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Model-based Networked Control for Finite-Time Stability of Nonlinear Systems: The Stochastic case

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Abstract— **In this paper we analyze model-based networked control systems for a discrete-time nonlinear plant model, operating in the presence of stochastic dropout of state observations. The dropout is modeled as a Markov chain, and sufficient conditions for finite-time stochastic stability are provided using the stochastic version of Lyapunov second method. In a companion paper we model the dropout as a deterministic sequence.**

I. INTRODUCTION

In several recent works, the problem of networked control systems (NCS) has been posed and partially investigated [2], [4], [5], [3]. This new problem deals with the possibility of controlling a system remotely via a communication network and as such, instantaneous and perfect signals between controller and plant are not achievable.

In [6] a model for the networked control of linear time invariant systems was proposed. The network is modelled as a sampler placed between the plant and sensors on one side, and the controller on the other side of the network. Utilizing an approximate model of the process at the controller's side, the controller can maintain stability while receiving only periodic updates of the actual state of the plant. Whenever a new update is received, the model plant is initialized with the new information. This idea was utilized in [1], where the system evolved in discrete-time, and state updates were either received or dropped at each sample due to the effect of the network. The characterization of such a dropout is achieved through the use of Markov chain that takes on values of 0 or 1 depending on whether a sample was lost or received, respectively. Recently in [7], the initial model for a continuous-time plant and a network modelled with a fixed rate sampler was extended to bounded yet random variable sample times driven by a Markov chain.

In this paper, we present an extension of the discrete-time systems in [1] into a nonlinear setting, i.e. our plant and the model used for state estimation are both nonlinear. We utilize the same model of packets being dropped according to a Markov chain, and obtain results that guarantee finite-time stability in a stochastic Lyapunov setting.

The paper is organized as follows: In Section II, we reformulate the model-based networked control problem in the nonlinear setting with stochastic packet dropouts. Section III briefly describe some finite-time stochastic stability results, that are then used in section IV in a networked control setting, sufficient conditions for finite-time stochastic stability of the networked control system are provided. Section V extends the previous result to design. Finally in Section VI we present some examples to illustrate our results, and our conclusion in section VII.

A companion paper titled "Model-based Networked Control for Finite-Time Stability of Nonlinear Systems: The Deterministic Case" has also been submitted to the conference and deals with the same problem using deterministic models. The first two sections of the current paper and the companion paper are identical in order to make them as self-contained as possible.

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II. PROBLEM FORMULATION

In [1] a discrete-time model-based control with observation dropouts is proposed for linear discrete-time systems. Our objective in this paper is to propose a similar framework in the case of nonlinear systems, and to study finite-time stochastic stability of the closed-loop system. As depicted in Figure 1, discrete-time model-based control is comprised of a plant with the network residing between the sensors of the plant and the actuators.

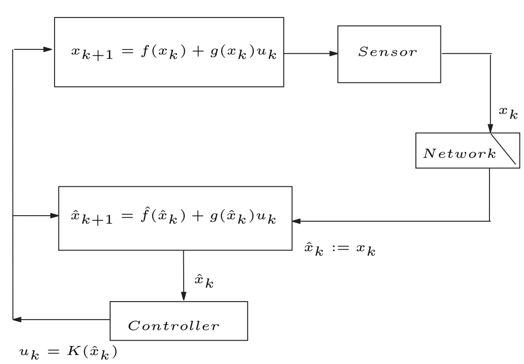

Fig. 1. Model-Based NCS

The network is modelled as a Markov chain θ_k, where a measurement is dropped if $\theta_k = 0$, and a measurement is received when $\theta_k = 1$. Due to our inability to receive an update of the plant's state at each discrete instant of time, we use an inexact model plant on the controllers side that provides us with the missing measurement. Such a model is given by

$$\hat{x}_{k+1} = \hat{f}(\hat{x}_k) + \hat{g}(\hat{x}_k)u_k. \quad (1)$$

In order to carry out the analysis, we define the estimation error as $e_k = x_k - \hat{x}_k$, and augment the state vector with e_k so that the closed-loop state vector is given by $z_k = \left(x_k^T; e_k^T\right)^T$. The closed-loop system evolves according to

$$\begin{aligned} z_{k+1} &= \begin{pmatrix} f(x_k) \\ (f(x_k) - \hat{f}(x_k)) + .. \\ \quad (1-\theta_k)((\hat{f}(x_k) - \hat{f}(\hat{x}_k))) \end{pmatrix} \\ &+ \begin{pmatrix} g(x_k)K(\hat{x}_k) \\ (g(x_k) - \hat{g}(x_k))K(\hat{x}_k) + .. \\ \quad (1-\theta_k)(\hat{g}(x_k) - \hat{g}(\hat{x}_k))K(\hat{x}_k) \end{pmatrix} . \end{aligned} \quad (2)$$

