Analytic Phase Margin Design

Peter Dorato, Domenico Famularo, and Chaouki T. Abdallah

***Abstract*—In [4] an algorithm is presented for analytic phase margin control design. Without special care, however, the compensator computed with this algorithm may not be a *real* rational function. The problem is evident when the plant has *real* unstable poles. In this case the algorithm in [4] requires a mapping of real points into complex values, and it is not clear that the resulting compensator has real coefficients. The purpose of this paper is to show how a *complex* mapping required in this algorithm can always be selected so that the compensator does have real coefficients.**

***Index Terms*—Analytic positive, interpolation, phase margin optimization, strict Schur.**

I. Introduction

In most introductory control textbooks, phase margin design is done by trial-and-error loop-shaping techniques [3], [7]. A one-point frequency design approach is sometimes presented as an "analytic" design technique, even though such an approach may lead to an unstable closed-loop system, as noted in [7]. In [4] and [5], true analytic procedures are presented for phase margin design. In particular, in [4, Sec. 11.4] the maximum possible phase margin for a given plant is derived, and an algorithm is given for the synthesis of a compensator which achieves any phase margin up to the maximum value. However, without special care in the case when the plant has real unstable poles, the resulting compensator $C(s)$ may not be a *real* rational function. The problem occurs in such a case because the algorithm involves finding an interpolating function with *complex* values at real points. The problem of a complex compensator does not arise in the gain margin optimization case (see [4, Sec. 11.3]), because in that case, real points are mapped into *real* interpolation values.

The purpose of this paper is to show how the nonreal interpolating function can be selected so that the resulting compensator is real. This paper also presents a convenient way to deal with plants that have zeros at infinity of multiplicity greater than one. Several examples are included to illustrate the design approach. It is also seen that for some plants, the maximal achievable phase margin is very small, and that in focusing on phase margin optimization, very fragile compensators may result [6]. The examples also illustrate that when the plant cannot be stabilized with a stable compensator, the usual trial-and-error loop-shaping techniques may be very difficult to apply.

This paper is organized as follows. Section II contains an outline of the problem and the design procedure. In Section III we present some illustrative numerical examples and give our conclusions in Section IV.

II. Outline of the Problem and Main Result

We define first some special functions needed in the sequel. The functions in question are assumed to be *rational* unless otherwise noted. We denote the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$ Also $\mathrm{Re}(s)$ denotes the real part of the complex number s, $\arg(s)$ denotes the argument of the complex number s, and $\|W(s)\|_\infty$ denotes the $\mathcal{H}_\infty$ norm of the function $W(s)$ [4]. Finally, we say that a transfer function $T(s)$ is *stable* if it is BIBO stable, i.e., $T(s)$ is proper and analytic in $\mathrm{Re}(s) \geq 0$.

1) A function $W(s)$ is a *strict Schur (SS) function* if it is analytic and $\|W(s)\|_\infty < 1$, for all $s{:}\, \mathrm{Re}(s) \geq 0$. Note that an SS function may have complex valued coefficients.
2) A function $V(s)$ is a *strictly bounded-real (SBR) function* if it is a *real* SS function, that is an SS function with only real coefficients.
3) A function $Z(s)$ is a *strictly positive (SP) function* if it is analytic and $-\pi/2 < \arg(Z(s)) < \pi/2$, or equivalently $\mathrm{Re}(Z(s)) > 0$, for all $s{:}\, \mathrm{Re}(s) \geq 0$.
4) A function $F(s)$ is an *analytic-positive (AP) function*, [1], if it is analytic and $-\pi < \arg(T(s)) < \pi$ for all $s{:}\, \mathrm{Re}(s) \geq 0$.

We define the *analytic phase margin design problem* as that of determining the maximum phase margin possible for a given plant $P(s)$ and synthesizing a feedback controller $C(s)$ which realizes an admissible phase margin. This problem is solved in [4] through conformal mappings and interpolation with SS functions. We summarize next the solution procedure given in [4]. First it is noted that the phase margin problem, for a given phase margin $\bar{\theta}$, involves finding a controller $C(s)$ such that the loop-gain $C(s)P(s)$ satisfies the following condition:

$$1 + e^{j\theta} C(s)P(s) \neq 0, \quad \text{for all } s, \theta{:}\, \mathrm{Re}(s) \geq 0, \qquad -\bar{\theta} \leq \theta \leq \bar{\theta}. \tag{1}$$

Condition (1) is then shown to be equivalent to the avoidance, for all $s{:}\, \mathrm{Re}(s) \geq 0$, by the closed-loop transfer function $T(s) = \frac{C(s)P(s)}{1+C(s)P(s)}$ of the following region of the complex plane (union of the boldface vertical lines in Fig. 1):

$$\mathbb{F} = \left\{ s \in \mathbf{C} \,\middle|\, s = \frac{1}{2} + j\frac{\sin(\theta)}{2(1-\cos(\theta))}, -\bar{\theta} \leq \theta \leq \bar{\theta} \right\}.$$

