Static Output Feedback: A Survey [1]

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Key Words: Output Feedback, Linear Systems, Pole Placement, Linear Quadratic Regulator, Linear Matrix Inequalities, Decision Methods, Computational Complexity.

Abstract

This paper reviews the static output feedback problem in the control of linear, time-invariant (LTI) systems. It includes analytical and computational methods and presents in a unified fashion, the knowledge gained in the decades of research into this important open problem. The paper shows that although many approaches and techniques exist to approach different versions of the problem, no efficient algorithmic solutions are available.

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[1] A preliminary version of this paper appeared in the Proceedings of the 33rd IEEE CDC, 1994, Lake Buena Vista, Fl., pp. 837–842

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1 Introduction

The static output feedback (SOF) problem is one of the most important open questions in control engineering, see for example the two recent surveys (Bernstein, 1992; Blondel et al., 1995). Simply stated, the problem is as follows: Given a linear, time-invariant system, find a static output feedback so that the closed-loop system has some desirable characteristics, or determine that such a feedback does not exist.

This paper attempts to survey the state of knowledge concerning the output feedback problem. The survey will encompass both Single-Input-Single-Output (SISO) and Multi-Input-Multi-Output (MIMO) systems. Even though the SISO case may be efficiently resolved using graphical techniques, we include it here because the fundamental question of the existence of static output controllers is still open, even in the scalar case.

This survey paper has two main parts. The first involves the study of the time-invariant plant described by

$$\dot{x}(t) = Ax(t) + Bu(t); \quad y(t) = Cx(t) \tag{1.1}$$

under the influence of static output feedback of the form

$$u(t) = Ky(t) + v(t). \tag{1.2}$$

The closed-loop system is

$$\dot{x} = (A + BKC)x(t) + Bv(t) \equiv A_c x(t) + Bv(t). \tag{1.3}$$

The problem of output feedback evolves around the selection of a constant feedback gain matrix K to achieve various closed-loop properties. We take the state $x(t) \in \mathbb{R}^{\mathrm{n}}$, the control input $u(t) \in \mathbb{R}^{\mathrm{m}}$, and the output $y(t) \in \mathbb{R}^{\mathrm{p}}$. The problem may also be studied in a transfer function setting where one is given a transfer matrix relationship between the input $u(s)$ and the output $y(s)$ (s denotes the Laplace transform variable) such that $y(s) = C(sI - A)^{-1}Bu(s) \equiv H(s)u(s)$ and the objective is to find K in the feedback law (1.2) so that the closed-loop system,

$$\begin{aligned} y(s) &= C(sI - A - BKC)^{-1}Bv(s) \\ &= [I - H(s)K]^{-1}H(s)v(s) \end{aligned} \tag{1.4}$$

satisfies some performance objectives.

The second part of this paper involves the solution of various coupled matrix design equations of the sort obtained in pole-placement and LQ design using output feedback, game theory, and elsewhere. Such coupled systems of equations are currently "solved" using iterative numerical techniques. The computational difficulty or cost of such numerical techniques have not been investigated until recently (Blondel and Tsitsiklis, 1995; Toker and Özbay, 1995).

We recall here a few mathematical definitions which will be used in this paper. We say that a rational function $H(s)$ is Bounded-Input-Bounded-Output-Stable (BIBO) stable or that it belongs to H^∞ if it is proper, with all its poles in the left-half-plane (LHP). We let $\mathcal{S}$ denote the set of matrices whose entries are in H^∞. A Unit in $\mathcal{S}$ is a member of $\mathcal{S}$ whose inverse is also in $\mathcal{S}$. We define a blocking zero of a transfer function $H(s)$ as any complex number $s = z_0$ such that $H(z_0) = 0$. In what follows, A^T denote the transpose of any matrix A, and the controller is $u = Ky + v$.

The paper basically concludes that the problem of static output feedback is still open despite the availability of many approaches and numerical algorithms. This statement is justified by the fact that no testable necessary and sufficient conditions exist to test the stabilizability of a given system using static output feedback, and that numerical algorithms can not be shown to be convergent in general. Moreover, recent results from the theory of computational complexity suggest that numerical algorithms which work well on small-sized problems, may be doomed as the problem size increases.