In the above model $\theta_k \in \{0, 1\}$ is a Markov chain that indicates the reception ($\theta_k = 1$) or the loss (θ_k=0) of the packet containing the state measurement x_k. If a packet is received, it is used as an initial condition for the next time step in the model, otherwise the previous state of the model is used. We then classify the NCS errors as follows:

(I). Model structure errors

$$\begin{aligned} e_{f1}(x_k) &= f(x_k) - \hat{f}(x_k) \quad (3) \\ e_{g1}(x_k) &= g(x_k) - \hat{g}(x_k). \quad (4) \end{aligned}$$

These are the errors between the plant and the model evaluated at the plant's state, and are therefore dependent on the system's structure.

(II). State dependent errors

$$\begin{aligned} e_{f2}(x_k, \hat{x}_k) &= \hat{f}(x_k) - \hat{f}(\hat{x}_k) \quad (5) \\ e_{g2}(x_k, \hat{x}_k) &= \hat{g}(x_k) - \hat{g}(\hat{x}_k). \quad (6) \end{aligned}$$

These represent the errors between the model evaluated at the plant's state and at its own state, i.e. the error introduced by the difference in the states.

(III). Structure and state dependent errors

$$\begin{aligned} e_{f3}(x_k, \hat{x}_k) &= f(x_k) - \hat{f}(\hat{x}_k) \quad (7) \\ e_{g3}(x_k, \hat{x}_k) &= g(x_k) - \hat{g}(\hat{x}_k), \quad (8) \end{aligned}$$

which include both model structure and state dependent errors.

With the new notation, the system (2) becomes

$$z_{k+1} = \begin{pmatrix} f(x_k) + g(x_k)K(\hat{x}_k) \\ e_{f1}(x_k) + e_{g1}(x_k)K(\hat{x}_k) + .. \\ \quad (1-\theta_k)(e_{f2}(x_k, \hat{x}_k) + e_{g2}(x_k, \hat{x}_k)K(\hat{x}_k)) \end{pmatrix}$$

Based on the value of θ_k we have two possible situations:

1. for $\theta_k = 1$ the system will be

$$z_{k+1} = \begin{pmatrix} f(x_k) + g(x_k)K(\hat{x}_k) \\ e_{f1}(x_k) + e_{g1}(x_k)K(\hat{x}_k) \end{pmatrix} \quad (9)$$

2. for $\theta_k = 0$

$$z_{k+1} = \begin{pmatrix} f(x_k) + g(x_k)K(\hat{x}_k) \\ e_{f3}(x_k, \hat{x}_k) + e_{g3}(x_k, \hat{x}_k)K(\hat{x}_k) \end{pmatrix} \quad (10)$$

For the remainder of this paper we use the following compact form to represent the system above, which also highlights the fact that θ_k represents packet dropouts,

$$z_{k+1} = H_1(z_k) + H_2(z_k)(1-\theta_k), \ k \geq 0 \quad (11)$$

with

$$\begin{aligned} H_1(z_k) &= F_1(z_k) + G_1(z_k)K(\hat{x}) \quad (12) \\ H_2(z_k) &= F_2(z_k) + G_2(z_k)K(\hat{x}). \quad (13) \end{aligned}$$

$$\begin{aligned} F_1(z_k) &= \begin{pmatrix} f(x_k) \\ e_{f1}(x_k) \end{pmatrix}, \ F_2(z_k) = \begin{pmatrix} 0 \\ e_{f2}(x_k, \hat{x}_k) \end{pmatrix} \\ G_1(z_k) &= \begin{pmatrix} g(x_k) \\ e_{g1}(x_k) \end{pmatrix}, \ G_2(z_k) = \begin{pmatrix} 0 \\ e_{g2}(x_k, \hat{x}_k) \end{pmatrix} \end{aligned}$$

While the control law has no access to the plant's state, we assume in the analysis of the global system full-state availability (i.e. both x_k and $\hat{x}_k$ available). Moreover, we assume that the control law $u_k = K(\hat{x}_k)$ stabilizes the model plant and in the case of full-state availability, it also stabilizes the plant.

Next we define a particular class of NCS for which we characterize the accuracy of the model in representing the plant's dynamics, and describe how the model discrepancy affects the NCS structure.

