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Robust $\mathscr{H}_\infty$ networked control for systems with uncertain sampling rates

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Abstract: This study investigates the problem of controller design for systems with uncertain sampling rates. The system is controlled through a communication network. The sampling period, within a given interval, is assumed to be time-varying and a simplified framework for the network-induced delay is considered. The overall system is thus described by an uncertain discrete-time model with time-varying parameters inside a polytope whose vertices are obtained by means of the Cayley–Hamilton theorem. A digital robust controller that minimises an upper bound to the $\mathscr{H}_\infty$ performance of the closed-loop networked control system (NCS) is determined. The design conditions rely on a particular parameter-dependent Lyapunov function and are expressed as bilinear matrix inequalities (BMIs) in terms of extra matrix variables, which may be explored in the search for a better system behaviour. Numerical examples illustrate the results.

1 Introduction

The control community has struggled for many decades to find solutions to problems concerned with the perfect operation of dynamical systems immersed in hostile environments. There is no denying that it is wise to seek better characterisations of model uncertainties, to guarantee not only stability but also robustness against practical disturbances and perturbations. Within this framework, networked control system (NCS) architecture has received considerable recent attention.

Technological advances have enabled the extensive use of communication channels in the control of dynamic systems [1, 2]. Using a real-time network to exchange information among control system components (sensors, actuators, controllers, and so on), NCSs are a good alternative to implementing distributed control and interconnected systems. To illustrate the usefulness of NCSs one can cite the following benefits: reduced system wiring, plug and play devices and ease of system diagnosis and maintenance [2]. Unfortunately there are also some drawbacks: systems with loops closed over communication networks become complex and require sophisticated control techniques. Among the main issues arising in NCSs deserving special attention are network-induced delay, packet dropouts, multiple-packet transmission and bandwidth requirements.

Network-induced delays occur whenever data are exchanged through a communication channel and, in general, can be broken into three parts: time delays at the source node, on the network channel, and at the destination node [3]. As pointed out in [2], their nature is related to the medium access control (MAC) protocol and may be constant, time varying or random. Packet dropouts may occur whenever more than one node tries to transmit simultaneously, leading to a message collision, or because of node failures. Although retransmission is an option, there are some cases where it may be a disadvantage or even impossible. Multiple-packet instead of single-packet transmission may be needed for many reasons, as, for instance, bandwidth and packet size constraints, which in some sense increase the chances of packet dropouts and network-induced delays. Bandwidth usage has a direct impact on system stability and performance. From the control point of view, it is known that a faster sampling rate is required to guarantee that the behaviour of sampled data models approximates that of continuous-time systems.

In NCSs this implies a high network load and consequently in larger bandwidth usage.

The study of control strategies to overcome these difficulties has received considerable recent attention [4–10]. Lyapunov theory, which has been one of the main tools for dealing with the stability analysis and synthesis of controllers, has begun to be used in the NCS framework. Recent works include [11], where a feedback controller is constructed for a discrete-time Markovian jump system with random delays via a set of linear matrix inequality (LMI) conditions; [12], where the control problem is solved for the multipoint-packet system using $\mathscr{H}_2$ optimisation techniques; [13], where stabilisation of an NCS is achieved by means of a packet-loss dependent Lyapunov function; and [14], where a Lyapunov–Krasovskii functional is used to design a state feedback controller for a time-delay sampled system. As can be seen from these works, much effort has been made to bring together advances in control theory and the benefits of communication networks.

Depending on the system to be controlled, some networks may be more suitable than others. For instance, Ethernet-based network solutions may be more appropriate for NCSs operating at low network loads, since in this case the induced time delay is very small, whereas ControlNet network solutions equipped with a token bus protocol perform well at high network loads when the percentage of packets discarded is at issue, as discussed in [3]. It is important to point out that control strategies based on a simplified framework, such as a constant delay or even zero delay, may display reliable behaviour when applied in specific cases. In any case, a controller design method that takes into account all the characteristics of a network which impact system stability still remains a challenge.

This paper addresses the design of robust controllers to stabilise NCSs subject to time-varying sampling rates. The stability of this type of system is important within the NCS framework, especially in the context of dynamic bandwidth allocation and bandwidth usage control. A simplified framework for the networked-induced delay is assumed. The uncertain sampling period is taken to lie inside a known interval. The sampled data system is represented by an uncertain discrete-time linear model with time-varying parameters lying inside a polytope whose vertices are determined through the Cayley–Hamilton theorem, without using approximations or truncation. The proposed approach complements and extends in some sense the results of [6, 8] with respect to two aspects: index of performance and the stability of sampled data systems with time-varying sampling periods. Specifically, the stability conditions of the closed-loop system are certified by a parameter-dependent Lyapunov function and the robustness of the controller by an $\mathscr{H}_\infty$ guaranteed cost, as proposed in the preliminary version of this paper [15].

An improved strategy is used in which a more general parameter-dependent Lyapunov function is applied to provide less conservative stability conditions. As shown in [16, 17], this class of *path-dependent* Lyapunov functions can provide necessary and sufficient conditions for robust stability analysis of arbitrarily time-varying discrete-time systems. Extra matrix variables are introduced in the bounded real lemma conditions, producing design conditions that are expressed in terms of bilinear matrix inequalities (BMIs). A robust memory controller is then obtained by the solution of an optimisation problem that minimises an upper bound to the $\mathcal{H}_\infty$ index of performance subject to a finite number of BMI constraints formulated only in terms of the vertices of a polytope. As illustrated by means of numerical examples, the use of BMIs could prove interesting in the search for better NCS performance. Furthermore, the conditions can be reduced to a set of LMIs by a convenient choice of the extra variables. At each step of the algorithm, a convex optimisation problem with LMI constraints is solved, providing non-increasing values for the bounds on the $\mathcal{H}_\infty$ index of performance. Even when no communication channel is considered, the proposed approach improves some of the results in the literature concerned with the robust control of time-varying discrete-time systems, for instance, those in [18, 19].

2 Preliminaries and problem statement

The NCS model considered is described in Fig. 1.