At this point the problem is then to find a stable function $T(s)$ which avoids the region $\mathbb{F}$. However, to preserve internal stability there cannot be unstable pole/zero cancellations in the loop-gain $C(s)P(s)$. This then requires that $T(s)$ satisfies the following interpolation conditions: 1) $T(a_i) = 1$, $i = 1, \ldots, n$ and 2) $T(b_i) = 0$, $i = 1, \ldots, m$, where a_i and b_i are the unstable poles and zeros, respectively, of the plant $P(s)$. To simplify the initial discussion, it is assumed that the unstable poles and zeros are all simple and that $P(s)$ is exactly proper. This problem of avoidance and interpolation via $T(s)$ is then converted, via conformal mappings, to that of finding a stable function $F(s)$

$$F(s) = \frac{1}{2} + \frac{a_2 - j\left(T(s) - \frac{1}{2}\right)}{a_2 + j\left(T(s) - \frac{1}{2}\right)} \tag{2}$$

where $a_2 = \frac{\sin(\bar{\theta})}{2(1-\cos(\bar{\theta}))} = \frac{1}{2}(\tan(\frac{\bar{\theta}}{2}))^{-1}$, which maps the forbidden region of $T(s)$ into the bold face line segment shown in Fig. 2. Note that if $F(s)$ is an AP function, the line segment in Fig. 2 is indeed avoided. On the other hand, an AP function can always be written as the square of an SP function, i.e.,

$$F(s) = Z^2(s) \tag{3}$$

Manuscript received January 22, 1998. Recommended by Associate Editor, S. Weiland. This work was supported in part by NASA under Contract NCCW-0087 and in cooperation with the NASA Center for Autonomous Control Engineering, The University of New Mexico.

P. Dorato and C. T. Abdallah are with the Department of Electrical and Computer Engineering, The University of New Mexico, Albuquerque, NM 87131-1356 USA (e-mail: peter@eece.unm.edu).

D. Famularo is with the Dipartimento di Elettronica, Informatica e Sistemistica, Università degli Studi della Calabria, Rende, I-87036, Italy.

Publisher Item Identifier S 0018-9286(99)07882-4.

0018–9286/99$10.00 © 1999 IEEE

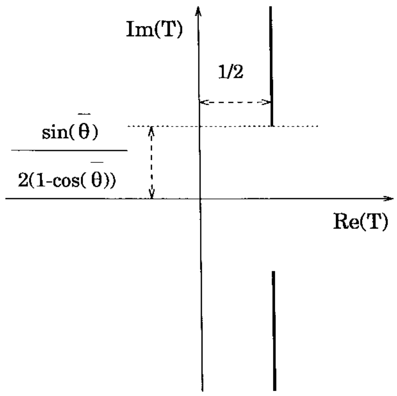

Fig. 1. Region of $\mathbf{C}$ to be avoided by $T(s)$ for all $s\colon \mathrm{Re}(s) \geq 0$.

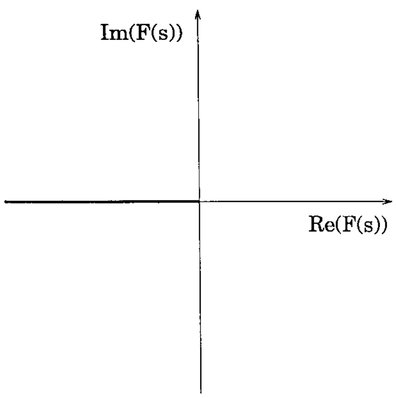

Fig. 2. Region to be avoided by $F(s)$ for all $s\colon \mathrm{Re}(s) \geq 0$.

where $Z(s)$ is an SP function. Finally the mapping

$$W(s) = \frac{Z(s) - e^{j\bar{\theta}/2}}{Z(s) + e^{-j\bar{\theta}/2}} \tag{4}$$

is used to convert the SP function $Z(s)$ into an SS function $W(s)$, with induced interpolation conditions

$$\begin{cases} W(a_i) = \beta = -j \sin\left(\dfrac{\bar{\theta}}{2}\right) e^{j\bar{\theta}/2}, & i = 1, \ldots, n \\ W(b_i) = 0, & i = 1, \ldots, m. \end{cases} \tag{5}$$

The phase-margin design problem is then reduced to a Nevanlinna–Pick interpolation problem for the computation of $W(s)$, with the phase margin $\bar{\theta}$ selected less than $\theta_{\max}$, where $\theta_{\max}$ is given by the following theorem.

Theorem 1 [4]: If $P(s)$ is stable or minimum phase, then $\theta_{\max} = \pi$; otherwise

$$\theta_{\max} = 2 \sin^{-1} \left(\frac{1}{\gamma_{\text{opt}}} \right) \tag{6}$$

where $\gamma_{\text{opt}} = \inf \|T(s)\|_\infty$, subject to interpolation conditions $T(a_i) = 1$, $T(b_i) = 0$.

The algorithm for the computation of $C(s)$ is then given in the following steps.

1) Compute

$$\inf \|T(s)\|_\infty = \gamma_{\text{opt}}$$

given $T(a_i) = 1$, $T(b_i) = 0$.

2) Pick $\bar{\theta} < \theta_{\max}$ where $\theta_{\max}$ is computed from (6).

3) Use the Nevanlinna–Pick interpolation algorithm to compute the strict Schur function $W(s)$ which satisfies the conditions (5); see [2] for details.