The paper is organized as follows: In section 2 we present a discussion on the relevance and motivation of the the static output feedback problem. Section 3 contains a discussion of stabilizability using static output feedback, including (non-testable) necessary and sufficient conditions. The section also includes design procedures such as the covariance assignment and the decision methods. The pole placement problem is presented in Section 4 and the eigenstructure assignment is discussed in section 5. Section 6 is devoted to the Linear Quadratic Regulator problem with output feedback. Section 7 reviews some recent results on the computational complexity of the SOF problem and our conclusions are presented in section 8.

2 Problem Relevance

The SOF problem is important in its own right, but also because many other problems are reducible to some variation of it. The problem is relevant for example when a simple controller must be used due to cost and reliability. It is also important in applications where a certain degree of tuning must be attempted in order to control a physical plants, and the designer needs to minimize the required number of parameters. See (Davison, 1965; Davison and Tripathi, 1978; Davison et al., 1973; Bengtsson and Lindahl, 1974) for a sample of applications of the SOF to physical plants.

The case where a dynamical output compensator of order $q \leq n$ is used may be brought back to the static output feedback case as follows (see for example (Nett et al., 1989; Mårtensson, 1985)): Suppose the dynamic compensator is given in state-space form as

$$\dot{x}_f(t) = A_f x_f(t) + B_f y(t) \tag{2.5}$$

$$u(t) = C_f x_f(t) + D_f y(t) + v(t) \tag{2.6}$$

Then, an augmented state-system is obtained when $u_f(t) = \dot{x}_f(t)$, and $y_f(t) = x_f(t)$ by

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_f(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix} + \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \begin{bmatrix} u_f(t) \\ u(t) \end{bmatrix} \\ \begin{bmatrix} y_f(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix} \end{aligned} \tag{2.7}$$

so that the feedback law is now static and given by

$$\begin{bmatrix} u_f(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \begin{bmatrix} y_f(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v(t) \tag{2.8}$$

or in a more compact description

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}\tilde{u}(t); \quad \tilde{y}(t) = \tilde{C}\tilde{x}(t); \quad \tilde{u}(t) = \tilde{K}\tilde{y}(t) + \tilde{v}(t) \tag{2.9}$$

where

$$\begin{aligned} \tilde{x} &= \begin{bmatrix} x \\ x_f \end{bmatrix}; \quad \tilde{u} = \begin{bmatrix} u_f \\ u \end{bmatrix}; \quad \tilde{y} = \begin{bmatrix} y_f \\ y \end{bmatrix}; \quad \tilde{v} = \begin{bmatrix} 0 \\ v \end{bmatrix} \\ \tilde{A} &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \\ \tilde{C} &= \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}; \quad \tilde{K} = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \end{aligned} \tag{2.10}$$

Fixed-order dynamic output feedback compensation of order $q < n$ is of paramount importance in control systems design. Modern control design techniques provide controllers of order equal to or grater than the order of the plant ($q = n$). However, for large order systems such as flexible structures these controllers are difficult or impossible to implement due to cost, reliability and hardware implementation constraints. For example, control computer hardware limitations of the Hubble Space Telescope require that the pointing error controller has order less or equal to 42 states (Buckley, 1995; Zhu et al., 1995). Model and controller order reduction techniques are often used to obtain low-order controllers for large-order systems but in this case stability of the closed-loop system is not guaranteed due to spillover effects (Balas, 1982).

3 Stabilizability By Static Output Feedback

In this section, we discuss the problem of stabilizing an open-loop unstable system with static output feedback. We present first some necessary conditions, then some sufficient ones for the solvability of this problem. We then discuss some approaches used to find a stabilizing gain K.

3.1 Necessary Conditions

We first identify the cases where static output feedback cannot stabilize an open-loop unstable system. This will at least provide us with necessary conditions, which when violated, tell us that dynamic feedback compensation is required. In order to state these conditions, we recall the following theorems.

Theorem 3.1 (Youla et al., 1974) **The Parity-Interlacing-Property (PIP)** A linear system $H(s)$ is stabilizable with a stable compensator $C(s)$ (i.e., *strongly* stabilizable) if and only if the number of real poles of $H(s)$, counted according to their McMillan degree, between any pair of real blocking zeros in the right-half-plane is even. A system which satisfies the pole-zero constraints is said to satisfy the PIP. ■

Note that in the SISO case, the PIP fails to hold for many real systems. On the other hand, as observed in (Hagander and Bernhardsson, 1990), (Vidyasagar, 1985) and (Youla et al., 1974), the PIP holds generically in the MIMO case.

