

TABLE II
VALUES OF g_k , G_k , AND H_k FOR EXAMPLE 2

k	g_k	G_k	H_k
1	-1	1.8	0.5556
2	0.2	1.6	0.2
3	0.45	1.15	0.0355
4	0.4	0.75	0.0008
5	0.2875	0.4625	0.0493
6	0.1875	0.2750	0.1670
7	0.1156	0.1594	0.3679
8	0.0688	0.0906	0.6941
9	0.0398	0.0508	1.2282
10	0.0227	0.0281	2.1184
...

According to Theorem 1, the closed-loop stability can be obtained if the prediction horizon is greater than or equal to eight in EHPC1. On the other hand, the closed-loop system designed via the EHPC2 strategy is asymptotically stable if the prediction horizon is larger than or equal to seven.

Example 2 (Example of [10]): Consider the following nonminimum phase system

$$(1 - 0.5q^{-1})^2 y(t) = (-1 + 1.2q^{-1})u(t - 1).$$

The values of g_k , G_k , and H_k are reported in Table II. Its gain μ is 0.8.

Proposition 2 of [10] points out that if for some $k > 0$, either $g_k > g_{k+1} > \dots > g_N > 0$ or $g_k < g_{k+1} < \dots < g_N < 0$, then the closed-loop system designed with the EHPC2 strategy with prediction horizon k is asymptotically stable. According to this criterion, the closed-loop stability can be obtained with $k = 3$ in EHPC2. But, by using Corollary 1 in this paper, when $k = 2$, since $g_1 \neq g_2$, and $g_2 = H_2 = 0.2$, the closed-loop stability can also be obtained with $k = 2$ for this overdamped plant. For EHPC1, since Proposition 1 of [10] is a special case of Theorem 1, the results are the same for Proposition 1 and Theorem 1, i.e., the closed-loop system designed via the EHPC1 strategy is asymptotically stable if the prediction horizon is greater than or equal to six.

VI. CONCLUSIONS

This paper has introduced some simple criteria for choosing the prediction horizon in extended horizon predictive control. The closed-loop system designed via these criteria is asymptotically stable, as long as the given plant is asymptotically stable despite its property of being overdamped or underdamped.

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Extreme-Point Robust Stability Results for Discrete-Time Polynomials

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Abstract—This paper provides some new extreme-point robust-stability results for discrete-time polynomials with special uncertainties in the coefficient space. The proofs, obtained using the bilinear transformation, are simple, and the results specialize to existing robust-stability results.

I. INTRODUCTION

The stability of uncertain polynomials has recently become an active area of research. The problem was elegantly solved in the continuous-time case by the Kharitonov theorem [1]. Such a solution does not exist for discrete-time polynomials [2], although results are available in special cases [3], [4], and [5]. The main contribution of the paper is to extend the class of uncertainties which may be dealt with in the discrete-time case. Fam [6] presented a geometric approach to the stability problem using the barycentric coordinates (BC). Bartlett *et al.* [3], in their edge theorem, have shown that the stability of the exposed edges of the polytope Γ is a necessary and sufficient condition for the stability of every member of Γ . The problem of checking such edges generated much interest, see, for example [7]. In some cases, however, the stability of the two extremes of an edge is sufficient to guarantee the stability of all polynomials on that particular edge [4] and [8]. These are known as "extreme-point" tests to which this paper is related.

Since the submission of this paper, Rantzer [8] published a paper addressing the general problem of stability of polytopes of polynomials. In addition, the results of [9] are similar in scope and

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were independently derived. We point out here the importance of extreme points or vertex results in the problem of stabilization of uncertain plants. Despite much of the recent work in that area, one still has to stabilize a number of plants (at least four) before the stability of the family is guaranteed. The problem of simultaneously stabilizing more than two plants is, however, an open problem. The vertex results allow us to stabilize two extreme plants while guaranteeing the stability of their convex combinations.

In Section II, we will formulate the problem and introduce our notation. New stability results are given in Section III, and some examples are presented in Section IV. Our conclusions are discussed in Section V.