Definition 1: A model-based NCS of the form (11), belongs to a class C_{B-NCS} with the bounds $(B_f, B_g, B_{efi}, B_{egi}; B_{h_i})$, $i = 1, 2$ if for all $k \in \mathbb{N}$ and for all $x_k \in S \subset \mathbb{R}^n$, the system structure and error norms are bounded as follows

$$\begin{aligned} &||f(x_k)|| \leq B_f, \ ||g(x_k)u(\hat{x}_k)|| \leq B_g(\hat{x}_k) \\ &||e_{f1}(x_k)|| \leq B_{ef1}, \ ||e_{f2}(x_k, \hat{x}_k)|| \leq B_{ef2}(\hat{x}_k) \\ &||e_{g1}(x_k)u(\hat{x}_k)|| \leq B_{eg1}(\hat{x}_k) \\ &||e_{g2}(x_k, \hat{x}_k)u(\hat{x}_k)|| \leq B_{eg2}(\hat{x}_k) \end{aligned}$$

where B_f, B_{ef1} are constant bounds and $B_g(\hat{x}_k)$, $B_{ef2}(\hat{x}_k)$, $B_{eg1}(\hat{x}_k)$, $B_{eg2}(\hat{x}_k)$ are bounds that depend on the model state. Such NCS are called bounded model-based NCS (B-MB-NCS).

The above definition describes the class of NCS, for which it is possible to define bounds on the plant and the NCS errors, and where such bounds depend only on the model's state.

Next we state a lemma that describes properties of class C_{B-NCS}. In particular the lemma describes how bounds on the norm of the B-MB-NCS errors imply bounds on the norm of the NCS dynamics.

Lemma 1: Consider the NCS (11) and assume the system belongs to class C_{B-NCS}. Then the following bounds hold on the norm of the NCS dynamics for

$i, j = \{1, 2\}$, $j \neq i$, $k \in \mathbb{N}$ and for all $x_k \in S \subset \mathbb{R}^n$,

$$H_i^T H_j \leq B_{H_{i,j}}(\hat{x}_k), \ H_i^T H_i \leq B_{H_i}(\hat{x}_k) \quad (14)$$

where the bounds on the vector functions are related to the bounds on the errors as follows:

$$\begin{aligned}
B_{H_1}(\hat{x}_k) = \quad & (B_f + B_g(\hat{x}_k)) + (B_{ef1} + B_{eg1}(\hat{x}_k)) \\
& +2(B_f B_g(\hat{x}_k)) + 2(B_{ef1} B_{eg1}(\hat{x}_k)) \\
B_{H_{1,2}}(\hat{x}_k) = \quad & (B_{ef1} B_{ef2}(\hat{x}_k) + B_{eg1}(\hat{x}_k) B_{eg2}^T(\hat{x}_k)) \\
& +B_f B_g(\hat{x}_k) + (B_{ef1} B_{eg2}(\hat{x}_k) \\
& +B_{eg1}(\hat{x}_k) B_{ef2}^T(\hat{x}_k)) \\
B_{H_2}(\hat{x}_k) = \quad & (B_{ef2} + B_{eg2}(\hat{x}_k)) + 2((B_{ef2} B_{eg2}(\hat{x}_k))
\end{aligned}$$

The proof of the above lemma can be found in [13].

Lemma 2: Consider the NCS (11), belonging to class $C_{B-NCS}(B_f, B_g, B_{efi}, B_{egi}; B_{h_i})$, $i = 1, 2$ then for all $x_k \in S \subset \mathbb{R}^n, \forall k \in \mathbb{N}$

$$||x_k|| \leq B_x(\hat{x}), \ ||e_k|| \leq B_e(\hat{x}), \ ||z_k|| \leq B_z(\hat{x}) \tag{15}$$

where

$$\begin{aligned}
B_x(\hat{x}) &= B_f + B_g(\hat{x}_k), \ B_z(\hat{x}) = B_x(\hat{x}) + B_e(\hat{x}) \\
B_e(\hat{x}) &= B_{ef1} + B_{eg1}(\hat{x}_k) + B_{ef2}(\hat{x}_k) + B_{eg2}(\hat{x}_k)
\end{aligned}$$

Proof: The first two inequalities just follow from (2), (14). The second part trivially follows from $||z_k|| = (||x_k|| + ||e_k||) \leq (B_x(\hat{x}) + B_e(\hat{x})) = B_z(\hat{x})$ ■

III. STOCHASTIC FINITE-TIME STABILITY

Next, we describe how finite-time stability, which was originally defined for deterministic systems may be extended to stochastic systems. Consider a discrete time, stochastic dynamical system

$$x_{k+1} = f(x_k, \theta_k), \, x \in \mathbb{R}^n, \, x(0) = x_0 \tag{16}$$

Where x is the system state, and $f : \mathbb{R}^n \times \mathcal{B} \to \mathbb{R}^n$ is a vector function, $\mathcal{B}$ is the family of Borel subsets of points on $\mathbb{R}$; also $\{\theta_k\}$ is a stationary independent random sequence, with mean $\mu_\theta = \mathbb{E}[\theta_k] = \mathbb{E}[\theta_k^2]$ and variance σ_θ, which makes x_k a Markov process in $\mathbb{R}^n$. In stochastic dynamical systems it is meaningful to consider the probability for the trajectory not to exceed a given bound over a finite time interval. Therefore we consider the following definitions