Theorem 3.2 (Wei, 1990) A linear system $H(s)$ is stabilizable with a stable compensator $C(s)$ which has no real unstable zeros if and only if: 1) $H(s)$ satisfies the PIP, and 2) The number of real blocking zeros of $H(s)$ between any two real poles of $H(s)$ is even. In this case we say that $H(s)$ satisfies the even PIP. ■

Using Theorem 3.2, the following necessary condition is obtained:
Necessary Condition 1: A necessary condition for static output stabilizability is that the plant $H(s)$ satisfies the even PIP. ■

Example 3.1 *The following example, suggested by a reviewer, illustrates the necessary conditions cited above. It also illustrates the SOF problems can be ill posed. Consider the plant*

$$H(s) = \frac{1-s}{(\epsilon s+1)(s-2)}$$

For $\epsilon > 0$ PIP is not satisfied, so that this plant is not SOF stabilizable. For $-0.5 < \epsilon \leq 0$, the even PIP is satisfied, and for small enough ϵ a simple root-locus analysis indicates the plant is SOF stabilizable.

3.2 Sufficient Conditions

We start out by noting the simple case of SISO systems, of relative degree $n^* \leq 1$, and which are minimum phase (the finite zeros are stable). A simple root-locus argument then shows that such systems are stabilizable with a large enough static output feedback. In fact, the minimum-phase and the relative-degree conditions are necessary and sufficient to make square (i.e. same number of inputs and outputs) systems Strictly-Positive-Real (SPR) using static output feedback as described for example in (Gu, 1990b) and (Abdallah et al., 1991).

3.3 Design Approaches and Limitations

In the case of SISO systems, graphical approaches (root-locus, Nyquist) are used to answer both the existence and the design questions of stabilizing static output controllers. In addition, there exist some necessary and sufficient algebraic tests (Helmke and Anderson, 1992), (Pérez et al., 1993) for the existence of stabilizing output feedbacks. These tests however, require some preliminary derivations (finding roots, eigenvalues) which are just as complicated as the graphical methods. In addition, they are not easily extendable to the MIMO case, although some specialized cases may be resolved using the Multivariable Nyquist criterion (Brockett and Byrnes, 1981). The work in (Byrnes and Crouch, 1985), also presents a complete characterization of strictly-proper SISO systems related to each other with static output feedback. In fact, it states that such systems must share the same zeros and the same breakaway points. This then leads to the open question, of finding at least one stable transfer function having the same zeros and the same breakaway points as our open-loop system $H(s)$.

In this section, we list some parameterization results that are potentially useful in solving the static output feedback problem. The idea basically is that a stabilizing static output feedback must be a member of the family of all stabilizing output feedback compensators.

3.3.1 Youla Parameterization Method

The following result parameterizes all stabilizing controllers in terms of a matrix in $\mathcal{S}$ (Vidyasagar, 1985).

Theorem 3.3 A compensator $C(s) = N_c(s)D_c(s)^{-1} = \tilde{D}_c(s)^{-1}\tilde{N}_c(s)$ where $N_c(s), \tilde{N}_c(s), \tilde{D}_c(s)$ and $D_c(s)$ are in $\mathcal{S}$, internally stabilizes the plant $H(s) = N_p(s)D_p(s)^{-1} = \tilde{D}_p(s)^{-1}\tilde{N}_p(s)$ if and only if either

1. $\tilde{N}_c(s)N_p(s) + \tilde{D}_c(s)D_p(s)$, or
2. $\tilde{N}_p(s)N_c(s) + \tilde{D}_p(s)D_c(s)$

is a Unit of $\mathcal{S}$. Moreover, the set of all stabilizing compensators of $H(s)$ is given by

$$\begin{aligned} \mathcal{C} &= \left\{C(s);\quad C(s) = [\tilde{N}_c(s) + D_p(s)Q(s)][\tilde{D}_c(s) - N_p(s)Q(s)]^{-1}\right\};\\ &\quad \left\{C(s);\quad C(s) = [D_c(s) - R(s)\tilde{N}_p(s)]^{-1}[N_c(s) + R(s)\tilde{D}_p(s)]\right\}; \end{aligned}$$

for any $Q(s), R(s) \in \mathcal{S}$ as long as $|\tilde{D}_c(s) - N_p(s)Q(s)| \neq 0$ and $|D_c(s) - R(s)\tilde{N}_p(s)| \neq 0$. ■