The continuous-time plant is given by the following equations, for $t \geq 0$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-\tau) \\ y(t) &= Cx(t) + Du(t) + D_d u(t-\tau) \\ x(0) &= 0, \;\; u(\vartheta) = 0, \;\; \vartheta \in \{-\tau, 0\} \end{aligned} \tag{1}$$

where τ represents the network-induced time delay, $x(t) \in \mathbb{R}^n$ is the state space vector, $u(t) \in \mathbb{R}^m$ is the control

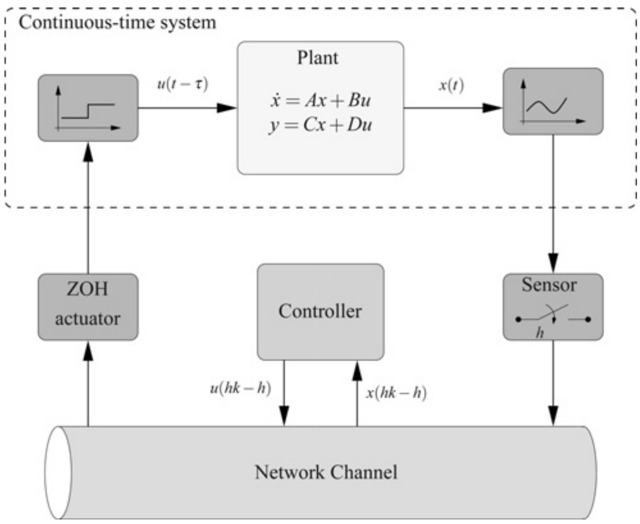

Figure 1 *NCS model*

signal and $y(t) \in \mathbb{R}^q$ is the output. All matrices are real, with appropriate dimensions.

The total networked-induced delay τ is broken into two parts: the delay that occurs when data are transmitted from the sensor to the controller τ_{sc} and the delay when the data are transmitted from the controller to the actuator τ_{ca}. As mentioned in [2], the delay due to computations in the controller can be modelled into either τ_{sc} or τ_{ca}. Note that the delays are not being used to model network scheduling. Depending on the MAC protocol of the network, network-induced delay can be constant, time varying or random. Under the assumption of a scheduling MAC protocol, the delays occur while waiting for the token, or time slot. In this case, it can be said that a scheduling network is an example of a situation in which the delay can be bounded and made constant by transmitting packets periodically [2]. Concerned with small delays, Ethernet-based networks experience almost no delay at low network loads [3]. Furthermore, if the controller is time invariant (such as the one addressed in this paper), these sources of delays can be added for analysis purposes, for example, $\tau = \tau_{sc} + \tau_{ca}$. For simplicity, τ is considered constant and known, a situation that may occur when static scheduling network protocols are implemented. Moreover, τ is supposed to be less than one sampling period. A buffer in the controller node is used to store the delayed information.

System (1) is sampled with a period $h > \tau$, yielding the discrete-time model [20], for $k \in \mathbb{Z}_+$, $x(0) = 0$, and $u(\vartheta) = 0$, $\vartheta \in \{-h, 0\}$

$$\begin{aligned} x(kh+h) &= A_s(h)x(kh) + B_{su0}(h,\tau)u(kh) \\ &\quad + B_{su1}(h,\tau)u(kh-h) + B_{sw}w(kh) \\ y(kh) &= C_s x(kh) + D_{su}u(kh) + D_{sd}u(kh-h) \\ &\quad + D_{sw}w(kh) \end{aligned} \tag{2}$$

where $w(kh) \in \mathbb{R}^r$ is an extra input, belonging to $l_2[0,\infty)$, used to model possible noise in the process. The system matrices $A_s(h)$, $B_{su0}(h,\tau)$, $B_{su1}(h,\tau)$, C_s, D_{su} and D_{sd} are given by

$$\begin{aligned} & A_s(h) = \exp(Ah), \quad B_{su0}(h,\tau) = \int_0^{h-\tau} \exp(As)\,\mathrm{d}sB, \\ & D_{sd} = D_d \\ & B_{su1}(h,\tau) = \exp(A(h-\tau)) \int_0^{\tau} \exp(As)\,\mathrm{d}sB, \quad C_s = C, \\ & D_{su} = D \end{aligned} \tag{3}$$

As discussed in [6, 8], the sampling period h may change its value at runtime for different reasons, for example, dynamic bandwidth allocation and scheduling decisions. By considering the sampling period as a time-varying parameter, it is possible to reduce the flow of information between sensor and actuator. Nevertheless, bounds on such variations can be determined, guaranteeing that the actual values of h at each instant k, namely, h_k, lie inside a finite discrete set as specified below

$$h_k \in \{h_{\min}, \ldots, h_{\max}\}, \; h_k = \kappa \cdot g, \; \kappa \in \mathbb{N} \tag{4}$$

It is assumed that the real values of h_k are not known at the instant of time k, but only that they belong to (4) and $h_{\min} \geq \tau$. The number of possible values of these sets depends on the processor/network clock granularity g, as discussed in [8]. The clock granularity is related to processor frequency and $\kappa \in \mathbb{N}$ is a function of time that specifies how many times g the sampling period h will be at instant k.

In order to guarantee the stability of the networked system shown in Fig. 1, a state feedback controller is designed. Using an extra state variable $z(kh) = u(kh - h)$ to store the last value of the control signal, the dynamics of system (2) can be represented by the following difference equations [20]

$$\begin{aligned} \tilde{x}(kh+h) &= \tilde{A}(h)\tilde{x}(kh) + \tilde{B}_u(h)u(kh) + \tilde{B}_w w(kh) \\ y(kh) &= \tilde{C}\tilde{x}(kh) + D_{su}u(kh) + D_{sw}w(kh) \end{aligned} \tag{5}$$

where $\tilde{x}(kh) = [x(kh)' \; z(kh)']'$ and

$$\begin{aligned} & \tilde{A}(h) = \begin{bmatrix} A_s(h) & B_{su1}(h,\tau) \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_u(h) = \begin{bmatrix} B_{su0}(h,\tau) \\ \mathbf{I} \end{bmatrix} \\ & \tilde{B}_w = \begin{bmatrix} B_{sw} \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} C_s & D_{sd} \end{bmatrix} \end{aligned} \tag{6}$$

In the case where there is no time delay ($\tau = 0$), the state space vector becomes $\tilde{x}(kh) = x(kh)$ and the augmented system matrices simplify in a standard way.