Definition 2: [9] Consider the dynamical stochastic system (16), the associated inclusion probability with respect to $(\alpha, \beta, N, ||.||)$ is defined as follows:

$$P_{in}(x_k; \alpha, \beta, N) = P\{||x_k|| \leq \beta : 0 \leq k \leq N; ||x_0|| \leq \alpha\}$$

Definition 3: Consider the dynamical stochastic system (16), the associated exit probability with respect to $(\alpha, \beta, N, ||.||)$ is defined as follows:

$$P_{ex}(x_k; \alpha, \beta, N) = P\{\sup_{N \geq k \geq 0} ||x_k|| > \beta; ||x_0|| \leq \alpha\}$$

Note that $P_{ex}(x_k; \alpha, \beta, N) = 1 - P_{in}(x_k; \alpha, \beta, N)$. Therefore, we define stochastic finite-time stability:

Definition 4: The dynamical system (16) is Finite Time Stochastic Stable (FTSS) with respect to $(\alpha, \beta, N, \lambda, ||.||)$ if $P_{in}(x_k; \alpha, \beta, N) \geq (1 - \lambda)$, $or P_{ex}(x_k; \alpha, \beta, N) < \lambda$.

We will show next how FTSS can be indirectly determined by studying the exit and inclusion probabilities associated with a function $V(x_k, k)$ defined for the dynamical system.

A. Bounds on Exit Probability

In order to analyze and to eventually design for the finite-time stability of a process, we provide in this section upper bounds on the exit probability of the process (16) and on the associated function V_k. These upper bounds will allow us to indirectly study the FTSS of the system. The first theorem we present is from [8].

Theorem 1: [8] Consider a discrete-time Markov process x_k, $k = 0, 1, \ldots$. Also consider the function $V(x_k, k) = V_k \geq 0$ and the open set $S_\gamma = \{x_k : V_k \leq \gamma\}$. If the following conditions are satisfied $\forall x_k \in S_\gamma$, $\phi_k \geq 0$

$$\begin{aligned}
&\mathbb{E}_{x_k}[V(x_{k+1}, k+1)] \leq \infty \, \forall x_k \in S_\gamma, \\
&\mathbb{E}_{x_k}[V(x_{k+1}, k+1) - V(x_k, k)] \leq \phi_{k+1}
\end{aligned}$$

Then for the initial condition $x(0) = x_0$ we have

$$P_{ex}(V_k; \gamma_0, \gamma, N) \leq \frac{[V_0 + \Phi_N]}{\gamma} \tag{17}$$

where $\Phi_N = \sum_{i=1}^N \phi_i$

Proof: See [8]. ■

The last theorem gives an upper bound for the exit probability of V_k. This upper bound depends on the initial conditions through V_0, on the desired bound through γ, and on the time interval and state dynamics indirectly through Φ_N. Next, we bound the exit probability of the state dynamics of (16) directly.

Theorem 2: Consider the dynamical system (16) and its exit probability with respect to $(\alpha, \beta, N, ||.||)$, $P_{ex}(x_k; \alpha, \beta, N)$, also consider the function V_k as described previously, we have the following upper bound

$$P_{ex}(x_k; \alpha, \beta, N) \leq \mathbb{E}\left[\frac{\sup_{N \geq k \geq 0} ||x_k||}{\beta}; ||x_0|| \leq \alpha\right]$$

Proof: The proof easily follows from Chebychev inequality [12]. In the following, I is the indicator function, for brevity $I = I_{\{\sup_{N \geq j \geq 0} ||x_j|| > \beta\}}$. Also recalling that $P(x \leq t) = \mathbb{E}[I_{x \leq t}]$, then

$$\begin{aligned}
P_{ex}(x_k; \alpha, \beta, N) &= P\{\sup_{N \geq k \geq 0} ||x_k|| > \beta; ||x_0|| \leq \alpha\} \\
&= \mathbb{E}\left[I(\sup_{N \geq j \geq 0} ||x_j||); ||x_0|| \leq \alpha\right] \\
&\leq \mathbb{E}\left[\frac{\sup_{N \geq k \geq 0} ||x_k||}{\beta}; ||x_0|| \leq \alpha\right]
\end{aligned}$$

■

Again the bound on $P_{ex}(x_k; \alpha, \beta, N)$ is directly related to the bounds on the state α, β, to the state dynamics, and to the time interval.