4) Compute

$$Z(s) = \frac{e^{j\bar{\theta}/2} + e^{-j\bar{\theta}/2} W(s)}{1 - W(s)}. \tag{7}$$

5) Compute

$$T(s) = \frac{1}{2} - j a_2 \frac{1 - Z^2(s)}{1 + Z^2(s)}. \tag{8}$$

6) Finally, compute

$$C(s) = \frac{T(s)}{P(s)(1 - T(s))}. \tag{9}$$

The above algorithm does not guarantee that $W(s)$ is a *real* function (in particular real unstable poles must interpolate to complex values β), and (7) and (8) are not *real-to-real* mappings. Thus, without special care, $T(s)$ in (8) will not be a real function, and hence the controller $C(s)$ in (9) will not be a rational function with *real* coefficients. A real $C(s)$ is, of course, required for physical realization. The following theorem ensures a real $C(s)$ when all the unstable poles and zeros are real. A simple extension of the theorem may then be used when complex poles and zeros occur in complex conjugate pairs.

Theorem 2: Let $V(s)$ be an SBR function which satisfies the following interpolation conditions:

$$\begin{cases} V(a_i) = |\beta| = \sin\left(\dfrac{\bar{\theta}}{2}\right), & i = 1, \ldots, n \\ V(b_i) = 0, & i = 1, \ldots, m \end{cases} \tag{10}$$

where $\bar{\theta} \leq \theta_{\max}$ and b_i and a_i are, respectively, the right half-plane zeros and poles of $P(s)$. Then the function

$$W(s) = -j e^{j\bar{\theta}/2} V(s) \tag{11}$$

which satisfies the conditions in (5) is the required SS function, and the resulting compensator $C(s)$, computed from (7)–(9), stabilizes the plant $P(s)$, guarantees a phase margin equal to $\bar{\theta} < \theta_{\max}$, and has real coefficients.

Proof: First note that $W(s)$ given in (11) is a function which satisfies the interpolation conditions in (5) and $\|W(s)\|_\infty = \|V(s)\|_\infty$. As shown in [4], if $\bar{\theta} < \theta_{\max}$, then there exists a strict Schur function which interpolates the points in (5), hence there exists an SBR function $V(s)$ which interpolates the points in (10).

We now show that $W(s)$ computed in (11) does result in a real $C(s)$. From (7) we have

$$Z(s) = e^{j\bar{\theta}/2} \frac{1 - j e^{-j\bar{\theta}/2} V(s)}{1 + j e^{j\bar{\theta}/2} V(s)}. \tag{12}$$

Now if this expression for $Z(s)$ is substituted back into (8) one obtains, after some algebra

$$T(s) = V(s) \frac{1 - \sin\left(\frac{\bar{\theta}}{2}\right) V(s)}{\sin\left(\frac{\bar{\theta}}{2}\right)(1 - V^2(s))} \tag{13}$$

which is a real rational function. The expression of the compensator obtained from (9) is then

$$C(s) = \frac{V(s)\left(1 - \sin\left(\frac{\bar{\theta}}{2}\right) V(s)\right)}{P(s)\left(\sin\left(\frac{\bar{\theta}}{2}\right) - V(s)\right)}. \tag{14}$$

□

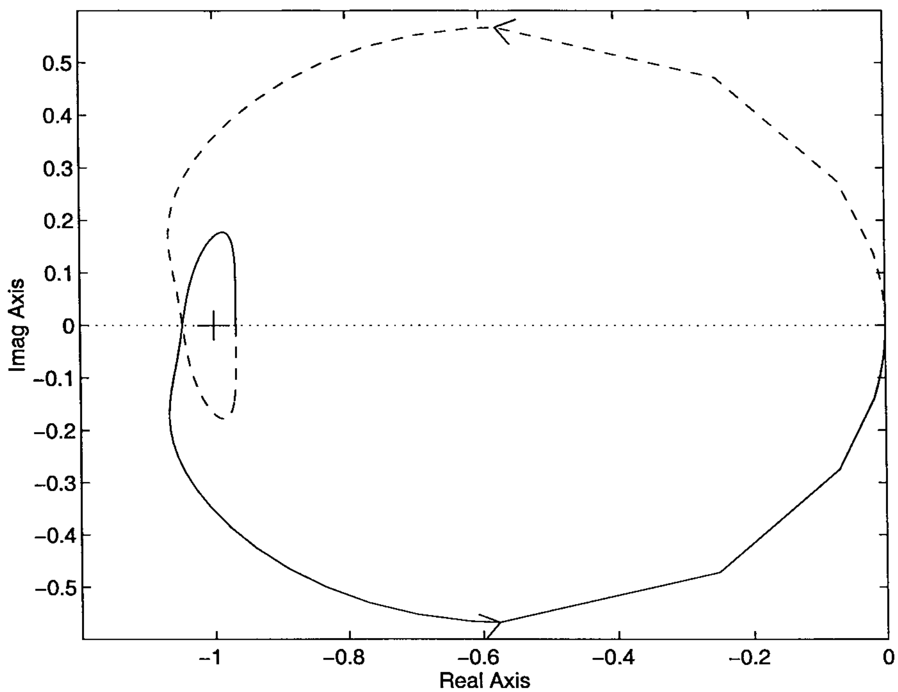

Fig. 3. Loop gain Nyquist diagram (Example 1).

Remark 1: If the plant $P(s)$ is not exactly proper, then the function $W(s)$ must contain a roll-off term of the type

$$\frac{1}{(\tau s+1)^k}$$

where k is the relative degree of $P(s)$. In this case $W(s)$ is modified as follows:

$$W(s) = -je^{j\bar{\theta}/2}\frac{1}{(\tau s+1)^k}\tilde{V}(s)$$

where the interpolation conditions on the function $\tilde{V}(s)$ are given as

$$\begin{cases}\tilde{V}(a_i) = (\tau a_i+1)^k|\beta|, & i=1,\dots,n,\\ \tilde{V}(b_i)=0, & i=1,\dots,m.\end{cases} \tag{15}$$

The parameter τ must be chosen small enough to ensure that $\tilde{V}(s)$ is an SBR function and that the guaranteed phase margin $\bar{\theta}$ is preserved.