II. DEFINITIONS AND PROBLEM STATEMENT

The set of all univariate real polynomials in the complex variable z is denoted by $\mathbf{R}[z]$. Each polynomial in $\mathbf{R}[z]$ has the form $P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$, where n is a nonnegative integer and each a_i is an element of \mathbf{R} . If $a_0 \neq 0$, the degree of $P(z)$, denoted by $\deg P(z)$, equals n . A polytope Γ of a finite set Z of polynomials $P^i(z)$ is the convex hull of that set. We now review the BC concept [10] and discuss the stability of $P(z)$. Consider the bilinear transformation $s = (z+1)/(z-1)$; $z = (s+1)/(s-1)$ which maps the unit disk of the z -plane to the left-half s -plane. Let us define the polynomials $A_i(s)$, $i = 0, \dots, n$ as $A_i(s) = (s+1)^i (s-1)^{n-i}$. Substituting z from the bilinear transformation into $P(z)$ and multiplying by $(s-1)^n$, we get

$$\begin{aligned}
 P(s) &= a_0 A_n(s) + a_1 A_{n-1}(s) + \dots + a_n A_0(s) \\
 &= b_0 s^n + b_1 s^{n-1} + \dots + b_n
 \end{aligned} \quad (1)$$

where b_0, b_1, \dots, b_n are the barycentric coordinates of $P(z)$ [10]. The following two lemmas describe $A_i(s)$ and will be useful in proving the robust stability results of the following sections.

Lemma 1: The polynomial $A_k(s) + A_{n-k}(s)$ for all n and $k < n$ is an even polynomial. \square

Proof: This follows from $(s+1)^k (s-1)^{n-k} + (s-1)^k (s+1)^{n-k} = (-s+1)^k (-s-1)^{n-k} + (-s-1)^k (-s+1)^{n-k}$.

Lemma 2: If n is even, then $A_{n/2}(s)$ is an even polynomial. \square

Proof: It follows directly from the previous lemma.

Lemma 3 [11]: Let $P(s, \lambda) = P_0(s) + \lambda P_d(s)$, where $P_d(s)$ is a polynomial with all even or all odd-degree terms and $0 \leq \lambda \leq \lambda_{\max}$. Then, $P(s, \lambda)$ is stable for all $0 \leq \lambda \leq \lambda_{\max}$ if and only if $P(s, 0)$ and $P(s, \lambda_{\max})$ are both stable. \square

The problem addressed in this paper is to deduce the stability of all polynomials in Γ from the stability of a subset of Γ .

III. NEW STABILITY RESULTS

In this section we consider that a particular coefficient a_i is known to fall in the following closed interval $a_i^- \leq a_i \leq a_i^+$. Let us also define the extreme polynomials

$$\begin{aligned}
 P_1(z) &= P(z, a_i^-, a_{n-i}^-) \\
 &= a_0 z^n + \dots + a_i^- z^{n-i} + \dots + a_{n-i}^- z^i + \dots + a_n.
 \end{aligned}$$

$$\begin{aligned}
 P_2(z) &= P(z, a_i^+, a_{n-i}^+) \\
 &= a_0 z^n + \dots + a_i^+ z^{n-i} + \dots + a_{n-i}^+ z^i + \dots + a_n.
 \end{aligned}$$

$$\begin{aligned}
 P_3(z) &= P(z, a_i^+, a_{n-i}^-) \\
 &= a_0 z^n + \dots + a_i^+ z^{n-i} + \dots + a_{n-i}^- z^i + \dots + a_n.
 \end{aligned}$$

$$\begin{aligned}
 P_4(z) &= P(z, a_i^-, a_{n-i}^+) \\
 &= a_0 z^n + \dots + a_i^- z^{n-i} + \dots + a_{n-i}^+ z^i + \dots + a_n. \quad (2)
 \end{aligned}$$

The first problem addressed is to obtain conditions under which the stability of an edge can be easily derived from the stability of the extreme polynomials. We first restate the following theorem which appeared in [12]

Theorem 1: Given the family of polynomials $\mathcal{P}(z) = \{P(z); a_i^- \leq a_i \leq a_i^+\}$, where for some i , $0 \leq i \leq n$, $a_i - a_i^- = a_{n-i} - a_{n-i}^-$. Then, the entire family of polynomials is stable if and only if $P_1(z)$ and $P_2(z)$ in (2) are stable. For the case when $a_i^+ - a_i = a_{n-i} - a_{n-i}^-$, the entire family of polynomials is stable if and only if $P_3(z)$ and $P_4(z)$ are stable. \square

Proof: The proof of this theorem may be found in [12].

It should be noted that Theorem 1 contains two important constraints: 1) the coefficients must be coupled as (a_i, a_{n-i}) and 2) the edges allowed in the pairwise variation have $\pi/4$ and $3\pi/4$ slopes in the parameter space. Now we will show that Theorem 1 and Theorem 2 in [5] are particular cases of Theorem 2 which allows a larger class of perturbations. To state Theorem 2, we introduce the polynomials

$$P_{1j}(z) = P(z, a_i^-, a_{n+j-i}^-); \quad P_{2j}(z) = P(z, a_i^+, a_{n+j-i}^+)$$

$$P_{3j}(z) = P(z, a_i^+, a_{n+j-i}^-); \quad P_{4j}(z) = P(z, a_i^-, a_{n+j-i}^+). \quad (3)$$

Similar definitions of $P_{kj}(s)$; $k = 1, 2, 3, 4$ will also be used, and we let $[x]$ be the integer part of the real number x .