B. Stochastic Finite-Time Stability Analysis

In the previous section we showed how the exit probability relative to the state dynamics x_k and to the associated function $V(x_k, k)$ can be bounded and how the bound depends on the parameters describing the finite-time stability objective. In this section we use the described bound to provide sufficient conditions for FTSS stability of system (16).

Theorem 3: Consider the dynamical system (16) and a function V_k such that for given δ_1, δ_2 we have $\delta_1||x_k|| \leq V(x_k,k) \leq \delta_2||x_k||$, and $\gamma = \beta\delta_1$, $\gamma_0 = \alpha\delta_2$, $V_0 \leq \gamma_0$, $\delta_1 > 0, \delta_2 > 0$. Then the system is finite-time stochastically stable with respect to $(\alpha, \beta, N, ||.||, \lambda)$, if any of the following three conditions is satisfied

(i) $\forall x_k \in S_\gamma$, $\phi_k \geq 0$

$$\mathbb{E}_{x_k}[V_{k+1}] \leq \infty,\ \mathbb{E}_{x_k}[\Delta V_k] \leq \phi_{k+1}$$

$$\frac{[\alpha\delta_2 + \Phi_N]}{\beta\delta_1} \leq \lambda,\ \Phi_N = \sum_{k=1}^{N} \phi_k,$$

(ii)

$$\mathbb{E}\left[\frac{\sup_{N\geq k\geq 0}||x_k||}{\beta}; ||x_0|| \leq \alpha\right] \leq \lambda \quad (18)$$

(iii) $\forall k = 0, \ldots, N$, $\forall x_k \in S_\gamma$, $\rho_k > -1$

$$P\{\Delta V_k \leq \rho_k V_k\} \geq (1-\lambda) \quad (19)$$

$$\frac{\gamma}{\gamma_0} \geq \sup_k \prod_{i=0}^{k-1}(1+\rho_i) \quad (20)$$

Proof: In order to prove the above statements we verify that $(i)-(iii)$ imply finite-time stability for the system. Finite-time stability easily follows from point (i) considering that for $\delta_1||x_k|| \leq V(x_k,k) \leq \delta_2||x_k||$, $\forall k = 0, \ldots, N$ and $\gamma_0 = \delta_2\alpha$, $\gamma = \delta_1\beta$ we have $P_{ex}(x_k; \alpha, \beta, N) \leq P_{ex}(V_k; \gamma_0, \gamma, N)$, and therefore from theorem 1 and (i) follows $P_{ex}(x_k; \alpha, \beta, N) \leq \lambda$. Now recalling that $P_{ex}(x_k; \alpha, \beta, N) + P_{in}(x_k; \alpha, \beta, N) = 1$ we have that finite-time stability for the system (16) with respect to $(\alpha, \beta, N, ||.||, \lambda)$ i.e. $P_{in}(x_k; \alpha, \beta, N) \geq (1-\lambda)$. For point (ii), from the upper bound on $P_{ex}(x_k; \alpha, \beta, N)$ provided in theorem 2, with the same principle as before directly follows that $P_{ex}(x_k; \alpha, \beta, N) \leq \lambda$, and therefore $P_{in}(x_k; \alpha, \beta, N) \geq (1-\lambda)$. Finally for the proof of point (iii) let us consider the following for $\rho_k > -1$ and $\forall k = 0, \ldots, N$, $P\{\Delta V_k \leq \rho_k V_k\} = P\{V_{k+1} - (1+\rho_k)V_k \leq 0\}$. Then iterating the partial difference inequalities and considering the upper bound on $V_0 \leq \gamma_0$ we get $\forall k = 0, \ldots, N$ $P\{\Delta V_k \leq \rho_k V_k\} \leq P\{V_k \leq \gamma_0 \prod_{i=0}^{k-1}(1+\rho_i)\}$. Then using the condition (20) from (iii) it follows that $\forall k = 0, \ldots, N$, $P\{\Delta V_k \leq \rho_k V_k\} \leq P\{V_k \leq \gamma\}$ and moreover $\forall k = 0, \ldots, N$ $(1-\lambda) \leq P\{\Delta V_k \leq \rho_k V_k\} \leq P\{V_k \leq \gamma\}$, that implies finite time stability with respect to $(\alpha, \beta, N, ||.||, \lambda)$ ■

The above results can be considered comparable, however a more detailed description of their relations can be found in [13].