Remark 2: The same proof remains valid when the unstable zeros and the poles appear in complex conjugate pairs. In this case the interpolation conditions for $V(s)$ are given by

b_1	$\bar{b}_1$	b_2	$\bar{b}_2$	$\dots$	b_q	$\bar{b}_q$	a_1	$\bar{a}_1$	a_2	$\bar{a}_2$	$\dots$	a_p	$\bar{a}_p$
0	0	0	0	$\dots$	0	0	$\lvert\beta\rvert$	$\lvert\beta\rvert$	$\lvert\beta\rvert$	$\lvert\beta\rvert$	$\dots$	$\lvert\beta\rvert$	$\lvert\beta\rvert$

the first line of the Fenyves array [2]. With the Nevanlinna–Pick algorithm [2] we obtain the following expression for the $2q$th line of the Fenyves array:

a_1	$\bar{a}_1$	a_2	$\bar{a}_2$	$\dots$	a_p	$\bar{a}_p$
$\gamma_1\lvert\beta\rvert$	$\bar{\gamma}_1\lvert\beta\rvert$	$\gamma_2\lvert\beta\rvert$	$\bar{\gamma}_2\lvert\beta\rvert$	$\dots$	$\gamma_p\lvert\beta\rvert$	$\bar{\gamma}_p\lvert\beta\rvert$

where

$$\gamma_i = \prod_{k=1}^{q}\frac{a_i+b_k}{a_i-\bar{b}_k}\frac{a_i+\bar{b}_k}{a_i-b_k}.$$

Now the interpolation values appear in complex conjugate pair so that a *real* interpolation function exists. The zeros $b_i, \bar{b}_i$ are easily interpolated by multiplying the interpolating function for the unstable poles by terms of the type

$$\frac{s-b_i}{s+\bar{b}_i}\frac{s-\bar{b}_i}{s+b_i}.$$

In the next section, Theorem 2 is used to compute *real* compensators for some phase margin design problems.

III. Examples

Example 1: This example is taken by [4]. In this case the plant is

$$P(s) = \frac{(s-1)}{(s+1)(s-p)}, \qquad p=\frac{5}{4}$$

and the value of $\gamma_{\rm opt}$, as computed from [4] is: $\gamma_{\rm opt} = |\frac{p+1}{p-1}| = 9$ with $\theta_{\max} = 12.7587°$. We select the guaranteed phase margin to be $\bar{\theta} = 10° = \frac{\pi}{18}$. This plant has a simple zero at infinity, hence a first-order roll-off term of the form $\frac{1}{\tau s+1}$ is required, with τ chosen small enough so that the $\mathcal{H}_\infty$-norm $W(s)$ remains less than one. In this case $W(s)$ is exactly

$$W(s) = -1440je^{j\pi/36}\sin\left(\frac{\pi}{36}\right)\frac{(s-1)}{(s+1)(4s+155)}, \qquad \left(\tau=\frac{4}{155}\right)$$

resulting in the controller

$$C(s) = 360\frac{4s^2+148.061\,58s+165.938\,41}{4s^2-1121s-49\,445}.$$

For this plant the parity interlacing property (*p.i.p.*) condition is not satisfied, so that an unstable controller is expected. In particular, the controller designed above has one unstable pole, so that for closed-loop stability the Nyquist diagram should encircle the -1 point twice (one unstable pole in the plant and one unstable pole in the controller). The Nyquist plot shown in Fig. 3 has the correct number of encirclements. The Bode plots of the loop gain are shown in Fig. 4 and the computed increasing gain and phase margins are $\mathrm{GM} = 0.3972$ dB, $\tilde{\theta} = 10.23° \geq \bar{\theta}$. Note that in order to meet a near-optimal phase margin, the Nyquist diagram is distorted in such

