section 2, our main results are given in section 3, while our conclusions are given in section 4.

2 Problem Statement

We consider the problem of stabilizing the SISO continuous-time, linear, time-invariant system described by the transfer function

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^l + \cdots + b_{l-1}s + b_l}{s^n + \cdots + a_{n-1}s + a_n},\ l < n \qquad (1)$$

connected in the standard feedback configuration, with the output feedback compensator $u = -ky + r$, so that the closed-loop system is described by

$$\begin{aligned} T(s) &= \frac{kG(s)}{1 + kG(s)} \\ &= \frac{k(b_0 s^l + b_1 s^{l-1} + \cdots + b_{l-1}s + l_n)}{p(s,k)} \end{aligned} \qquad (2)$$

where $p(s,k) = a(s) + kb(s)$. Let us decompose $p(s,k)$, $a(s)$ and $b(s)$ into their even and odd parts

$$\begin{aligned} p(s,k) &= p_e(s^2,k) + sp_o(s^2,k) \\ a(s) &= a_e(s^2) + sa_o(s^2) \\ b(s) &= b_e(s^2) + sb_o(s^2) \end{aligned} \qquad (3)$$

The approach we consider is to determine first the jw axis crossings w_i of the roots of $p(s,k)$, solve for the corresponding gains k_i and then determine whether a particular crossing is from the Left-Half-Plane (LHP) to the Right-Half-Plane (RHP) or vice-versa. By keeping track of the number and the direction of crossings, we will be able to answer the stabilizability question for a given $G(s)$.

3 Main Results

Let us then consider the closed-loop characteristic equation $0 = p(s,k)$, which becomes along the jw axis

$$\begin{aligned} 0 &= a(jw) + kb(jw) \qquad (4) \\ &= a_e(-w^2) + jwa_o(-w^2) + k[b_e(-w^2) + jwb_o(-w^2)] \\ &= [a_e(-w^2) + kb_e(-w^2)] + jw[a_o(-w^2) + kb_o(-w^2)] \\ &= [a_R(-w^2) + kb_R(-w^2)] + j[a_I(-w^2) + kb_I(-w^2)] \end{aligned}$$

where $x_R(jw) = x_e(jw)$ and $x_I(jw) = wx_o(jw)$. Therefore, setting both real and imaginary parts to zero, we can eliminate k and obtain

$$Y(-w^2) = a_R(-w^2)b_I(-w^2) - a_I(-w^2)b_R(-w^2) = 0 \qquad (5)$$

The positive real roots of this equation $w_i,\ i = 1, \cdots, m$ represent the positive jw axis crossings. We can then find the corresponding gains as in [2]

$$\begin{aligned} k_i &= -a(jw_i)/b(jw_i);\ i = 1, \cdots, m \\ &= -a_I(-w_i^2)/b_I(-w_i^2);\ i = 1, \cdots, m \\ &= -a_R(-w_i^2)/b_R(-w_i^2);\ i = 1, \cdots, m \end{aligned} \qquad (6)$$

and order them as $k_1 < k_2 < \cdots < k_m$. Let us assume that $a(s)$ has at least one root in RHP. Otherwise, a small enough value of k which stabilizes $p(s,k)$ always exists. The closed-loop system will be stabilized if, at some k, all n roots are in the LHP. We then have the following results.

Lemma 1 *The output stabilizbility problem is solvable if and only if at least one of the m polynomials $p(s,k_s)$ is stable, where k_s is any value which satisfies $k_{i-1} < k_s < k_i$; $i = 1, \cdots, m$.*

Proof: Obvious. ■

Lemma 2 *Suppose that $p(s)$ has a single root at $s = jw_i - \epsilon$, for a small real $\epsilon > 0$. Then the angle of $p(jw)$ is a strictly increasing function of w at w_i, i.e., $\frac{\partial}{\partial w}\arg\{p(jw)\}\mid_{w=w_i} > 0$.*

Proof: The proof can be obtained by writting $p(jw) = (jw + \epsilon - jw_i)R(jw)$, $R(jw_i - \epsilon) \neq 0$ and deriving its argument. ■

We will next present a lemma and its proof for the special case where only one branch of the root locus crosses the jw axis at a particular k_i. The more general case where q roots cross the jw axis admits an indentical test and its proof may be found in [3].