IV. Finite-Time Stochastic Stability for Networked Control Systems

Consider the model-based NCS subject to the random loss of packets as described previously

$$z_{k+1} = H_1(z_k) + H_2(z_k)\varphi_k, \quad z_k \in \mathbb{R}^{2n}, k = 0, 1, \ldots \quad (21)$$

in which the dropping sequence $\varphi_k = (1-\theta_k)$ is a stationary independent random sequence, with mean $\mu_\varphi = (1-p) = q$ and $\mu_{\varphi^2} = q$, where q is the probability of dropping a packet.

We defined finite time stochastic stability for a generic discrete-time dynamical system with respect to $(\alpha, \beta, N, \lambda, ||.||)$. Here we reformulate the FTSS definition for NCS (11). In the following we let $||z_k|| = \sqrt{z_k^T z_k}$ be the Euclidian norm.

Definition 5: The NCS (11) is FTSS with respect to $(\alpha, \beta, N, \lambda, ||.||)$ if

$$P_{in}(z_k; \alpha, \beta, N) = \quad (22)$$
$$P\{z_k^T z_k < \beta : k \in [0, N] \mid z_0^T z_0 \leq \alpha\} \geq (1-\lambda)$$

Let $V(z_k, k) = z_k^T M(k) z_k$ be a quadratic function where $M(k) = \left[\begin{array}{c|c} m_1(k) & m_2(k) \\ \hline m_3(k) & m_4(k) \end{array}\right]$, is a given $2n \times 2n$ time-varying real-valued matrix, with $m_i(k) \in \mathbb{R}^{n\times n}$, $m_2(k)^T = m_3(k)$, $M(k) > 0$. Then consider the following definition

Definition 6: The NCS (11) is quadratically FTSS with respect to $(\alpha, \beta, N, \lambda, M)$ if for the quadratic function $V(z_k, k) = z_k^T M(k) z_k$ the following holds

$$P_{in}(V_k; \gamma_0, \gamma, N) =$$
$$P\{z_k^T M(k) z_k < \gamma : k \in [0, N] \mid z_0^T M(k) z_0 \leq \gamma_0\}$$
$$\geq (1-\lambda) \quad (23)$$

where $\delta_1||z_k||^2 \leq V(z_k, k) \leq \delta_2||z_k||^2$, $\delta_1(k) = \lambda_{min}\{M(k)\}$, $\delta_2(k) = \lambda_{max}\{M(k)\}$ are the minimum and maximum eigenvalue of $M(k)$ respectively. In addition we have $\delta_2(k)\alpha \geq \gamma_0$ and $\delta_1(k)\beta \geq \gamma$.

It is easily proved that quadratic finite-time stability implies finite-time stability.

Next we denote the sets of states with bounded V as follows

$$S_\gamma = \{z_k : V_z(z_k, k) \leq \gamma\},\ S_\beta = \{x_k : V_x(x_k, k) \leq \beta\}$$

We aim to study the behavior of the system over a finite time in the presence of packet dropping. In particular, assuming that with full information available, the system's state is constrained within a bound β over a finite time N, we want to find conditions for which the state remains within the given bound over the time interval when packets are being dropped. Moreover, we want these conditions to depend on the model's state and on the amount of packets dropped.

We are now ready to state the following theorem that considering a class C_{B-NCS} NCS, gives sufficient conditions on the bounds defined on the NCS for which FTSS holds.

Theorem 4: Consider the NCS (11), and assume it belongs to class C_{B-NCS}, also consider the function $V_z(z_k, k) = z_k^T M(k) z_k$, in which $M(k)$, is a real-valued $2n \times 2n$ matrix, where $m_1(k) > 0$, $m_4(k) > 0$. Assume that $\forall z_k \in S_\gamma$ and $k \in [0, N]$

$$B_{H_1}(\hat{x}_k) + 2B_{H_{1,2}}(\hat{x}_k)q + B_{H_2}(\hat{x}_k)q \leq \phi_{k+1}$$

$$\frac{\alpha\delta_2 + \Phi_N}{\beta\delta_1} \leq \lambda \quad (24)$$

where $\Phi_N = \sum_{k=1}^{N} \phi_k$. Then the system is FTSS with respect to $(\alpha, \beta, N, M(k), \lambda)$.

Proof. The proof follows from theorem 1 and using lemma 1. The conditions in the theorem

$$\begin{aligned}
&\mathbb{E}_{z_k}[\Delta V_z(z_k,k)] = \mathbb{E}_{z_k}[(H_1(z_k) + H_2(z_k)\varphi_k)^T \\
&M(k+1)(H_1(z_k) + H_2(z_k)\varphi_k) - z_k^T z_k] \leq \\
&B_{H_1}(\hat{x}_k) + 2B_{H_{1,2}}(\hat{x}_k)q + B_{H_2}(\hat{x}_k)q\phi_{k+1}, \\
&\forall k = 0, \ldots, N,\ z_k \in S_\gamma \qquad (25)
\end{aligned}$$

and

$$\frac{\alpha\delta_2 + \Phi_N}{\beta\delta_1} \leq \lambda \quad (26)$$

from which FTSS follows.