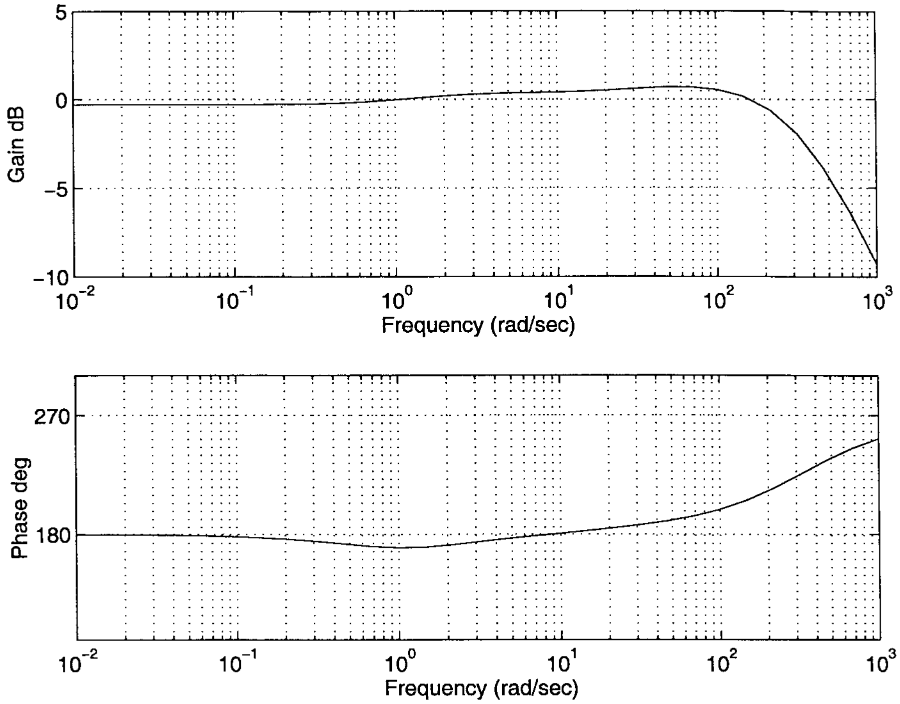

Fig. 4. Loop gain Bode diagram (Example 1).

a way that a very small gain margin results. This implies a very fragile/nonrobust controller with respect to gain perturbations and illustrates the robustness and fragility problems that typically result when a single optimization criterion is used for design.

To show how an arbitrary choice of the Schur function may result in a complex coefficients compensator for the plant in this example, consider the following SS function $W(s)$:

$$W(s) = -310\frac{(s-1)}{(s+1)} \times \frac{(-319+288e^{j\pi/36})s+1285-1440e^{j\pi/36}}{(4s+155)((412+288je^{-j\pi/36})s-205-360je^{-j\pi/36})} \tag{16}$$

This SS function interpolates to

$$\begin{aligned} W(5/4) &= \beta = -j\sin\left(\frac{\pi}{36}\right)e^{j\pi/36} \\ W(1) &= 0 \end{aligned} \tag{17}$$

as required by the conditions in (5), and has the proper roll-off at infinity. The compensator computed from this $W(s)$ is given by

$$\begin{aligned} C(s) &= \frac{c_1 s^4 + c_2 s^3 + c_3 s^2 + c_4 s + c_5}{c_6 s^4 + c_7 s^3 + c_8 s^2 + c_9 s + c_{10}} \\ c_1 &= (3.514\,564 + 8.105\,417j)10^2 \\ c_2 &= (1.528\,751 + 3.010\,686j)10^4 \\ c_3 &= (8.787\,351 + 5.006\,403j)10^4 \\ c_4 &= (116.6633 - 9.382\,413j)10^3 \\ c_5 &= (3.751\,727 - 3.313\,033j)10^4 \\ c_6 &= (1.198\,356 + 1.886\,006j) \\ c_7 &= (-2.507\,776 - 6.588\,968j)10^2 \\ c_8 &= (-1.326\,417 - 2.880\,513j)10^4 \\ c_9 &= (-6.851\,272 - 1.641\,024j)10^4 \\ c_{10} &= (-3.143\,535 + 2.617\,595j)10^4. \end{aligned}$$

Note that the there are no common poles/zeros so that the complexity of the controller cannot be reduced. When the compensator is complex, the frequency response does not have the usual symmetry properties for positive and negative frequencies. Thus, the Nyquist plot is no longer symmetric about the real axis, but the phase-margin design will still meet the phase-margin design specifications.

Example 2: Let us consider the following linear plant:

$$P(s) = \frac{(s-1)(s-3)}{(s-2)(s-4)}. \tag{18}$$

Note that this plant does not satisfy the *p.i.p.* [8] and hence cannot be stabilizable by a stable compensator. The first step in phase margin design is to evaluate the maximum possible phase margin for the

---

$$W(s) = -je^{j\pi/360}5\sin\left(\frac{\pi}{360}\right) \times \frac{(s-1)(s-3)\left(\left(-13+525\sin^2\left(\frac{\pi}{360}\right)\right)s+38+1050\sin^2\left(\frac{\pi}{360}\right)\right)}{(s+1)(s+3)\left(\left(-1+1025\sin^2\left(\frac{\pi}{360}\right)\right)s-2-2750\sin^2\left(\frac{\pi}{360}\right)\right)}$$

$$C(s) = \frac{-59.422\,28s^4 - 209.759\,06s^3 + 382.152\,62s^2 + 1763.113\,73s + 1270.300\,95}{58.892\,06s^4 + 310.111\,31s^3 + 315.265\,45s^2 - 414.702\,61s - 478.748\,81}$$