The control signal is given by

$$\begin{aligned} u(kh) &= K_x x(kh) + K_d u(kh-h) = \begin{bmatrix} K_x & K_d \end{bmatrix} \begin{bmatrix} x(kh) \\ z(kh) \end{bmatrix} \\ &= K\tilde{x}(kh) \end{aligned} \tag{7}$$

A discrete-time polytopic model is used to represent the set of all possible matrices in system (5) due to the uncertain time-varying sampling periods h_k given by (4). More specifically, the system matrices $(\tilde{A}(h), \tilde{B}_u(h))$, for any $k \geq 0$, are described as a convex combination of well-defined vertices $(\tilde{A}_j, \tilde{B}_{uj})$. The main difficulty in defining the vertices is related to the exponential terms in (3), which need to be computed for all h_k in (4). By using the Cayley–Hamilton

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theorem [21], these terms can be written as

$$\exp(Ah) = \sum_{i=0}^{n-1} \rho_i(h)A^i \tag{8}$$

$$\begin{aligned}\int_0^{h-\tau} \exp(As)\,\mathrm{d}s &= \int_0^{h-\tau}\left(\sum_{i=0}^{n-1}\rho_i(s)A^i\right)\mathrm{d}s \\ &= \sum_{i=0}^{n-1}\left(\int_0^{h-\tau}\rho_i(s)\,\mathrm{d}s\right)A^i = \sum_{i=0}^{n-1}\eta_i(h)A^i\end{aligned} \tag{9}$$

where

$$\eta_i(h) = \int_0^{h-\tau}\rho_i(s)\,\mathrm{d}s$$

The coefficients $\rho_i(h)$ and $\eta_i(h)$ can be determined for each value of h_k by solving a set of linear equations defined in terms of the eigenvalues of matrix A. For instance, the first block of the matrix $\tilde{A}(h_k)$ in (6) is given by

$$\exp(Ah) = \sum_{i=0}^{n-1}\rho_i(h)A^i = \sum_{i=0}^{n-1}\theta_i(h_k)\Omega_i \tag{10}$$

where the coefficients $\theta_i(h_k)$, $i = 0, \ldots, n-1$, are obtained from the modes associated with the eigenvalues of A and matrices $\Omega_i \in \mathbb{R}^{n\times n}$ are determined by collecting terms in the above equality. Similarly, $B_{su1}(h_k, \tau)$ and $B_{su0}(h_k, \tau)$ can be computed as a linear combination of matrices, following (3), (8) and (9), and in some cases can be described in terms of the same parameters $\theta_i(h_k)$, $i = 0, \ldots, n-1$.

Since $\rho_i(h)$, $i = 0, \ldots, n-1$, are written as linear combinations of terms $h^k\exp(\lambda h)$, where λ is an eigenvalue of matrix A, and h_k satisfies (4), the minimum and maximum values of $\theta_i(h_k)$, $i = 0, \ldots, n-1$, can be determined in such a way that

$$\underline{\theta}_i \le \theta_i(h_k) \le \overline{\theta}_i, \quad i = 0, \ldots, n-1$$

All possible outcomes for $\tilde{A}(h_k)$ and $\tilde{B}_u(h_k)$ are then given by

$$\tilde{A}(\alpha(k)) = \sum_{j=1}^{N}\alpha_j(k)\tilde{A}_j, \quad \tilde{B}_u(h_k) = \sum_{j=1}^{N}\alpha_j(k)\tilde{B}_{uj}$$

with $N = 2^n$ and the time-varying vector $\alpha(k)$ lying inside the unit simplex

$$\mathscr{U} = \left\{\alpha \in \mathbb{R}^N : \sum_{i=1}^{N}\alpha_i = 1,\ \alpha_i \ge 0,\ i = 1, \ldots, N\right\} \tag{11}$$

for all $k \ge 0$. The vertices $(\tilde{A}_j, \tilde{B}_{uj})$ of the polytope are obtained by all possible combinations of $\underline{\theta}_j$ and $\overline{\theta}_j$ in (10).

The uncertain polytopic closed-loop system is then given by

$$\begin{aligned}\tilde{x}(k+1) &= \tilde{A}_{cl}(\alpha(k))\tilde{x}(k) + \tilde{B}_w w(k) \\ y(k) &= \tilde{C}_{cl}\tilde{x}(k) + \tilde{D}_w w(k)\end{aligned} \tag{12}$$

with

$$\tilde{A}_{cl}(\alpha(k)) = \tilde{A}(\alpha(k)) + \tilde{B}_u(\alpha(k))K, \quad \tilde{C}_{cl} = \tilde{C} + \tilde{D}_u K \tag{13}$$

and the uncertain matrices $(\tilde{A}(\alpha(k)), \tilde{B}_u(\alpha(k)))$ belong to the polytope

$$\mathscr{P} \triangleq \left\{(\tilde{A}(\alpha(k)), \tilde{B}_u(\alpha(k))) = \sum_{j=1}^{N}\alpha_j(\tilde{A}_j, \tilde{B}_{uj}), \quad \alpha \in \mathscr{U}\right\} \tag{14}$$

for all $k \ge 0$.

The control problem to be dealt with can be stated as follows.

Problem 1: Find constant matrices $K_x \in \mathbb{R}^{m\times n}$ and $K_d \in \mathbb{R}^{m\times n}$ of the state feedback control (7) such that the closed-loop system (12) is asymptotically stable and an upper bound γ to the $\mathscr{H}_\infty$ performance is minimised, that is

$$\sup_{w\neq 0}\frac{\|y\|_2^2}{\|w\|_2^2} < \gamma^2 \tag{15}$$

with $w \in l_2[0, \infty)$.

In the literature, an LMI characterisation of such an $\mathscr{H}_\infty$ disturbance attenuation for a precisely known closed-loop system is given by the discrete-time version of the bounded real lemma [22], with extensions to uncertain systems [23] and to the time-varying case [24]. A slightly modified version, motivated by a quadratic in the state path-dependent Lyapunov function [17] is presented in the next lemma.