Roughly speaking, the theorem restates the conditions for FTSS described in theorem 3, in a NCS context. Moreover, in order to make the analysis dependent only on the model's state that is assumed to be always available, it uses the fact that the NCS belongs to class C_{B-NCS}. Finally those bounds are used to specify FTSS conditions.

V. Finite-Time Stochastic Stability Design

In the previous section we presented sufficient conditions for FTSS of the NCS in the presence of packet dropping. We now investigate the possibility of designing a controller to guarantee the FTSS of the system. We therefore consider a network model in which the input function $u_k = K(\hat{x}_k)$ is not fixed i.e.

$$\begin{aligned}
z_{k+1} &= (F_1(z_k) + F_2(z_k)\varphi_k) + (G_1(z_k) + G_2(z_k)\varphi_k)u_k, \\
&\quad k \geq 0 \qquad (27)
\end{aligned}$$

Where the functions F_1, F_2, G_1, G_2 were previously defined and $u_k : \mathbb{R}^n \to \mathbb{R}$ is a scalar input. Although we will only focus on the case of scalar inputs, the results may be easily extended to multidimensional inputs.

Theorem 5: The class C_{B-NCS} NCS (27), is quadratically finite-time stochastically stabilizable with respect to $(\alpha, \beta, M, N, \lambda)$ and $\phi_k = \phi = \frac{\gamma\lambda - \gamma_0}{N}$ if for the function $V(z_k,k) = z_k^T M(k) z_k$, where $M(k)$ is as in definition 5, there exists an input law $u_k = K(\hat{x}_k)$ such that

1. The system is FTSS with respect to $(\alpha, \beta, M, N, \lambda)$ for the time in which the input cannot affect it, i.e. if

$$\begin{aligned}
&\mathbb{E}_{z_k}[(G_1(z_k) + G_2(z_k)q)^T M(k+1) \\
&(G_1(z_k) + G_2(z_k)q)] = 0, \qquad (28) \\
&\mathbb{E}_{z_k}[(F_1(z_k) + F_2(z_k)q)^T M(k+1) \\
&(G_1(z_k) + G_2(z_k)q)] = 0 \\
&\Rightarrow \mathbb{E}_{z_k}[(F_1(z_k) + F_2(z_k)q)^T M(k+1) \\
&(F_1(z_k) + F_2(z_k)q) - z_k^T M(k) z_k] \leq \phi_k
\end{aligned}$$

2. We have for all $\hat{x}_k \in S_\beta$

$$\begin{aligned}
&\mathbb{E}_{z_k}[\Delta V_{z_k}(z_k,k)] = (B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)q)u_k^2 \\
&2((B_{F1G2}(\hat{x}_k) + B_{F2G1}(\hat{x}_k) + \\
&B_{F1G2}(\hat{x}_k))q + B_{F1G1}(\hat{x}_k))u_k \\
&+(B_{F2}(\hat{x}_k))q + 2(B_{F1F2}(\hat{x}_k)q) + B_{F1}(\hat{x}_k) \\
&\leq \frac{\gamma\lambda - \gamma_0}{N}
\end{aligned}$$

The set of controllers is given by:

$$\begin{aligned}
&u_1(\hat{x}_k) \leq u(\hat{x}_k) \leq u_2(\hat{x}_k), \\
&for\ (B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)) \neq 0 \qquad (29) \\
&u = 0, \\
&for\ (B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)) = 0 \qquad (30)
\end{aligned}$$

$$\begin{aligned}
&B_{FG} = ((B_{F1G2}(\hat{x}_k) + B_{F2G1}(\hat{x}_k) + \\
&B_{F1G2}(\hat{x}_k))q + B_{F1G1}(\hat{x}_k)) \\
&B_F = B_{F2}(\hat{x}_k))q + 2(B_{F1F2}(\hat{x}_k)q) + B_{F1}(\hat{x}_k)
\end{aligned}$$

$$\begin{aligned}
u_{1,2} &= \frac{-|B_{FG}|}{(B_{G1}(\hat{x}_k) + B_{G2}q)} \\
&\frac{\pm\sqrt{(B_{FG})^2 - (B_{G1}(\hat{x}_k) + B_{G2}q)(B_F - \frac{\gamma\lambda-\gamma_0}{N}))}}{(B_{G1} + B_{G2}q)}
\end{aligned}$$

with

$$\begin{aligned}
0 &\leq (B_{FG})^2 - \\
&\quad [B_{G1} + B_{G2}q][(B_F - \hat{x}_k^T m_4(k)\hat{x}_k - \frac{\gamma\lambda - \gamma_0}{N})] \\
0 &\neq B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)q
\end{aligned}$$

Proof.