Theorem 2: Consider the family of polynomials $P(z) = P(z, a_i, a_{n+j-i}) = a_0 z^n + a_1 z^{n-1} + \dots$ where for some i , $0 \leq i \leq (n+1)/2$ and some $0 \leq j \leq i$

$$a_i - a_i^- = a_{n+j-i} - a_{n+j-i}^- \quad (4)$$

Then, the entire family of polynomials is stable if and only if the corresponding corner polynomials $P_{1j}(z)$ and $P_{2j}(z)$ are stable. Similarly, when

$$a_i^+ - a_i = a_{n+j-i} - a_{n+j-i}^- \quad (5)$$

the stability of the whole family is equivalent to that of $P_{3j}(z)$ and $P_{4j}(z)$. \square

Proof: We provide the proof for (4), easily adapted to (5). The Schur stability of $P(z)$ does not change if we multiply this polynomial by z^j , $0 \leq j \leq i$ to obtain $P'(z) = a'_0 z^{n+j} + a'_1 z^{n+j-1} + \dots + a'_{n+j}$, where $a'_i = a_i$; $i \leq n$ and $a'_i = 0$ otherwise. The proof is then completed by Theorem 1.

If we set $j = 2i - n$, $i > [(n-1)/2]$, we find the allowed variation $a_i^- \leq a_i \leq a_i^+$. Combining this with the Edge Theorem in [3], we obtain Theorem 2 of [5]. The next theorem will relax constraint 2) above by allowing variations in the sector $\pi/4 \leq \theta \leq 3\pi/4$.

Theorem 3: Consider the family of polynomials $P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n$ where for some i , $0 \leq i \leq (n+1)/2$ and some β , $0 \leq \beta < 1$

$$a_i - a_i^- = \beta(a_{n-i} - a_{n-i}^-). \quad (6)$$

Then, the entire family of polynomials is stable if and only if $P_1(z)$ and $P_2(z)$ in (2) are stable. Similarly, if

$$a_i^+ - a_i = \beta(a_{n-i} - a_{n-i}^-) \quad (7)$$

then, the entire family of polynomials is stable if and only if $P_3(z)$ and $P_4(z)$ in (2) are stable. \square

Proof: We will derive the result for case (6), since case (7) is similar. The Schur stability of $P(z)$ is maintained if we multiply it by the term $(z^k + \beta)$, where $0 \leq \beta < 1$ and $k = n - 2i$. In this case, a new polynomial $P'(z) = (z^k + \beta)P(z)$ or $P'(z) = a_0 z^{n+k} + \dots + a_i z^{n+k-i} + \dots + a_{n-i} z^i + \dots + \beta a_n$ is obtained, where we have used the fact that $(n+k)/2 = n - i$. Let

$$\lambda = a_i - a_i^- = \beta(a_{n-i} - a_{n-i}^-) \quad (8)$$

so that $0 \leq \lambda \leq a_i^+ - a_i^- = \beta(a_{n-i}^+ - a_{n-i}^-)$. If we write $P'(z) = a_0' z^{n+k} + a_1' z^{n+k-1} + \dots + a_{n+k}'$, it becomes clear that the only coefficients of $P'(z)$ that are varying with λ are a_{n+k-i}' , $a_{(n+k)/2}'$ and a_i' . Identifying terms and using (8), the variations are

$$a_i' = a_i = a_i^- + \lambda = a_i'^- + \lambda$$

$$a_{(n+k)/2}' = a_{n-i} + \beta a_i = a_{(n+k)/2}'^- + \left(\frac{\beta^2 + 1}{\beta}\right) \lambda$$

$$a_{n+k-i}' = \beta a_{n-i} = \beta a_{n-i}^- + \lambda = a_{n+k-i}'^- + \lambda. \quad (9)$$

If we denote by $P'(s)$ the result of applying the bilinear transformation to $P'(z)$, we can write

$$P'(s) = \sum_{i=0}^{n+k} a_i A_{n+k-i}(s). \quad (10)$$

Substituting (9) into (10), we can write

$$P'(s) = P'(s, a_i'^-, a_{(n+k)/2}'^-, a_{n+k-i}'^-) + \lambda \left(A_i(s) + A_{n+k-i}(s) + \frac{\beta^2 + 1}{\beta} A_{(n+k)/2}(s) \right).$$

Then, using Lemmas 1 and 2 and noting that the degree of $P'(s)$, $n+k$ is always even, we obtain $P'(s) = P'(s, a_i'^-, a_{(n+k)/2}'^-, a_{n+k-i}'^-) + \lambda P_d(s)$ where $P_d(s)$ is a polynomial with only even powers of s . The proof is then completed by Lemma 3.