Lemma 3 *A complex conjugate pair crosses the jw axis*

1. *From the LHP to the RHP at $\pm jw_i$ if and only if*

$$\frac{\partial}{\partial w}[Y(-\omega^2)] \mid_{w=w_i} > 0$$

2. *From the RHP to the LHP at $\pm jw_i$ if and only if*

$$\frac{\partial}{\partial w}[Y(-\omega^2)] \mid_{w=w_i} < 0$$

Finally, the roots stay in one half-plane if

$$\frac{\partial}{\partial w}[Y(-\omega^2)] \mid_{w=w_i} = 0;$$

Proof: We will only prove case 1) for the case where one branch of the root locus crosses the jw axis at $k = k_i$. At $k = k_i - \epsilon$, for a small $\epsilon > 0$, we have a pair of complex conjugate roots in the LHP, but close to the jw axis. Then, by Lemma 2,

$$\frac{\partial}{\partial w} arg\{a(jw) + (k_i - \epsilon)b(jw)\} \mid_{w=w_i} > 0$$

In the following, we drop the explicit dependence on w and w_i, to obtain

$$\begin{aligned}
& \frac{\partial}{\partial w} arg\{a(jw) + (k_i - \epsilon)b(jw)\} \mid_{w=w_i} > 0 \\
\Longleftrightarrow\; & \frac{\partial}{\partial w} Arctan\{\frac{a_I + k_i b_I - \epsilon b_I}{a_R + k_i b_R - \epsilon b_R}\} > 0 \\
\Longleftrightarrow\; & [a_I' + (k_i - \epsilon)b_I'][a_R + (k_i - \epsilon)b_R] \\
& > [a_R' + (k_i - \epsilon)b_R'][a_I + (k_i - \epsilon)b_I] \\
\Longleftrightarrow\; & [a_I' + (k_i - \epsilon)b_I'][-\epsilon b_R] - [a_R' + (k_i - \epsilon)b_R'][-\epsilon b_I] > 0 \\
\Longleftrightarrow\; & -(a_I' + k_i b_I')b_R + (a_R' + k_i b_R')b_I - \epsilon(b_I' b_R - b_R' b_I) > 0
\end{aligned}$$

then, since ϵ is arbitrarly small, and using (6),

$$\begin{aligned}
& (a_R' + k_i b_R')b_I - (a_I' + k_i b_I')b_R > 0 \\
\Longleftrightarrow\; & a_R' b_I - a_I b_R' - a_I' b_R + a_R b_I' > 0 \\
\Longleftrightarrow\; & \frac{\partial}{\partial w}[a_R(jw)b_I(jw) - a_I(jw)b_R(jw)] \mid_{w=w_i} > 0
\end{aligned}$$

∎

Lemma 4 *A complex conjugate pair crosses the jw axis from the LHP to the RHP at $\pm jw_i$ if and only if*

$$k_i \frac{\partial}{\partial w} arg\{b(jw)/a(jw)\} \mid_{w=w_i} < 0$$

Proof: Consider

$$k_i \frac{\partial}{\partial w}[arg(b(jw)/a(jw))] \mid_{w=w_i} < 0$$

In the following, we drop the explicit dependence on w and w_i, to obtain

$$\begin{aligned}
& k_i \frac{\partial}{\partial w}[arg(b(jw)/a(jw))] \mid_{w=w_i} < 0 \\
\Longleftrightarrow\; & k_i \frac{\partial}{\partial w}[arg\{ba^*\}] < 0 \\
\Longleftrightarrow\; & k_i \frac{\partial}{\partial w}(\frac{-b_R a_I + b_I a_R}{b_I a_I + a_R b_R}) < 0 \\
\Longleftrightarrow\; & k_i(-b_R a_I + b_I a_R)'(b_I a_I + a_R b_R) \\
& < k_i(b_I a_R + a_R b_R)'(-b_R a_I + b_I a_R)
\end{aligned}$$

but using (6),

$$\begin{aligned}
-b_R a_I + b_I a_R &= \frac{1}{k_i}(b_R b_I - b_R b_I) = 0 \\
b_I a_I + a_R b_R &= \frac{1}{k_i}(-a_I^2 - a_R^2) \qquad (7)
\end{aligned}$$

therefore, (7) is satisfied if an only

$$(b_I a_R - b_R a_I)' < 0 \qquad (8)$$

which is condition 1) in Lemma 3. Therefore, the lemma is proven. ∎

4 Conclusions

In this paper we have provided algebraic conditions for the stabilizability of SISO systems with constant gains. The conditions are simple, testable, and may be extended to the robust stabilizability problem as will be reported on in a future paper.

References

[1] M. Prakash and A. Fam, "A geometric approach to stabilization by output feedback," *Int.J. Control*, vol. 37, no. 1, pp. 11–125, 1983.

[2] H. Chapellat and S. Bhattacharyya, "An alternative proof of Kharitonov's theorem," *IEEE Trans. Auto. Control*, vol. AC-34, pp. 448–450, 1989.

[3] F. Pérez, C. Abdallah, P. Dorato, and D. Docampo, "Output stabilizability," *Technical Report, UNM: EECE93-005, Albuquerque, NM, 87131*, 1993.