Tracking Control of Uncertain Systems

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Abstract

This paper deals with the problem of designing output
tracking controllers for uncertain systems. The sys-
tems we consider may be non-minimum phase but are
restricted to be linear. The problem is motivated by
control applications where a desired output trajectory
is specified, and the corresponding input to the system
is to be found.

1 Introduction

Mathematically, the output-tracking problem consid-
ered may be studied as an inverse problem. The output
tracking has a long history but more recently, Devasia
et. al. [1] initiated a new line of research by studying
stable inversion as a viable approach to exact-output
tracking. This research has also benefited from con-
tributions by Hunt et. al. [2, 3]. At this stage, it is
understood that the stable inversion of non-minimum
phase systems (linear or nonlinear, time invariant or
time varying) is possible at least for a large class of
trajectories. The basic idea is that rather than finding
an inverse for all trajectories, one only need be con-
cerned with the inverse of the particular trajectory to
be tracked. The resulting input then has a noncausal
component and is of the feedforward variety.

This paper uses a different approach to study a simi-
lar but more general problem in the linear system case.
The previously mentioned results assume given the sys-
tem dynamics even though some robustness issues have
been studied in [4, 5]. In many cases, the system under
study is highly uncertain and what we are given are
samples of the desired output trajectory, rather than a
closed-form time-dependent trajectory. Our approach
relies on concepts from statistical learning theory and
on the results of a recent paper by Efromovich and

Koltchinski [6]. As such, our approach has much in
common with the recent book by Vidyasagar [7], and
other recent papers which attempt to answer some con-
trol questions statistically.

2 Mathematical Set-Up

The authors in [6] were interested in estimating a func-
tion $f \in L_2([0,1])$ which is observed in Gaussian white
noise modeled by the stochastic differential equation
$dY^\epsilon(t) = (Hf)(t)dt + \epsilon dW(t)$ where H is a linear op-
erator, $\epsilon > 0$ and W is a standard Brownian motion.
This is again an inverse problem and applications of
the approach to signal processing and communication
theory is described by the authors of [6]. The novelty
of their results lie in the facts that H may be unknown
and that the problem may be ill-posed due to the non-
invertibility of H. In the setting of tracking control,
the output is the desired trajectory to be tracked, H
is an unknown system and f corresponds to the input
to be found. The non-invertibility of H may be due
to it being non-minimum phase. In order to use the
setup of [6], we first assume that samples of the desired
output are given even though they may contain ran-
dom errors, and that the system is unknown. We then
use statistical learning theory [7] to generate training
input functions along with the corresponding (possi-
bly noisy) outputs. Based on this data, a data-driven
learning machine is obtained to estimate the desired
input for the original tracking problem.

More specifically, we review the mathematical setup of
[6]: Let $f \in \mathcal{F} \subseteq \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space, and
let H be a linear operator from $\mathcal{H}$ into another Hilbert
space $\mathcal{R}$ such that our unknown system satisfies the
linear operator equation $Hf = g$. We do not assume
that the system H is known but rather that the func-
tion class $\mathcal{F}$ is given. The problem we study is then to
find the input f such that the output g tracks a given
trajectory, when we only have available noisy samples
of the desired trajectory.

Suppose an orthonormal basis is given in $\mathcal{H}$ by $\{e_i; i \geq 1\}$ and an orthonormal basis of $\mathcal{R}$ is $\{\psi_j; j \geq 1\}$.

Both bases will be assumed known. We denote inner products over a Hilbert space by $\langle.,.\rangle$. The operator H defines the infinite matrix $(h_{ij})_{i,j=1}^{\infty}$, where $h_{ij} = \langle He_j, \psi_i\rangle$. Note that one can approximate the possibly infinite-dimensional H by the $n \times n$ matrixes $H_n = (h_{ij})_{i,j=1}^{n}$. Suppose that we are given a noisy sequence of the desired outputs $Y_j = \langle Hf, \psi_j\rangle + \epsilon\eta_j$; $j = 1, 2, \cdots$ and that we can generate a training set of data $X_{(ji)} = h_{ji} + \sigma\zeta_{ji}$; $i, j = 1, 2, \cdots$, where the random variables $\{\eta_j, \zeta_{ji}; i, j = 1, 2, \cdots\}$ are i.i.d and Normal, and where ϵ, σ are nonnegative real numbers. The results of [6] then place conditions on the different mathematical objects so that we can obtain an asymptotic or a minimax estimate of the desired tracking input as ϵ and σ tend to zero. The studied risk is the minimax integrated squared error

$$R(\hat{f}, f) = \inf_{\hat{f}} \sup_{f \in \mathcal{F}} \mathbb{E}||\hat{f} - f||^2 \qquad (1)$$