The proof follows from theorem 1. In particular the control law with the conditions above imply

$$\begin{aligned}
\mathbb{E}_{z_k}[\Delta V(z_k,k)] &\leq \phi_k \\
\frac{\alpha\delta_2 + \Phi_N}{\beta\delta_1} &\leq \lambda, \qquad (31) \\
\forall k = 0, \ldots, N,\ z_k \in S_\gamma &
\end{aligned}$$

and therefore FTSS follows.

The theorem uses the FTSS analysis result to set sufficient conditions for the NCS, to generate a control law that will satisfy those conditions, and therefore will stochastically stabilize the NCS in a finite time with respect to the specified conditions.

VI. Examples

This section provides a set of examples of NCS, for which we study FTSS for different amounts of dropped packets. Though we only analyzed the case of scalar inputs, the results presented can be easily extended to the vector input case. In this section we present vector inputs examples. In particular, we consider

$$\begin{aligned}
x_1(k+1) &= x_1(k) + u_1(k) \\
x_2(k+1) &= x_2(k) + u_2(k) \qquad (32) \\
x_3(k+1) &= x_3(k) + (x_1(k)u_2(k) - x_2(k)u_1(k))
\end{aligned}$$

which is the discrete-time version of the non-holonomic integrator proposed by Brockett in [15]. Also consider its approximate model

$$\begin{aligned} \hat{x}_1(k+1) &= 10\hat{x}_1(k) + 3u_1(k) \\ \hat{x}_2(k+1) &= 50\hat{x}_2(k) + 7u_2(k) \\ \hat{x}_3(k+1) &= 50\hat{x}_3(k) - 8(\hat{x}_1(k)u_2(k) + 7\hat{x}_2(k)u_1(k)) \end{aligned} \tag{33}$$

We want to FT stabilize the system trough the network by using a class (a) linear controller defined as $u(k) = -\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix}$, where $a_1 = 1.7$, $a_2 = 2.3$. At first we check using the multi-input version of theorem 3 that the proposed controller FT stabilizes the system trough the network. With $M = I_{3\times 3}$ we obtain $\mathbb{E}_{z_k}\Delta V(z_k, k) \leq \frac{\lambda\gamma - \gamma_0}{N} = 0.0585$, where z_k is given as in (2) with θ_k being an independent random sequence. In Figures 2, 3 and 4 we show simulations of the system controlled across a network, using a linear class (a) controller in which $a_1 = 1.7$, $a_2 = 2.3$, and with packets loss of $0\%, 20\%, 50\%$ respectively. Note how in the case of a class (a) controller with full

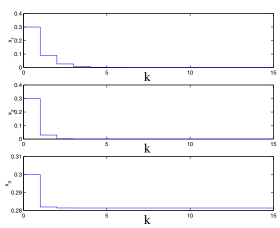

Fig. 2. Brockett integrator controlled through the network with linear class (a) controller with $a_1 = 1.7$, $a_2 = 2.3$ and 0% packets lost.

information available, finite-time stability is guaranteed for every set of parameters $(\alpha, \beta, N, M, \lambda)$ since the system is contracting. This property is however lost when the network starts dropping packets.

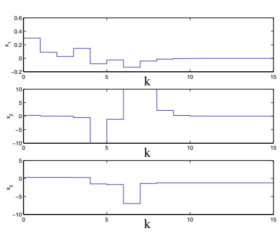

Fig. 3. Brockett integrator controlled through the network with linear class (a) controller with $a_1 = 1.7$, $a_2 = 2.3$ and 20% packets lost.

VII. Conclusions

Finite-time stochastic stability of model based NCS has been studied. In particular sufficient conditions for FTSS of the NCS were given. We showed how the FTSS of the system depends on three main factors: the stability of the closed-loop system in the case of available full information, the received information (packets transmitted), and the accuracy of the model and the initial conditions.

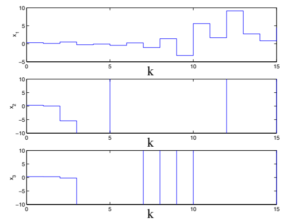

Fig. 4. Brockett integrator controlled through the network with linear class (a) controller with $a_1 = 1.7$, $a_2 = 2.3$ and 50% packets lost.

We also investigated the possibility of designing for FTSS, and a set of admissible controllers were proposed for a specific system. In particular we presented a class of controllers that only depends on the model state. Since the conditions used for design are only sufficient, the set of controllers might be conservative. Future work will be focused on characterize the FTSS of the system in terms of the amount of dropped packets.

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