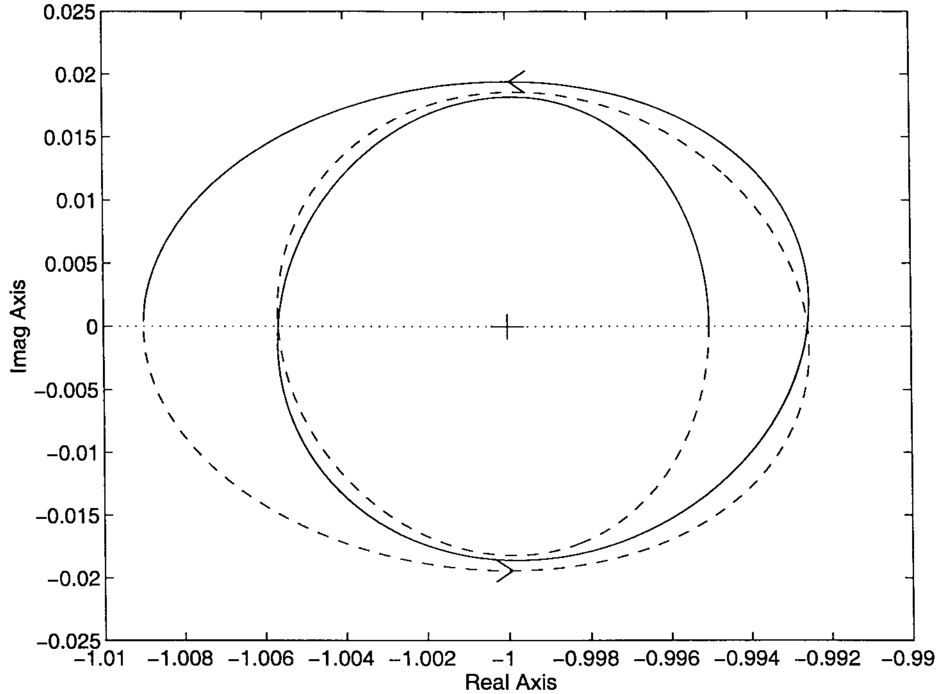

Fig. 5. Loop gain Nyquist diagram (Example 2).

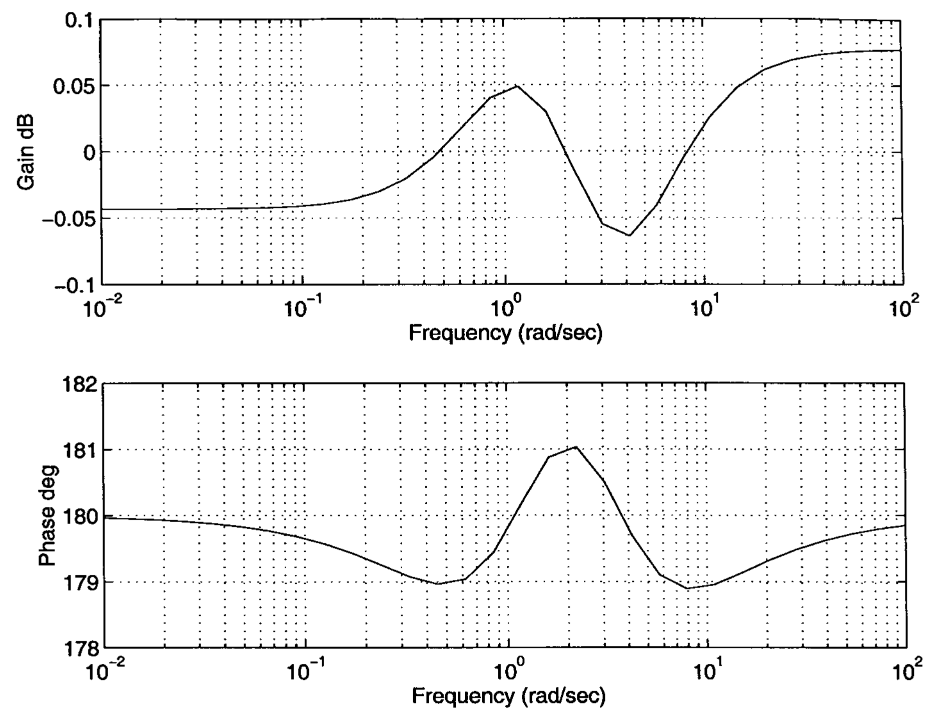

Fig. 6. Loop gain Bode diagram (Example 2).

given plant. Using the $\mathcal{H}_\infty$ optimization techniques in [4] one obtains for the given plant $\gamma_{\text{opt}} = 77.7492$ which leads to the maximum possible phase margin $\theta_{\max} = 2\sin^{-1}(1/\gamma_{\text{opt}}) = 1.4739°$. It is interesting to note how small the maximum phase margin is for this particular plant. We select a design value of $\bar{\theta} = 1° = \frac{\pi}{180}$. The SS function which interpolates the points (5) for this case is given by the equation shown at the bottom of the previous page. We list the exact value of $W(s)$ so that levels of precision in the realization of $C(s)$ may be studied. Using $W(s)$ and the mappings (7)–(9), $C(s)$ is given by the other equation shown at the bottom of the previous page. The compensator is, as expected, unstable. For closed-loop stability the Nyquist plot should encircle the -1 point three times (two unstable poles in the plant and one in the controller) in the counterclockwise sense. The Nyquist plot in Fig. 5 verifies that this is the case. Bode

