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Guaranteed-Cost Control of Polynomial Nonlinear Systems

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Abstract

This paper deals with the control of polynomial nonlinear systems which are affine in the control. We use Bernstein polynomials and Polytopic Linear Differential Inclusion (PLDI) to design gain-scheduled controllers using a guaranteed cost framework.

1 Introduction

One popular method of dealing with nonlinear systems is to use linear robust control methodologies. In such approach, the nonlinearities are assumed to be uncertain parameters. If the nonlinearities appear in a special form, the nonlinear system is a Polytopic Linear Differential Inclusion. However, approximating a nonlinear system as a PLDI usually requires over-bounding the nonlinearities with sector bounds. This could result in potentially severe conservatism, since there are many trajectories of the PLDI which are not trajectories of the nonlinear system, in addition to the conservatism introduced due to using a single quadratic Lyapunov function. Fortunately, the first type of conservatism can be overcome for a special class of nonlinear systems as done in this paper.

This paper is organized as follows: a brief introduction to PLDIs is presented in section 2 along with a discussion of Bernstein polynomials and their properties. In section 3, stability conditions for PLDIs with gain-scheduled state feedback is presented followed by the guaranteed-cost LQ design. In section 4, we present a numerical example and our conclusions are presented in section 5.

2 PLDIs

A PLDI can be written as, $\dot{x} = A(t,x)x$ where $A(t,x)$ is given by:

$$A(t,x) = \sum_{i=1}^{r} \alpha_i(t,x) A_i \qquad (1)$$

and where $\{A_1, \cdots, A_r\}$ are known matrices and $\alpha_1, \ldots, \alpha_r$ are positive scalars which satisfy $\sum_{i=1}^{r} \alpha_i(t,x) = 1$. Using global linearization [1], we can use PLDIs to study properties of nonlinear time varying systems. In fact, consider the system, $\dot{x} = f(t,x,u)$ and let the Jacobian of the system matrix $A(t,x) = \frac{\partial f}{\partial x}$ lie in the convex hull defined in (1), then every trajectory of the nonlinear system is also a trajectory of the LDI defined by Ω (See [1] for more details).

2.1 Bernstein Polynomials

Bernstein polynomials of degree n are defined as:

$$B_{i,n}x = \left(\begin{array}{c} n \\ i \end{array} \right) x^i (1-x)^{n-i}$$

for $i = 1, \cdots n$ and where

$$\left(\begin{array}{c} n \\ i \end{array} \right) = \frac{n!}{i!(n-i)!}.$$

Note that there are $n+1$ nth-order Bernstein polynomials and that we set $B_{i,n} = 0$ if $i < 0$ or $i > n$. Bernstein polynomials are given recursively by

$$B_{k,n}(x) = (1-x)B_{k,n-1}(x) + xB_{k-1,n-1}(x) \qquad (2)$$

The important property of Bernstein polynomials which makes them useful in the context of PLDIs is the fact that they are all non-negative over the interval $[0,1]$, and one can show that $\sum_{i=0}^{n} B_{i,n}(x) = 1$, i.e., they form a partition of unity. Most importantly, Bernstein polynomials of order n form a basis for polynomials of degree less than or equal to n. Although we limit our discussion to univariate Bernstein polynomials, the results presented can be extended to multivariate Bernstein Polynomials with minor modifications.

Our control approach is based on normalizing the states to the closed interval $[0,1]$, and then writing the nonlinear system as a convex combination of linear systems with Bernstein polynomials being the coefficients of these convex combinations. We can then use LMI methods to design gain-scheduling controllers for the nonlinear system. The overall controller for the original nonlinear system is obtained by aggregating the linear parts. Stability conditions for these systems were first given in [6] in the context of model-based fuzzy systems and later relaxed and transformed into Linear Matrix Inequalities which are efficiently solvable using interior-point convex optimization methods [1, 5]. While the approach of the authors in [7] is based on approximation of the nonlinear system as a fuzzy blending of local linear models, the representation of the polynomial nonlinear systems as convex combination of linear models is exact. The techniques used for guaranteed-cost performance are based on [4].

3 LMI-Based Designs

As mentioned earlier, let

$$\dot{x} = f(x) + g(x)u = A(x)x + g(x)u$$

where all entries of $A(x)$ and $B(x)$ are polynomials. Furthermore, we assume that the states components are already normalized to the closed interval [0,1]. We first write (3) as

$$\dot{x} = \sum_{i=1}^{r} \alpha_i(x)(A_i x + B_i u) \tag{3}$$

where $\alpha_i(x) > 0$ are Bernstein polynomials, and $\sum_{i=1}^{r} \alpha_i(x) = 1$. The following structure is chosen for the gain-scheduled controller:

$$u = -\sum_{j=1}^{r} \alpha_j(x) K_j x. \tag{4}$$

We then obtain the following closed-loop system:

$$\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i(x)\alpha_j(x)(A_i - B_i K_j)x \tag{5}$$

3.1 Stability

Stability conditions are given in the following theorem.

Theorem 1 *[7]:The closed-loop system (5) is globally asymptotically stable, if the pairs (A_i, B_i) are stabilizable , and there exist a common positive-definite matrix P which satisfies:*

$$\begin{aligned} (A_i - B_iK_i)^T P + P(A_i - B_iK_i) &< 0 \quad i = 1, \cdots, r \\ G_{ij}^T P + PG_{ij} &< 0 \quad j < i \le r \\ P &> 0 \end{aligned} \tag{6}$$

where G_{ij} is defined as $G_{ij} = A_i - B_iK_j + A_j - B_jK_i$.