Lemma 1: The closed-loop system (12) is asymptotically stable with an $\mathcal{H}_\infty$ disturbance attenuation given by $\gamma > 0$ if there exists a symmetric parameter-dependent matrix $P(\alpha(k), \alpha(k+1))$ such that [the symbol $(\star)$ indicates symmetric blocks in the LMIs] (see (16))

Proof: Note that the feasibility of (16) assures $P(\alpha(k), \alpha(k+1)) > 0$. Multiply on the left and on

$$\begin{bmatrix} P(\alpha(k+1), \alpha(k+2)) & \tilde{A}_{cl}(\alpha(k))'P(\alpha(k), \alpha(k+1)) & \tilde{C}_{cl}' & 0 \\ (\star) & P(\alpha(k), \alpha(k+1)) & 0 & P(\alpha(k), \alpha(k+1))\tilde{B}_w \\ (\star) & (\star) & \gamma\mathbf{I} & \tilde{D}_w \\ (\star) & (\star) & (\star) & \gamma\mathbf{I}\end{bmatrix} > 0 \tag{16}$$

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the right in (16) by $\text{diag}\{P(\alpha(k+1), \alpha(k+2))^{-1}, P(\alpha(k), \alpha(k+1))^{-1}, \mathbf{I}, \mathbf{I}\}$ and apply the Schur complement to obtain (see equation at the bottom of the page)

which is the discrete-time version of the bounded real lemma for time-varying systems. As a matter of fact, the above condition can be obtained by defining the Lyapunov function

$$v(x(k)) = x(k)'P(\alpha(k), \alpha(k+1))^{-1}x(k) \qquad (17)$$

and imposing

$$\Delta v(x(k)) + \gamma^{-1}y(k)'y(k) - \gamma w(k)'w(k) < 0$$

to the dual of system (12). □

Lemma 2: For a given $\gamma > 0$, if there exist a symmetric parameter-dependent matrix $P(\alpha(k), \alpha(k+1)) > 0$ and a parameter-dependent matrix $\mathcal{X}(\alpha(k), \alpha(k+1))$ such that (see (18))

where

$$\mathcal{B}(\alpha(k)) = \begin{bmatrix} -\mathbf{I} & \tilde{A}_{cl}(\alpha(k))' & \tilde{C}_{cl}' \end{bmatrix}$$

then the closed-loop system (12) is asymptotically stable with an upper bound $\gamma > 0$ to the $\mathcal{H}_\infty$ performance.

Proof: Suppose there exist $P(\alpha(k), \alpha(k+1))$ and $\mathcal{X}(\alpha(k), \alpha(k+1))$ such that (18) is verified. Then, multiply (18) by $(\mathcal{B}^\perp(\alpha(k)))'$ on the left and by $\mathcal{B}^\perp(\alpha(k))$ on the right with

$$\mathcal{B}^\perp(\alpha(k)) = \begin{bmatrix} \tilde{A}_{cl}(\alpha(k))' & \tilde{C}_{cl}' \\ \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix}, \quad \mathcal{B}(\alpha(k))\mathcal{B}^\perp(\alpha(k)) = 0$$

Considering the dual system (i.e. $\tilde{A}_{cl} = \tilde{A}_{cl}'$, $\tilde{B}_w = \tilde{C}_{cl}'$, $\tilde{C}_{cl} = \tilde{B}_w'$ and $\tilde{D}_w = \tilde{D}_w'$) and applying the Schur complement, inequality (16) follows in a straightforward way. □

Lemma 2 provides a sufficient condition that assures robust asymptotic stability with γ disturbance attenuation to the uncertain time-varying closed-loop system (12) in terms of the existence of a symmetric parameter-dependent matrix $P(\alpha(k), \alpha(k+1))$ and an extra variable $\mathcal{X}(\alpha(k), \alpha(k+1))$ that must verify inequality (18) for $\alpha(k) \in \mathcal{U}$, $\alpha(k+1) \in \mathcal{U}$. As has been presented, Lemma 2 cannot be used to solve Problem 1, since the decision variables do not have a known structure, the control gains K_x and K_d in the time-varying closed-loop matrix $\tilde{A}_{cl}(\alpha(k))$ appear in non-linear terms, and the parameter-dependent condition (18) must be tested for all $\alpha(k) \in \mathcal{U}$, $k \geq 0$.

The main purpose of this paper is to provide finite-dimensional LMI-based conditions, formulated in terms of the vertices of the polytope $\mathcal{P}$, to solve Problem 1. For that, two main facts are exploited:

- The time-varying parameter of the polytopic model $\alpha(k)$ is allowed to vary arbitrarily fast inside the polytope, that is, $\alpha(k+1) \in \mathcal{U}$ is independent of $\alpha(k) \in \mathcal{U}$.

- Lemma 2 provides a sufficient condition for the closed-loop system asymptomatic stability with γ disturbance attenuation independently of matrix $\mathcal{X}(\alpha(k), \alpha(k+1))$, which represents an important degree of freedom. The result can be viewed as an extension of Finsler's lemma [25]. Several different sufficient conditions could be derived by imposing particular choices to $\mathcal{X}(\alpha(k), \alpha(k+1))$. As an example, the particular choice

$$\mathcal{X} = \begin{bmatrix} F(\alpha(k))' & 0 & 0 \end{bmatrix}'$$

produces a result which is similar to the one in [24, Theorem 1], but with inconvenient products of terms depending on $\alpha(k)$. To avoid the product of parameter-dependent terms occurring at the same instant of time, some blocks could be made constant, zeroed out or constrained to depend only on $\alpha(k+1)$.

By making $\alpha(k+2) = \delta(k) \in \mathcal{U}$, $\alpha(k+1) = \beta(k) \in \mathcal{U}$ and by imposing a special structure to the the extra variable $\mathcal{X}(\alpha(k), \alpha(k+1)) = \mathcal{X}(\beta(k))$ in Lemma 2, BMI conditions assuring the existence of K_x and K_d that solve Problem 1 are given in the next section.

$$\begin{bmatrix} P(\alpha(k), \alpha(k+1))^{-1} - \tilde{A}_{cl}(\alpha(k))P(\alpha(k+1), \alpha(k+2))^{-1}\tilde{A}_{cl}(\alpha(k))' - \gamma^{-1}\tilde{B}_w\tilde{B}_w' & \tilde{A}_{cl}(\alpha(k))P(\alpha(k+1), \alpha(k+2))^{-1}\tilde{C}_{cl}' + \gamma^{-1}\tilde{B}_w\tilde{D}_w' \\ (\star) & \gamma\mathbf{I} - \tilde{C}_{cl}P(\alpha(k+1), \alpha(k+2))^{-1}\tilde{C}_{cl}' - \gamma^{-1}\tilde{D}_w\tilde{D}_w' \end{bmatrix} > 0$$

$$\begin{bmatrix} P(\alpha(k+1), \alpha(k+2)) & 0 & 0 \\ (\star) & -P(\alpha(k), \alpha(k+1)) + \gamma^{-1}\tilde{B}_w\tilde{B}_w' & \gamma^{-1}\tilde{B}_w\tilde{D}_w' \\ (\star) & (\star) & \gamma^{-1}\tilde{D}_w\tilde{D}_w' - \gamma\mathbf{I} \end{bmatrix} + \mathcal{X}(\alpha(k), \alpha(k+1))\mathcal{B}(\alpha(k)) + \mathcal{B}(\alpha(k))'\mathcal{X}(\alpha(k), \alpha(k+1))' < 0 \qquad (18)$$

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3 Main results

Theorem 1 ($\mathcal{H}_\infty$ robust controller): For a given $\gamma > 0$, if there exist matrices $L \in \mathbb{R}^{m\times(n+m)}$, $H_i \in \mathbb{R}^{q\times(n+m)}$, F, G_i, $P_{ij} = P'_{ij} > 0 \in \mathbb{R}^{(n+m)\times(n+m)}$, $i = 1, \ldots, N$ and $j = 1, \ldots, N$, such that (see (19))

then the memory state feedback control gain that solves Problem 1 is given by

$$K = [K_x \quad K_d] = L(F')^{-1} \tag{20}$$

assuring that the closed-loop system (12) is asymptotically stable with an upper bound γ to the $\mathcal{H}_\infty$ performance.