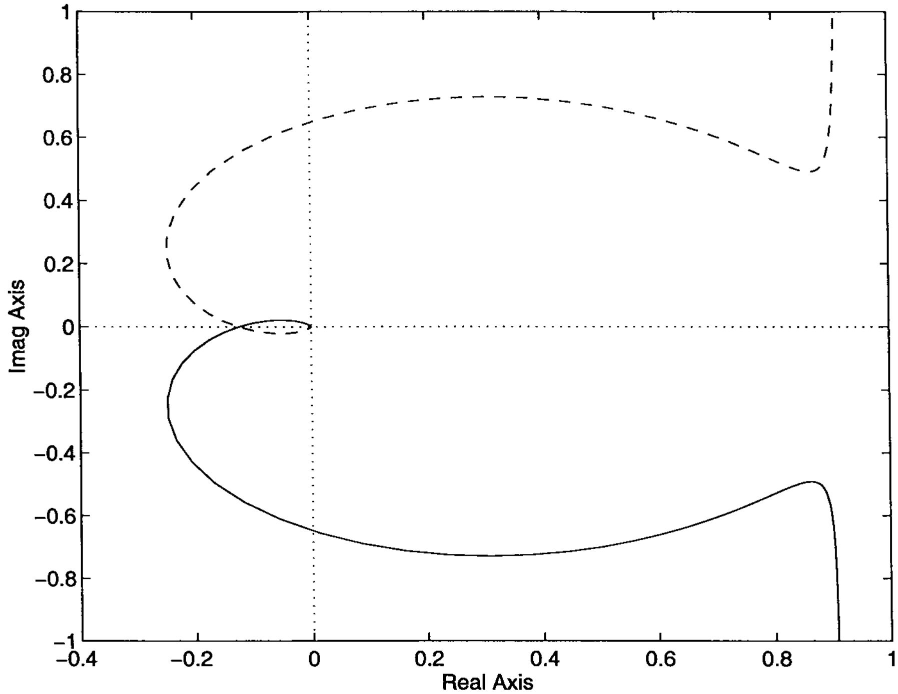

Fig. 7. Loop gain Nyquist diagram (Example 3).

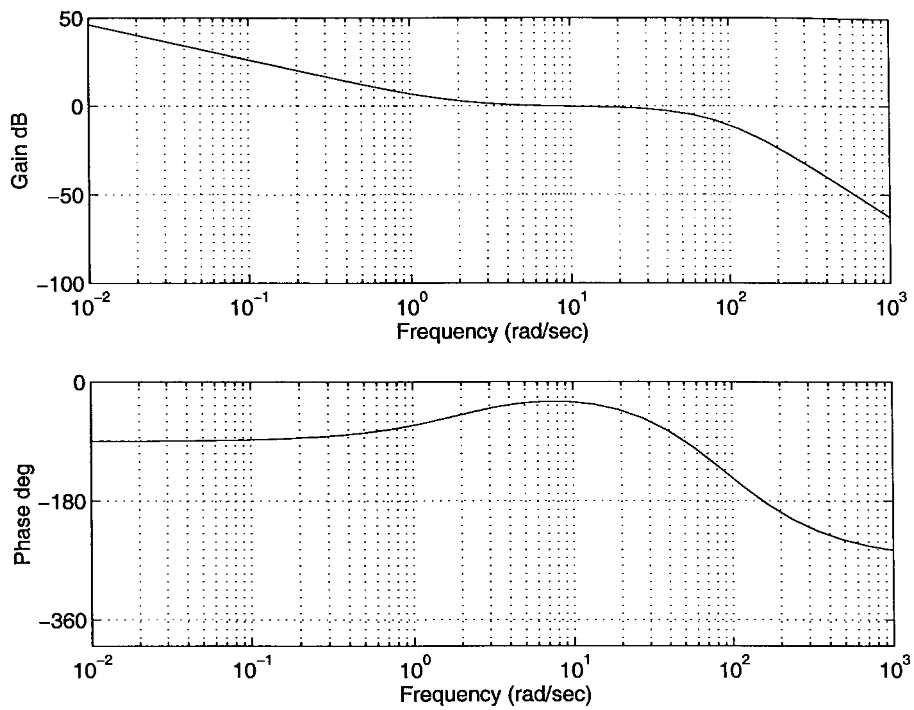

Fig. 8. Loop gain Bode diagram (Example 3).

plots of the loop gain are shown in Fig. 6. The computed gain and phase margins are $\mathrm{GM} = 0.048\,94$ dB, $\tilde{\theta} = 1.043^\circ \geq \bar{\theta}$. The actual phase margin is a bit larger than the guaranteed design value, but of course less that the maximum possible value. Trial-and-error phase margin design would be difficult in this case because an unstable controller is required. Because of the inherently small phase margin for this plant, this design is extremely *fragile* [6] with respect to any possible time delays in the controller.

Example 3: The following phase margin design problem is taken from [7, Example 9.2]. The plant is given by

$$P(s) = \frac{4}{s(s+1)(s+2)}.$$

$$C(s) = \frac{((s^2+3s+2)(4\tau^2(1+\sqrt{3})s^3 + 12\tau^2(1+\sqrt{3})s^2 + 12\tau(1+\sqrt{3})s - 1 + \sqrt{3})}{16(1+\sqrt{3})\tau(\tau s+1)^3(\tau^2 s^2 + 3\tau s + 3)}$$

$$C(s) = \frac{13\,824 s^5 + 3.756\,10^6 s^4 + 3.438\,87\,10^8 s^3 + 1.671\,10^9 s^2 + 2.661\,72\,10^9 s + 1.330\,86\,10^9}{7.695\,15\,10^{-2} s^5 + 41.3549 s^4 + 9.260\,32\,10^3 s^3 + 1.050\,62\,10^6 s^2 + 5.943\,38\,10^7 s + 1.330\,86\,10^9}$$