It can then be argued that a necessary and sufficient condition for the static output stabilizability problem is that there exists a $Q(s) \in \mathcal{S}$ such that

$$K = [N_c(s) + D_p(s)Q(s)][D_c(s) - N_p(s)Q(s)]^{-1} \tag{3.11}$$

is a constant matrix. In fact, such an approach is advocated in (Gu et al., 1993), where a search is conducted to find a $Q(s)$ to reduce the order of the compensators. Unfortunately, this and other so-called necessary and sufficient conditions are non-testable and as such they cannot be used to answer the existence question.

3.3.2 Inverse Linear Quadratic Approach

In (Trofino-Neto and Kučera, 1993) another necessary and sufficient condition was found for the stabilizability of a linear system using output feedback in terms of the solvability of a modified LQR problem. In fact, the authors in (Trofino-Neto and Kučera, 1993), state and prove the following result.

Theorem 3.4 Given the system (1.1), and let $E_i = C^\dagger C$, where superscript "†" denotes the Moore-Penrose inverse. Then, the system is stabilizable with static output feedback $K = -R^{-1}(L + B'P)E_i$ if and only if there exist matrices $Q > 0$, $R > 0$ and L of compatible dimensions such that the algebraic equation

$$A^T P + PA - E_i(PB + L^T)R^{-1}(B^T P + L)E_i + Q = 0 \tag{3.12}$$

has a unique solution $P > 0$. ■

The problem resides in the fact that one can not easily choose the matrices $Q > 0$, $R > 0$ and L, nor can we easily solve for P in (3.12). Moreover, this can only provide one solution to the SOF problem as opposed to the later result in Theorem 3.9 which provides all stabilizing SOF gains. A related (non-testable) necessary and sufficient condition is given in (Kučera and de Souza, 1995).

3.3.3 Covariance Assignability by Output Feedback

The basic idea behind the covariance control theory is to provide a characterization of all assignable covariance matrices and, in addition, a parameterization of all controllers which assign a particular covariance (Hotz and Skelton, 1987), (Yasuda et al., 1993), (Skelton and Iwasaki, 1993). Given a stochastic system

$$\begin{aligned} \dot{x} &= Ax + Bu + \Gamma w \\ y &= Cx \end{aligned} \tag{3.13}$$

where $w(t)$ is a zero-mean white-noise disturbance of intensity W, the steady-state covariance matrix of the state vector $x(t)$ is defined by

$$X = \lim_{t \to \infty} \mathcal{E}\{x(t)x(t)^T\} \tag{3.14}$$

where $\mathcal{E}$ denotes the expectation operator. For a static output feedback control law $u = Ky$ it is well known that X solves the Lyapunov equation

$$(A + BKC)X + X(A + BKC)^T + \Gamma W \Gamma^T = 0\,. \tag{3.15}$$

A matrix $X > 0$ is called an assignable covariance if there exists a controller gain K such that (3.15) is satisfied. If (A,B) is stabilizable and (A,Γ) is controllable then, from Lyapunov stability theory, $X > 0$ is equivalent to stability of the closed-loop system. The following result parameterizes all assignable covariances by static output feedback (Yasuda et al., 1993).

Theorem 3.5 A matrix $X > 0$ is an assignable covariance by static output feedback if and only if X satisfies

$$(I - BB^\dagger)(AX + XA^T + \Gamma W \Gamma^T)(I - BB^\dagger) = 0 \tag{3.16}$$

$$(I - C^\dagger C)X^{-1}(AX + XA^T + \Gamma W \Gamma^T)X^{-1}(I - C^\dagger C) = 0 \tag{3.17}$$

$$(I - \Delta\Delta^\dagger)(I - C^\dagger C)X^{-1}(AX + XA^T + \Gamma W \Gamma^T) = 0 \tag{3.18}$$

where

$$\Delta = (I - C^\dagger C)X^{-1}BB^\dagger \tag{3.19}$$

■

A parameterization of all static output feedback gains that stabilize the system and assign a particular assignable covariance is obtained as follows (Yasuda et al., 1993).