Proof: Multiplying (19) by α_i, β_j and δ_k, summing for $i, j, k = 1, \ldots, N$, and making $L = KF'$, one obtains (see (21))

which is exactly the parameter-dependent condition (18) of Lemma 2 with

$$\alpha(k) = \alpha \in \mathcal{U}, \quad \alpha(k+1) = \beta \in \mathcal{U}, \quad \alpha(k+2) = \delta \in \mathcal{U}, \quad \forall k \geq 0$$

$$P(\alpha(k), \alpha(k+1)) = P(\alpha, \beta) = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \beta_j P_{ij}$$

$$P(\alpha(k+1), \alpha(k+2)) = P(\beta, \delta) = \sum_{j=1}^{N}\sum_{k=1}^{N} \beta_j \delta_k P_{jk}$$

$$\mathcal{X}(\alpha(k), \alpha(k+1)) = \sum_{k=1}^{N} \beta_j \begin{bmatrix} F \\ G_j F \\ H_j F \end{bmatrix} = \begin{bmatrix} F \\ G(\beta)F \\ H(\beta)F \end{bmatrix}$$

and the closed-loop matrices $\tilde{A}_{cl}(\alpha)$ and $\tilde{C}_{cl}$ as in (12). Finally, the control gain is obtained from the change of variables $L = KF'$, yielding $K = L(F')^{-1}$. □

Corollary 1: The minimum γ attainable by the conditions of Theorem 1 is given by the optimisation problem

$$\min \gamma \ \ s.t. \ (19) \tag{22}$$

3.1 Remarks and extensions

The first important remark is that by fixing $G = H = 0$, the conditions of Theorem 1 reduce to LMIs. Consequently, Corollary 1 in this case is a convex optimisation problem that can be efficiently handled by semidefinite programming algorithms, for instance, SeDuMi [26]. Although several methods could be applied in the solution of the BMI problem (22), the following algorithm is suggested. Fix the variables $H_i = 0$ and $G_i = 0$ and minimise γ with respect to F, L and P_{ij}. Then, fix the variables F, L and P_{ij}, minimise γ with respect to H_i and G_i, and obtain the new values of H_i and G_i. Repeat this procedure until no significant changes in the value of γ occur. This algorithm is sometimes called the alternating semidefinite programming method and consists of fixing some variables and solving for others in such a way that at each step one has a convex optimisation problem. Despite the fact that there is no guarantee of convergence to a local minimum in a general BMI setting, these methods are easy to implement and provide good results in many cases, as illustrated by the examples presented in Section 4.

It is important to emphasise at this point that the BMI conditions are used to improve the quality of the $\mathcal{H}_\infty$ attenuation level γ, that is, to make it tighter. At each step of the proposed algorithm, a *convex* optimisation problem with LMI constraints is solved. More specifically, in the first step of this algorithm the matrices $G(\cdot)$ and $H(\cdot)$ are

$$\begin{bmatrix} P_{jk} - F - F' & F\tilde{A}'_i + L'\tilde{B}'_{ui} - F'G'_j & F\tilde{C}' + L'\tilde{D}_u - F'H'_j & 0 \\ (\star) & G_jF\tilde{A}'_i + \tilde{A}_iF'G'_j + G_jL'\tilde{B}'_{ui} + \tilde{B}_{ui}LG'_j - P_{ij} & G_jF\tilde{C}' + G_jL'\tilde{D}'_u + \tilde{A}_iF'H'_j + \tilde{B}_{ui}LH'_j & \tilde{B}_w \\ (\star) & (\star) & H_jF\tilde{C}' + \tilde{C}F'H'_j + H_jL'\tilde{D}'_u + \tilde{D}_uLH'_j - \gamma\mathbf{I} & \tilde{D}_w \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0$$
$$i = 1, \ldots, N, \ j = 1, \ldots, N, \ k = 1, \ldots, N \tag{19}$$

$$\begin{bmatrix} P(\beta,\delta) - F - F' & F(\tilde{A}(\alpha) + \tilde{B}_u(\alpha)K)' - F'G(\beta)' & F(\tilde{C} + \tilde{D}_uK)' - F'H(\beta)' & 0 \\ (\star) & G(\beta)F(\tilde{A}(\alpha) + \tilde{B}_u(\alpha)K)' + (\tilde{A}(\alpha) + \tilde{B}_u(\alpha)K)F'G(\beta)' - P(\alpha,\beta) & G(\beta)F(\tilde{C} + \tilde{D}_uK)' + (\tilde{A}(\alpha) + \tilde{B}_u(\alpha)K)F'H(\beta)' & \tilde{B}_w \\ (\star) & (\star) & H(\beta)F(\tilde{C} + \tilde{D}_uK)' + (\tilde{C} + \tilde{D}_uK)F'H(\beta)' - \gamma\mathbf{I} & \tilde{D}_w \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \tag{21}$$

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set to zero and the initial solution is obtained from a convex LMI optimisation problem (Corollary 1). The start of the algorithm with zeroed matrices turns out to be a good option since it reproduces the convex controller design conditions that appeared in the literature for discrete-time systems with time varying parameters [18]. Other choices for initial values $G(\cdot)$ and $H(\cdot)$, although possible, do not have a correspondence with existing conditions. Moreover, by fixing some variables while searching for others, one is always solving LMI problems that assure non-increasing values of γ. As can be seen from the numerical examples, the algorithm provides very good results.