Remark 1 *If $B_i = B$ for all values of i, the second set of inequalities in terms of G_{ij} are redundant.*

Pre-multiplying and post-multiplying both sides of the inequalities in (6) by P^{-1} and using the following change of variables

$$Y = P^{-1}; \quad X_i = K_i Y$$

we obtain the following LMIs [4]:

$$\begin{aligned} YA_i^T + A_iY - B_iX_i - X_i^TB_i^T &< 0 \\ Y(A_i + A_j)^T + (A_i + A_j)Y - M_{ij} - M_{ij}^T &< 0 \\ Y &> 0 \\ i = 1, \cdots, r; \quad j < i \le r \end{aligned}$$

where M_{ij} is defined as:

$$M_{ij} = B_iX_j + B_jX_i \tag{7}$$

The feasibility of the above LMIs guarantees stability, but in most practical problems, stability is just a primary goal and performance is also usually required. Next, we develop a guaranteed-cost framework for the design of nonlinear controllers [4].

3.2 Guaranteed-Cost Design

Consider the problem of minimizing the quadratic performance index:

$$J = \mathbb{E}_{x_0} \int_0^\infty (x(t)^T Q x(t) + u(t)^T R u(t)) dt \tag{8}$$

subject to : The PLDI in (3). It was shown in [4] that this problem can be transformed into the following optimization problem:

Min $tr(P)$

Subject to:

$$(A_i - B_iK_i)^T P + P(A_i - B_iK_i) + Q + \sum_{i=1}^{r} K_i^T R K_i < 0$$

$$G_{ij}^T P + PG_{ij} + Q + \sum_{i=1}^{r} K_i^T R K_i < 0$$

$$i = 1, \cdots, r \quad j < i \le r$$

where M_{ij} is the same as in (7). Using the change of variables in (7) and the Schur Complement lemma [1, 2], the inequalities above can be transformed into the following LMIs:

$$\begin{bmatrix} N_i & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ Q^{1/2}Y & -I_{n\times n} & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & -I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & \cdots & -I_{m\times m} \end{bmatrix} < 0$$

$$\begin{bmatrix} O_{ij} & YQ^{1/2} & X_1^T R^{1/2} & \cdots & X_r^T R^{1/2} \\ Q^{1/2}Y & -I_{n\times n} & 0 & \cdots & 0 \\ R^{1/2}X_1 & 0 & -I_{m\times m} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R^{1/2}X_r & 0 & 0 & \cdots & -I_{m\times m} \end{bmatrix} < 0$$

$$Y > 0$$

$$i = 1, \cdots, r \quad j < i \le r \tag{9}$$

where N_i and O_ij are defined as follows:

$$\begin{aligned} N_i &= YA_i^T + A_iY - B_iX_i - X_i^TB_i^T \\ O_{ij} &= Y(A_i + A_j)^T + (A_i + A_j)Y \\ &\quad -M_{ij} - M_{ij}^T \end{aligned} \quad (10)$$

To obtain the least possible upper-bound using a quadratic Lyapunov function, we have to solve the following optimization problem

Min $tr(Y^{-1})$
Subject To: LMIs in(9)
This is a convex optimization problem which can be solved in polynomial time [5] using any of the available LMI toolboxes. To make it possible to use *Matlab*'s LMI Toolbox, we introduce an artificial variable Z, which is an upper bound on Y^{-1}, and minimize $tr(Z)$ instead, i.e, we recast the problem in the following form

Min $tr(Z)$
Subject To LMIs in(9), and

$$\begin{bmatrix} Z & I_{n\times n} \\ I_{n\times n} & Y \end{bmatrix} > 0 \quad (11)$$

If the above LMIs are feasible, we can calculate the controller gains as $K_i = X_iY^{-1}$ The global controller can then be obtained as in (4).

4 Numerical Example

Consider the following third-order nonlinear system:

$$\begin{aligned} \dot{x} &= A(x)x + Bu; \quad u = -k(x)x \\ A(x) &= \begin{bmatrix} -2x_1^2 & -x_1^2 - 2(x_1 + 1 + x_2^2) + x_2 & 0 \\ -\frac{3x_1}{4} & -2.75x_1^2 + 2x_1 - 0.7x_2 - \frac{x_2^2}{2} & -1 \\ 0 & 0 & 0 \end{bmatrix} \\ B &= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \end{aligned} \quad (12)$$

Since the entries in the $A(x)$ are all polynomials of degree at most 2, we use the following basis:

$$\{\frac{1}{2}x_1^2, x_1(1-x_1), \frac{1}{2}(1-x_1)^2, \frac{1}{2}x_2^2, x_2(1-x_2), \frac{1}{2}(1-x_2)^2\}$$

Since B is a constant vector, there are only six LMIs which need to be solved. By solving the corresponding optimization problem discussed in the previous section, we obtain values for $\{K_i\}_{i=1}^6$. After obtaining values for controller gains, the control action is computed using (4). Simulation results will be shown at the presentation.

5 Conclusions

In this paper we developed a guaranteed-cost framework for designing controllers for a class of polynomial nonlinear systems. Using a Bernstein polynomial expansion the nonlinear plant was represented as a PLDI. Gain-scheduling type controllers were designed using LMI based optimization methods which would minimize an upper bound on a quadratic performance index. The results can be extended to the case where the entries of the $A(x)$ matrix are multivariate polynomials using the multivariate expansion of Bernstein polynomials. The proposed method suffers from potentially severe conservatism due to the choice of a single quadratic Lyapunov function and to the requirement that all state variables are normalized to the [0,1] interval.

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