Theorem 3.6 Let $X > 0$ be an assignable covariance matrix. Then all static output feedback gains that assign X to the closed loop system are parameterized by

$$K = -\frac{1}{2}B^\dagger(AX + XA^T + \Gamma W \Gamma^T + \Sigma)X^{-1}C^\dagger + Z - B^\dagger BZCC^\dagger \tag{3.20}$$

where

$$\begin{aligned}
\Sigma &= \left[\Phi^\dagger \Psi + (I - \Phi^\dagger \Phi)S\right](I - \Phi^\dagger \Phi) - (\Phi^\dagger \Psi)^T && (3.21)\\
\Phi &= \begin{bmatrix} I - BB^\dagger \\ (I - C^\dagger C)X^{-1} \end{bmatrix} && (3.22)\\
\Psi &= \begin{bmatrix} -(I - BB^\dagger) \\ (I - C^\dagger C)X^{-1} \end{bmatrix}(AX + XA^T + \Gamma W \Gamma^T) && (3.23)
\end{aligned}$$

and Z is an arbitrary matrix and S is an arbitrary skew-symmetric matrix. ∎

Conditions (3.16)-(3.23) can be interpreted as a state-space parameterization of all stabilizing static output feedback gains in terms of the state covariance matrix X. The major difficulty in covariance control theory is to test if the coupled covariance assignability equations (3.16)-(3.18) have a common solution $X > 0$, and to obtain such a solution if one exists. Once a common solution $X > 0$ is found, the parameterization (3.20)-(3.23) provides all static gains that stabilize the system and assign X as a closed-loop covariance. A deterministic interpretation of the covariance control theory is given in (Yasuda et al., 1993).

3.3.4 Output Structural Constraint Approach

The static output feedback problem can be viewed as a state feedback problem where the feedback gain is subject to a structural constraint. In particular, $A + BKC$ is stable if and only if $A + BL$ is stable where $LY = 0$ and Y is an orthonormal basis of the null space of C. Defining the augmented matrices

$$F = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}, \quad \mathrm{G} = \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathbb{R}^{(n+m)\times m} \tag{3.24}$$

and the functions

$$\begin{aligned}
\Theta(W) &= FW + WF^T && (3.25)\\
f(W) &= trace\left[{W_2}^T {W_1}^{-1} W_2 - {W_2}^T C^T (CW_1C^T)^{-1} CW_2\right] && (3.26)
\end{aligned}$$

where W is the $(n+m)\times(n+m)$ symmetric matrix

$$W = \begin{bmatrix} W_1 & W_2 \\ {W_2}^T & W_3 \end{bmatrix} \tag{3.27}$$

with $W_1 > 0$, a necessary and sufficient condition for output stabilization can be expressed as follows (Peres et al., 1993).

Theorem 3.7 There exists a stabilizing static output feedback gain if and only if

$$\mathcal{C} \bigcap \mathcal{W} \neq \emptyset \tag{3.28}$$

where

$$\begin{aligned}
\mathcal{C} &= \left\{W \,:\, v^T\Theta(W)v \le 0 \;\; \forall\, v \in \mathcal{N}(G^T)\right\} && (3.29)\\
\mathcal{W} &= \{W \,:\, f(W) \le 0\} && (3.30)
\end{aligned}$$

where $\mathcal{N}(G)$ denotes the null space of G. The set of all stabilizing static output feedback gain is parameterized by

$$K = {W_2}^T C^T (C W_1 C^T)^{-1} \tag{3.31}$$

where $W \in \mathcal{C} \bigcap \mathcal{W}$. ■

The set $\mathcal{C}$ is convex, however $\mathcal{W}$ is non-convex making the condition (3.28) difficult to check. Although f is not guaranteed to be convex, a supporting hyperplane to the epigraph of f can be calculated in a large subset of the domain of f (Geromel et al., 1993). Based on that, cutting plane algorithms have been proposed to obtain stabilizing output feedback gains, but convergence of these algorithms is not guaranteed (Geromel et al., 1993), (Peres et al., 1993).