We know define a few more terms to facilitate the statement of the main results: Let $\mathcal{H}_n$ denote the subspace of $\mathcal{H}$ spanned by $\{e_1, \cdots e_n\}$ and $\mathcal{R}_n$ be the subspace of $\mathcal{R}$ spanned by $\{\psi_1, \cdots \psi_n\}$. Let P_n and Π_n denote the orthogonal projections on $\mathcal{H}_n$ and $\mathcal{R}_n$, and $f_n = P_n f$. Let $||A||$ be the operator norm of A and $||A||_H = \sqrt{\text{Tr}(A^T A)}$ be the Hilbert norm of A. We let $H_n = \Pi_n H P_n$ be an operator from $\mathcal{H}$ into $\mathcal{R}$. If we restrict H_n to be an operator from $\mathcal{H}_n$ to $\mathcal{R}_n$, we obtain a matrix $[H]_n$ which is the $n \times n$ principal submatrix of H_n. We assume throughout that for a large enough n, $[H]_n$ is invertible. Then, we let $T_n = H_n^{-1} H$ be a projection operator from $\mathcal{H}$ onto $\mathcal{H}_n$. Next, let

$$X_n = (X_{(ji)})_{j,i=1}^{n}; \quad \Xi_n = (\zeta_{ji})_{ji=1}^{n} \qquad (2)$$

so that $X_n = [H]_n + \sigma\Xi_n$. The projection estimate is then given by $\hat{f}_n = \hat{H}_{n,\gamma}^{-1} Y^{(n)}$ where

$$\begin{aligned} Y^{(n)} &= \sum_{j=1}^{n} Y_j \psi_j; \quad = \Pi_n H f + \epsilon\eta^{(n)} \\ &= \Pi_n H f + \epsilon\sum_{j=1}^{n} \eta_j \psi_j \qquad (3) \end{aligned}$$

and

$$\hat{H}_{n,\gamma}^{-1} = \begin{cases} X_n^{-1} & \text{if } X_n \text{ is nonsingular and } ||X_n^{-1}|| \leq \gamma \\ 0 & \text{otherwise} \end{cases}$$

In particular, we consider the following form for γ, $\gamma = 2\frac{\alpha(\epsilon)}{\epsilon\sqrt{n}}$ where $\alpha(\epsilon)$ is a nondecreasing function of $\epsilon > 0$ such that $\alpha(\epsilon) = o(\log^{-1/2}(\epsilon^{-1}))$. Finally, we denote

$$A_n = ||H_n^{-1}||; \quad B_n = ||H_n^{-1}||_H; \quad \delta(\sigma, n) = \sigma\sqrt{n}A_n \quad (4)$$

The following theorem is then obtained:

Theorem 1 *[6] Suppose that*

$$\sup_{\delta(\sigma,n)\leq\delta} \log(\sigma^{-1}B_n^{-1} = o(\delta^{-2})$$

$$\gamma \geq 2A_n$$

$$\sup_{\delta(\sigma,n)\leq\delta} \log\gamma = o(\delta^{-2})$$

Then, for some $\beta > 0$ and for any $C > 1$, there exists $\delta_0 > 0$ such that for all σ, ϵ, and n satisfying

$$\delta(\sigma, n) \leq \frac{\delta_0}{\sqrt{\log(2 + ||Hf||)}}; \quad n \leq \frac{1}{\epsilon^2}$$

the following holds

$$\mathbb{E}||\hat{f}_n - T_n f|| \leq CB_n^2(\sigma^2||T_n f||^2 + \epsilon^2) + e^{-\beta\delta^2(\sigma,n)}$$

Moreover, if $\sigma \leq c\epsilon$, the following holds

$$\mathbb{E}||\hat{f}_n - T_n f|| \leq CB_n^2(\sigma^2||T_n f||^2 + \epsilon^2)$$

The above theorem illustrates the fact that as n increases, one is able to obtain a more accurate estimate of the desired tracking input $\hat{f}$. In fact, one can find the minimum number of samples needed to achieve a certain accuracy in the estimate of $\hat{f}$. Note also that if ϵ and σ are zero (i.e. deterministic case), one can guarantee the convergence of the estimate to zero as n increases. In the full version of the paper, we illustrate the main theorem using numerical examples and present other relevant results from paper [6] which may be applied to control problems.

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