The fact that the conditions of Theorem 1 need to be satisfied by constant matrices L and F guarantees the existence of a robust state feedback gain $K = L(F')^{-1}$. Other choices could be used, resulting in different structures for $\mathscr{X}(\alpha(k), \alpha(k+1))$ that would, in general, lead to parameter-dependent feedback gains. In particular, the choices made in Theorem 1 assure that the extra variable $\mathscr{X}(\alpha(k), \alpha(k+1))$ depends only on $\alpha(k+1)$, in other words, that all the products between the uncertain time-varying matrix $\tilde{A}(\alpha(k))$ and $G(\cdot)$, $H(\cdot)$ in Lemma 2 occur at different instants of time. Products of time-varying matrices at the same instant of time $\alpha(k)$ in Lemma 2 would require more involved manipulations, such as, for instance, the ones proposed in [27].

A state feedback controller has been chosen to illustrate the potentialities of the proposed approach. Sufficient conditions for decentralised or output feedback control could be obtained by imposing block diagonal structures to the matrices L and F in Theorem 1, following the lines depicted in [28, 29].

The results of Lemma 2 and Theorem 1 could be improved by considering a larger *path* in the Lyapunov function (17) of Lemma 2, that is, $P(\alpha(k), \ldots, \alpha(k+L))$. Larger paths (not necessarily of the same size) and other structures could also be used in the extra matrix $\mathscr{X}(\cdot)$ of Lemma 2. At the expense of a larger computational effort, lower values for γ can be obtained. Note that the LMI conditions for a path of size $L+1$ provide at least the same values of γ obtained with L.

On the other hand, simpler design conditions based on a Lyapunov matrix $P(\alpha(k))$ can be obtained as a particular case of Theorem 1. This preliminary result, stated in the next corollary, appears in [15].

Corollary 2: A sufficient condition for the existence of a memory state feedback control gain that solves Problem 1 is obtained by solving Theorem 1 with matrices $P_{ij} = P_i$ and $P_{jk} = P_j$, that is

$$P(\alpha(k)) = \sum_{i=1}^{N} \alpha_i P_i, \quad P(\alpha(k+1)) = \sum_{j=1}^{N} \beta_j P_j, \quad \alpha, \beta \in \mathscr{U}$$

Finally, it is important to emphasise that using the Cayley–Hamilton theorem to deal with the matrix exponential in (3) provides a systematic way to obtain the vertices of polytope (14). It is also helpful when bounded rates of variation are involved, since in this case an explicit expression for the variation rate may be obtained. Additionally, the use of a polytope to model the time-varying parameter h_k represents an interesting strategy for solving Problem 1. First, it does not require a knowledge of the processor/network clock granularity g, since the only information used to derive the polytopic model is the extreme values of sets (4). Second, the time-varying uncertainties, introduced during the sampling stage, can be completely modelled by a polytope of the form (14). Once one has defined the vertices of the closed-loop polytope, there will exist a vector $\alpha(k)$ such that (14) holds for each instant of time $k \geq 0$. The only condition on vector $\alpha(k)$ is that it belongs to the unit simplex $\mathscr{U}$ for all $k \geq 0$. Furthermore, the number of values in the set (4) does not influence the computational burden; in other words, a larger number of h_k does not imply a greater computational effort, which allows clock granularity to be as small as possible.

3.2 More complex NCS scenarios

The controller design method addressed here is mainly concerned with a time-varying sampling period motivated by applications to reduce bandwidth usage. As pointed out in [6], the bandwidth may be reduced by controlling the values that h_k assumes as time evolves in order to reduce the flow of information between the sensor and the controller/actuator. Since robust control is at issue, the sampling period is considered to be uncertain and Lyapunov theory is used for the purpose of synthesis. Although the proposed approach simplifies, or even neglects, some aspects of the NCS (the assumptions here being constant time delay, no packet dropouts, single-packet transmission and infinite sensor precision), some ideas are proposed on how to deal with more complex scenarios.

When the time delay is considered constant and longer than h, system (5) has to be slightly modified and more state variables are used to describe the delay, as proposed in [20]. In this case, the matrices in (6) become

$$\tilde{A}(h) = \begin{bmatrix} A_s(h) & B_{su1}(h, \tau) & 0 & \ldots & 0 \\ 0 & 0 & \mathbf{I} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \mathbf{I} \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix},$$

$$\tilde{B}_u(h) = \begin{bmatrix} B_{su0}(h, \tau) \\ 0 \\ \vdots \\ 0 \\ \mathbf{I} \end{bmatrix}, \quad \tilde{B}_w(h) = \begin{bmatrix} B_{sw}(h, \tau) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} C_s & D_{sd} & 0 & \ldots & 0 \end{bmatrix}$$

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Assuming an event-driven controller and actuator, Theorem 1 could be applied when the delay is time varying within an interval, but in this case the information on the bounds of τ_k is used to derive the vertices of the polytope – in this case, $\eta_i(\cdot)$ in (8) would be a function of both h_k and τ_k. Whenever possible, the use of a memory controller through the simplified analysis presented here is suggested, but the method could be adapted to use more complex Lyapunov functions, such as Lyapunov–Krasovskii functionals.

Data packet dropout and multiple-packet transmission in NCS can be modelled as an asynchronous dynamical system (ADS) with rate constraints on events [2]. A simplified ADS with rate constraints can be written as a set of difference equations, as proposed in [2]

$$x(k+1) = f_s(x(k)), \quad s = 1, 2, \ldots, N \tag{23}$$

where each discrete state $f_s(\cdot)$ occurs in a fraction of time r_s, $\sum_{s=1}^{N} r_s = 1$. The stability of such a class of systems is studied in [30], as reproduced in the following lemma.

Lemma 3 ([30]): Given an ADS as (23), if there exist a Lyapunov function $V(x(k)) : \mathbb{R}^n \to R_+$ and scalars ξ_1, ξ_2, $\ldots$, ξ_N corresponding to each rate such that

$$\xi_1^{r_1}\xi_2^{r_2}\cdots\xi_N^{r_N} > \xi > 1 \tag{24}$$

$$V(x(k+1)) - V(x(k)) \le (\xi_s^{-2} - 1)V(x(k)), \quad s = 1, 2, \ldots, N \tag{25}$$

then the ADS remains exponentially stable, with a decay rate greater than ξ.

By using Lemma 3, Theorem 1 can be extended to deal with packet dropout and multiple-packet transmission. The NCS is modelled by a set of difference equations activated by a switch that closes at a certain rate r. The packet dropout effect, or the multiple-packet transmission, is then represented by an augmented system, as done in [2], and Lemma 3 is applied in the study of stability.