The controller $C(s)$ must achieve a phase margin $\bar{\theta} \geq 50°$, with a DC gain of one, i.e., $C(0) = 1$. In [7] this problem is solved with a simple *lag* controller. The condition $C(0) = 1$ requires the interpolation condition $T'(0) = -\lim_{s \to 0} \frac{1}{sC(s)P(s)} = -\frac{1}{2}$. The problem of finding $\gamma_{\rm opt}$ is complicated by the additional condition on the derivative of $T(s)$, but it can be shown that $\gamma_{\rm opt} = 1$, leading to $\theta_{\max} = \pi$. We select a design phase margin much larger than in [7], but below the maximum value $\theta_{\max}$; in particular we select $\bar{\theta} = 150° = \frac{5\pi}{6}$. The condition $T'(0) = -\frac{1}{2}$ is not enforced in the interpolation procedure but is satisfied at the end when an expression of $T(s)$ with a free parameter is obtained. In this case we can select $W(s)$

$$W(s) = -je^{j(5\pi)/12} \sin\left(\frac{5\pi}{12}\right) \frac{1}{(\tau s+1)^3}$$

where the constant time τ in the roll-off term is any positive real number. We will use τ to meet the derivative interpolation condition. The expression of the controller, computed from (7)–(9), with variable τ, is given by the equation shown at the top of the page. The condition $C(0) = 1$ then yields $\tau = 0.011\,164\,5$. Finally the controller $C(s)$ is given by the other equation shown at the top of the page.

The above controller is stable (*p.i.p.* is satisfied for this plant). Since the plant has no poles inside the right half-plane and the controller is stable, the Nyquist diagram for the compensated system should have no encirclements of the -1 point for closed-loop stability. This is verified by the Nyquist plot shown in Fig. 7. Fig. 8 shows Bode plots for the loop gain of the compensated system. The computed gain and phase margins are $\mathrm{GM} = 18.13$ dB, $\tilde{\theta} = 150.5° \geq \bar{\theta}$. It is interesting to note that the analytic 150 degree phase margin design algorithm produced a fifth order *phase-lead* controller, compared to the first-order phase-lag design in [7] for a 50-degree phase margin. As it turns out, it is impossible to get a phase margin greater than 90 degrees for this plant with lag compensation. What is interesting is that the analytic procedure automatically selected the "right" type of compensator.

IV. Conclusion

We have shown how a *nonreal* interpolating function $W(s)$ can be chosen so that the analytic phase margin design algorithm developed in [4] can be used to design a compensator with *real* coefficients. While phase margin design is only one approach to robust design, it is commonly used in practice and is the only robust design approach discussed in most introductory control texts, where the problem is generally solved with trial-and-error procedures. A very significant part of the analytic design algorithm developed in [4] is that a maximum achievable phase margin is determined for any given plant.

The examples included here illustrate the limitations placed on phase margin design for given plants. In particular some plants may allow almost no phase margin at all, and to guarantee closed-loop stability very complicated Nyquist diagrams may be required. Finally, the examples illustrate that optimal single-objective design can cause serious problems in robustness and fragility.

References

[1] C. T. Abdallah, P. Dorato, F. Pérez, and D. Docampo, "Controller synthesis for a class of interval plants," *Automatica*, vol. 31, no. 2, pp. 341–343, 1995.

[2] P. Dorato, L. Fortuna, and G. Muscato, *Robust Control for Unstructured Perturbations—An Introduction*, Berlin, Germany: Springer-Verlag, 1992.

[3] R. C. Dorf, and R. H. Bishop, *Modern Control System*, 8th ed. Reading, MA: Addison-Wesley, 1997.

[4] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*. New York: McMillan, 1992.

[5] P. P. Khargonekar and A. Tannenbaum, "Non-Euclidean metrics and the robust stabilization of systems with parameter uncertainty," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 1005–1013, Oct. 1985.

[6] L. H. Keel and S. P. Bhattacharyya, "Robust, fragile, or optimal?," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1098–1105, Aug. 1997.

[7] C. L. Phillips and R. D. Harbor, *Feedback Control of Dynamic Systems*, 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[8] D. C. Youla, J. J. Bongiorno, and C. N. Lu, "Single-loop feedback stabilization of linear multivariable dynamical systems," *Automatica*, vol. 10, no. 3, pp. 159–173, 1974.

Correction to "Optimal Control of Perturbed Linear Static Systems"

Roy Smith and Andy Packard

***Abstract*—This paper presents a typographical correction to the proof of Theorem 4 in the above-mentioned paper.**

***Index Terms*—Robust control synthesis, structured singular value.**

I. The Correction

Equation (6)[1] should be replaced by the following:

$$\lambda_{\max}(zV_A + V_B) < 0 \tag{6}$$

where

$$V_A = V_\perp\left(\begin{bmatrix} P_{11}^T \\ P_{12}^T \end{bmatrix} [P_{11} \quad P_{12}] - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{bmatrix}\right) V_\perp^T$$
$$V_B = V_\perp\left(\begin{bmatrix} P_{21}^T \\ P_{22}^T \end{bmatrix} [P_{21} \quad P_{22}] - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix}\right) V_\perp^T.$$

Manuscript received June 30, 1998. Recommended by Associate Editor, A. Tesi. This work was supported by the NSF under Grants ECS-93-8917 and CTS 9057420.

R. Smith is with the Electrical and Computer Engineering Department, University of California, Santa Barbara, CA 93106 USA.

A. Packard is with the Mechanical Engineering Department, University of California, Berkeley, CA 94720 USA.

Publisher Item Identifier S 0018-9286(99)07883-6.

[1] R. Smith and A. Packard, *IEEE Trans. Automat. Contr.*, vol. 41, pp. 579–584, Apr. 1996.