Corollary 1: The family of polynomials $P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n$ where for some i , $0 \leq i \leq n$, some j , $\max\{2i-n, 0\} \leq j \leq i$ and some β , $0 \leq \beta \leq 1$, $a_i - a_i^- = \beta(a_{n+j-i} - a_{n+j-i}^-)$ is stable if and only if $P_{1j}(z)$ and $P_{2j}(z)$ in (3) are stable. Similarly, if, $a_i^+ - a_i = \beta(a_{n+j-i}^+ - a_{n+j-i}^+)$, then the stability of the entire family of polynomials is equivalent to the stability of $P_{3j}(z)$ and $P_{4j}(z)$ in (3). \square

Proof: The proof is obtained by applying Theorems 2 and 3 to $P(z)$ successively.

By combining the previous results, we obtain our most general test.

Corollary 2: Consider the polytope in the coefficient's space where each pair (a_i, a_k) , $0 \leq i \leq \lfloor (n+1)/2 \rfloor$, $n-i \leq k \leq n$ is varying inside a polygon with edges sloped in the closed interval $[\pi/4, 3\pi/4]$ and where each a_i can only be combined with one a_k and vice-versa, i.e., the pairwise variations (a_i, a_k) and (a_i, a_l) , $k \neq l$ are not allowed simultaneously. Then, every polynomial in the polygon will be stable if and only if all the polynomials obtained by combining all the polygon corners are stable. \square

Proof: The result is a direct consequence of the Edge Theorem [3] and Corollary 1.

Note that the "Weak-Kharitonov Theorem" [3] is a particular case of this corollary, where the polygons are "rotated" boxes and the coefficients are combined as in Theorem 1.

IV. NUMERICAL EXAMPLES

Example 1: Consider the family $P(z) = a_0 z^4 - 3.5z^3 - a_2 z^2 - 0.3z + 1$, where the couple (a_0, a_2) is varying as $a_0 = -2.7 + \lambda$; $a_2 = -2.7 + \lambda$, with $0 \leq \lambda \leq 6.6$. Note this variation would correspond to $i = 0$ and $j = -2$ in (4). Both extreme polynomials are stable, while for $\lambda = 0.5$, $P(z)$ is unstable. This proves that the results in Theorem 2 can not be extended to any j , $i-n \leq j < 0$. \square

Example 2: Consider the family $P(z) = a_0 z^4 + 1.2z^3 - 0.025z^2 + 1.75z + a_4$, where the couple (a_0, a_4) is varying as $a_0 = 2.7 - \lambda$; $a_4 = 1.98 + \lambda/1.5$, with $0 \leq \lambda \leq 15.4$. Note that this corresponds to $i = 0$ and $\beta = 1.5$ in (7). It can be verified that while the extreme polynomials are stable, the polynomial obtained for $\lambda = 1$ is not. \square

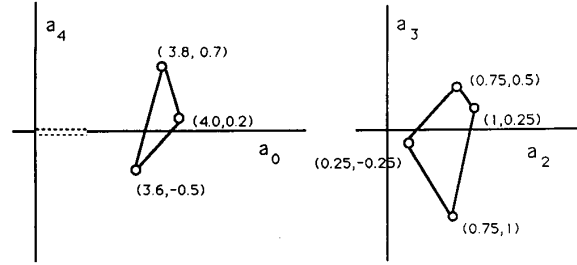


Fig. 1. Variation Regions for Example 3.

Example 3: Consider the family $P(z) = a_0 z^4 + 3.2z^3 + a_2 z^2 + a_3 z + a_4$, where the pairs (a_0, a_4) and (a_2, a_3) are varying as in Fig. 1. It can be seen that the 12 extreme polynomials represented by all the possible corner combinations are stable and that the conditions of Corollary 2 are satisfied, so that every polynomial belonging to the family is stable. \square

V. CONCLUSIONS

In this paper we have showed that for special coefficient variations in the sector bounded by the $\pi/4$ and the $3\pi/4$ sloped lines, the stability of the family of polynomials may be reduced to a stability check of the corner polynomials. It is obvious by now that a counterpart of Kharitonov's Theorem (with necessary and sufficient conditions) does not exist in the discrete polynomial case. We have then attempted to obtain necessary and sufficient conditions for the stability of special polynomials, with simple tests in the spirit of Kharitonov's Theorem.

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