Finally, concerning the infinite sensor precision, the effect of quantisers can be modelled by using the sector bound approach. This strategy treats the quantisation error as a non-linearity that lies inside a sector bound. It is a simple and classic approach to study quantisation effects and is closely related to absolute stability theory [31]. The approaches discussed in [32] could be explored in this direction.

It is worth mentioning that the extensions proposed in this section may be involved or introduce some conservatism in the results. The aim here is to point out that Theorem 1 is not restricted to a simplified framework and may be adapted to deal with different situations. These topics are under investigation by the authors.

4 Numerical experiments

Example 1: The aim here is to illustrate the potentialities of the proposed method and to show in detail the steps based on the Cayley–Hamilton theorem to obtain the vertices of the polytopic model.

This example, borrowed from [21], is a simplified model of an armature voltage-controlled DC servo motor consisting of a stationary field and a rotating armature and load. All effects of the field are neglected. The aim is to design $\mathscr{H}_\infty$ robust memory control of the speed of the shaft. All information is sent through a communication network. The behaviour of the DC servo motor shown in Fig. 2 can be described by the differential equations

$$\begin{bmatrix} \ddot{\varphi} \\ \dot{\rho}_a \end{bmatrix} = \begin{bmatrix} -\dfrac{b}{J} & \dfrac{K_T}{J} \\ -\dfrac{K_\varphi}{L_a} & -\dfrac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \rho_a \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{L} \end{bmatrix} e_a(t) \tag{26}$$

where e_a is the externally applied armature voltage, ρ_a is the armature current, R_a is the resistance of the armature winding, L_a is the armature winding inductance, e_m is the back electromotive force (emf) voltage induced by the rotating armature winding ($e_m = K_\varphi\dot{\varphi}$, $K_\varphi > 0$), b is the viscous damping due to bearing friction, J is the moment of inertia of the armature and load, and φ is the shaft position. The torque generated by the motor is given by $T = K_T i_a$ and $J = 0.01\,\text{kgm}^2/\text{s}^2$, $b = 0.1\,\text{Nms}$, $K_T = K_\varphi = 0.01\,\text{Nm/Amp}$, $R_a = 1\,\Omega$, and $L_a = 0.5\,\text{H}$.

System (26) was also studied in [8], assuming zero delay, time-varying sampling rates in the sensor and no index of performance. Although this system is already stable, Corollary 1 was applied in order to provide a gain matrix that guarantees robustness against unmodelled $l_2[0, \infty)$ perturbations by minimising the $\mathscr{H}_\infty$ index of performance of the closed-loop system. Furthermore, a non-zero delay is considered, $\tau = 0.5$ ms, and the sampling rate is allowed to vary within the interval $h_k \in [0.001\ 0.099]$.

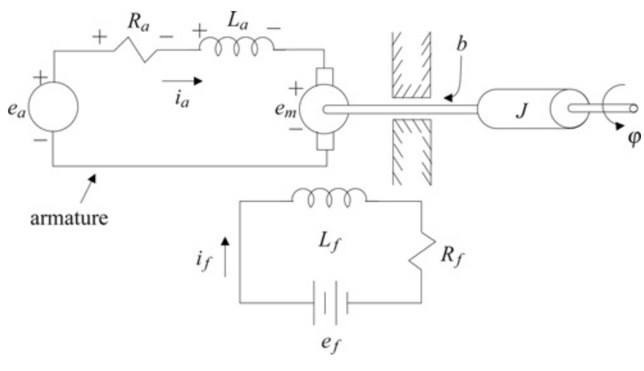

Figure 2 *DC servo motor as presented in [21]*

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Closing the loop with (7), one can express system (26) by the polytope (14) with four vertices ($N = 4$) obtained through the Cayley–Hamilton theorem as follows. First, to compute $A_s(h) = \exp(Ah)$, obtain $\rho_0(h)$ and $\rho_1(h)$ by solving the linear system

$$\begin{bmatrix} 1 & -9.9975 \\ 1 & -2.0025 \end{bmatrix}\begin{bmatrix} \rho_0(h) \\ \rho_1(h) \end{bmatrix} = \begin{bmatrix} \theta_1(h) \\ \theta_2(h) \end{bmatrix}$$

$$\theta_1(h) = \exp(-9.9975h), \quad \theta_2(h) = \exp(-2.0025h)$$

Then, express $\exp(Ah)$ as

$$\exp(Ah) = \rho_0(h)\mathbf{I} + \rho_1(h)A = \begin{bmatrix} 1.0003 & -0.1251 \\ 0.0025 & -0.0003 \end{bmatrix}\theta_1(h) + \begin{bmatrix} -0.0003 & 0.1251 \\ -0.0025 & 1.0003 \end{bmatrix}\theta_2(h)$$

By evaluating $\theta_1(h)$ and $\theta_2(h)$ at the extreme values of h, one has

$$0.3717 \le \theta_1(h) \le 0.9901, \quad 0.8202 \le \theta_2(h) \le 0.9980$$

and the polytopic model with $N = 4$ vertices (obtained by collecting terms) is given by

$$\exp(Ah) = \alpha_1\begin{bmatrix} 0.3715 & 0.0561 \\ -0.0011 & 0.8203 \end{bmatrix} + \alpha_2\begin{bmatrix} 0.9901 & -0.0212 \\ 0.0004 & 0.8201 \end{bmatrix} + \alpha_3\begin{bmatrix} 0.3715 & 0.0783 \\ -0.0016 & 0.9982 \end{bmatrix} + \alpha_4\begin{bmatrix} 0.9900 & 0.0010 \\ -0.0000 & 0.9980 \end{bmatrix}$$

Similarly, to compute $B_{su0}(h, \tau)$, use Cayley–Hamilton to obtain $\eta_0(h)$ and $\eta_1(h)$ such that

$$\int_0^{h-\tau} \exp(As)\,\mathrm{d}s = \eta_0(h)\mathbf{I} + \eta_1(h)A$$

by solving the linear system

$$\begin{bmatrix} \eta_0(h) \\ \eta_1(h) \end{bmatrix} = \begin{bmatrix} 0.0252 & -0.6251 \\ 0.0126 & -0.0625 \end{bmatrix}\begin{bmatrix} \theta_1(h) \\ \theta_2(h) \end{bmatrix} - \begin{bmatrix} -0.5994 \\ -0.0500 \end{bmatrix}$$

Then, using the extreme values for $\theta_1(h)$ and $\theta_2(h)$ above and collecting terms, one obtains

$$B_{su0}(h, \tau) = \int_0^{h-\tau} \exp(As)\,\mathrm{d}sB = \alpha_1\begin{bmatrix} 0.0067 \\ 0.1788 \end{bmatrix} + \alpha_2\begin{bmatrix} 0.0222 \\ 0.1788 \end{bmatrix} + \alpha_3\begin{bmatrix} -0.0155 \\ 0.0010 \end{bmatrix} + \alpha_4\begin{bmatrix} 0 \\ 0.0010 \end{bmatrix}$$

Matrix $B_{su1}(h, \tau)$ can be evaluated from $\exp(Ah)$ since

$$B_{su1}(h, \tau) = \exp(Ah)(\exp(-A\tau)\int_0^{\tau} \exp(As)\,\mathrm{d}sB)$$

yielding (from similar steps) the vertices

$$B_{su1_1} = \begin{bmatrix} 0.0001 \\ 0.0008 \end{bmatrix}, \quad B_{su1_2} = \begin{bmatrix} 0 \\ 0.0008 \end{bmatrix}$$
$$B_{su1_3} = \begin{bmatrix} 0.0001 \\ 0.0010 \end{bmatrix}, \quad B_{su1_4} = \begin{bmatrix} 0 \\ 0.0010 \end{bmatrix}$$

The polytopic model for $\tilde{A}(\alpha)$ is then given by

$$\tilde{A}_1 = \left[\begin{array}{cc|c} 0.3715 & 0.0561 & 0.0001 \\ -0.0011 & 0.8203 & 0.0008 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$\tilde{A}_2 = \left[\begin{array}{cc|c} 0.9901 & -0.0212 & 0 \\ 0.0004 & 0.8201 & 0.0008 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$\tilde{A}_3 = \left[\begin{array}{cc|c} 0.3715 & 0.0783 & 0.0001 \\ -0.0016 & 0.9982 & 0.0010 \\ \hline 0 & 0 & 0 \end{array}\right]$$

$$\tilde{A}_4 = \left[\begin{array}{cc|c} 0.9900 & 0.0010 & 0 \\ 0 & 0.9980 & 0.0010 \\ \hline 0 & 0 & 0 \end{array}\right]$$

System (26) is then rewritten as in (5) with matrices $D_{sw} = [1]$, $B'_{sw} = [0.1\ 0]$ and $D_{sd} = [0]$.

Corollaries 1 and 2 are solved by using alternating semidefinite programming. Each iteration consists of two steps. First, the problem is solved with $G(\cdot) = 0$ and $H(\cdot) = 0$ (in this case, the problem is convex) and, second, $G(\cdot)$ and $H(\cdot)$ are explored in the search for a better $\mathscr{H}_\infty$ upper bound γ. The results after five iterations are shown in Table 1.

Sufficient conditions for the existence of a decentralised or a static output feedback control gain can be obtained from Theorem 1 by simply imposing to matrices L and F in (20) a fixed structure, following the lines in [28, 29]. For instance, suppose that the first state variable is not available

Table 1 $\mathscr{H}_\infty$ robust memory controller for Example 1

Method	$\mathscr{H}_\infty$ Upper bound γ	Gain matrix K
Corollary 1	10.87	$[-1.8822\ -9.6684\ -0.0117]$
Corollary 2	10.90	$[-1.5670\ -9.8076\ -0.0150]$

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Table 2 $\mathscr{H}_\infty$ robust memory controller for Example 2

Method	Iteration	γ	Improvement, %	Time (s)
[33]	–	67.33	–	0.09
Corollary 1	1	31.37	53.40	0.17
Corollary 1	2	23.10	65.69	0.33
⋮	⋮	⋮	⋮	
Corollary 1	10	19.21	71.47	1.55
Corollary 2	1	30.39	54.87	0.20
Corollary 2	2	17.10	74.61	0.39
⋮	⋮	⋮	⋮	
Corollary 2	10	11.25	83.29	1.72

for feedback. By imposing

$$L = \begin{bmatrix} 0 & \ell_2 & \ell_3 \end{bmatrix}, \ F = \begin{bmatrix} f_{11} & 0 & 0 \\ 0 & f_{22} & f_{23} \\ 0 & f_{32} & f_{33} \end{bmatrix}$$

to matrices L and F in Theorem 1 the following result is obtained (after five iterations)

$$K = \begin{bmatrix} 0 & 0.8002 & -7.5613 \times 10^{-6} \end{bmatrix}, \quad \gamma = 11.09$$

Example 2: This example is intended to point out the quality of the proposed method when no communication channel is considered. Consider an uncertain time-varying discrete-time system with vertices given by

$$\tilde{A}_1 = \begin{bmatrix} 0.28 & -0.315 \\ 0.63 & -0.84 \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} 0.52 & 0.77 \\ -0.7 & -0.07 \end{bmatrix},$$

$$\tilde{B}_{u1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \tilde{B}_{u2} = \begin{bmatrix} 9 \\ 21 \end{bmatrix}$$

$$\tilde{B}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad \tilde{D}_w = \tilde{D}_u = \begin{bmatrix} 0 \end{bmatrix}$$

This system is also studied in [19] but for a simpler case, where the parameters of matrix $\tilde{B}_u$ are time invariant. Here, the results from Theorem 1 are compared to [33 Remark 2]. In order to illustrate the efficiency of the proposed method due to the use of a path-dependent Lyapunov function, Corollary 1 is contrasted with Corollary 2. The results can be seen in Table 2.

5 Conclusion

This paper addressed the $\mathscr{H}_\infty$ robust controller for NCSs with uncertain time-varying sampling rates. A new state space variable, representing the buffer of the controller, was added to model a time-delay in the control signal. A polytope with vertices determined by the Cayley–Hamilton theorem was used to model the system. Using an approach based on path-dependent Lyapunov functions, theoretical conditions were formulated for the existence of a state feedback control assuring an $\mathscr{H}_\infty$ attenuation level for the closed-loop system. Then, sufficient conditions for the existence of the memory controller are derived in terms of BMIs described only at the vertices of the polytope. An algorithm exploiting appropriate choices of the extra variables is used to solve the problem through a sequence of convex optimisation procedures, providing lower levels for the $\mathscr{H}_\infty$ performance of the closed-loop system. When no communication channel is considered, the proposed conditions can also provide better results when compared to other methods in the literature dealing with time-varying discrete-time systems. Some remarks on possible extensions to more complex NCS scenarios were presented and numerical experiments were provided to illustrate different aspects of the proposed approach.

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