3.3.5 Coupled Linear Matrix Inequality Formulation

Necessary and sufficient conditions for static output feedback can be obtained in terms of coupled Linear Matrix Inequalities following a quadratic Lyapunov function approach. From Lyapunov stability theory we know that the closed-loop system matrix $A + BKC$ is stable if and only if K satisfies the following matrix inequality

$$(A + BKC)P + P(A + BKC)^T < 0 \tag{3.32}$$

for some $P > 0$. For a fixed P, the inequality (3.32) is a **Linear Matrix Inequality** (LMI) in the matrix K, see (Boyd et al., 1994). The LMI (3.32) is convex in K so that convex programming techniques can be used to numerically find a K whenever $P > 0$ is given. Necessary and sufficient conditions for static output feedback stabilization are obtained by finding the solvability conditions of (3.32) in terms of K (Iwasaki and Skelton, 1995), (El-Ghaoui and Gahinet, 1993).

Theorem 3.8 There exists a stabilizing static output feedback gain if and only if there exists $P > 0$ such that

$$\begin{aligned} B^{\perp}(AP + PA^T)(B^{\perp})^T &< 0 && (3.33)\\ (C^T)^{\perp}(A^T P^{-1} + P^{-1}A)((C^T)^{\perp})^T &< 0 && (3.34) \end{aligned}$$

where $B^{\perp}$ and $(C^T)^{\perp}$ are full-rank matrices, orthogonal to B and C^T respectively. ■

Inequality (3.33) follows from (3.32) by simple multiplication on the left by $B^{\perp}$ and on the right by $(B^{\perp})^T$. Inequality (3.34) follows from (3.32), by multiplying on the left and right by P^{-1} and then multiplying on the left by $(C^{\perp})^T$, and on the right by $((C^T)^{\perp})^T$. In (Iwasaki and Skelton, 1995) and (El-Ghaoui and Gahinet, 1993) it is shown that the converse is also true, that is if there exists a $P > 0$ which satisfies inequalities (3.33) and (3.34), then there exists a stabilizing static output feedback K. A parameterization of all static output feedback gains that correspond to a feasible solution P of (3.33)-(3.34) is provided in (Iwasaki and Skelton, 1995).

Theorem 3.9 All stabilizing static output feedback gains are parameterized by

$$K = -R^{-1}B^T P Q^{-1} C^T (C Q^{-1} C^T)^{-1} + S^{1/2} L (C Q^{-1} C^T)^{-1/2} \tag{3.35}$$

where

$$\begin{aligned} S &= R^{-1} - R^{-1}B^T P Q^{-1}[Q - C^T (C Q^{-1} C^T)^{-1} C]^{-1} Q P B R^{-1} > 0 && (3.36)\\ Q &= PBR^{-1}B^T P - PA - A^T P && (3.37)\\ R^{-1} &> B^{\dagger}\left[\Phi - \Phi (B^T)^{\perp}(B^{\perp}\Phi(B^{\perp})^T)^{-1}B^{\perp}\Phi\right] B^{\dagger T} && (3.38)\\ \Phi &= P^{-1}A^T + AP^{-1} && (3.39) \end{aligned}$$

where P is any positive definite matrix which satisfies (3.33) and (3.34), and L is any matrix with $\| L \| < 1$. ■

Notice that (3.33) is an LMI on P and (3.34) is an LMI on P^{-1}, but finding such a $P > 0$ is a difficult task since the two inequalities are **not** convex in P. Computational methods based on iterative sequential solutions of the two convex LMI problems with respect to P and P^{-1} have been proposed to find stabilizing static output feedback gains, but convergence of the algorithms is not guaranteed (Iwasaki et al., 1994), (Geromel et al., 1994). Also, in (Grigoriadis and Skelton, 1996) alternating projection methods are suggested to solve fixed-order and output feedback control problems described by LMIs but with no guaranteed convergence. It is interesting to note that many other static output feedback control problems, such as suboptimal H^∞ control, suboptimal Linear Quadratic control and μ-synthesis with constant scaling can be formulated in terms of coupled LMIs as in (3.33)-(3.34), see (El-Ghaoui and Gahinet, 1993), (Gahinet and Apkarian, 1994), (Iwasaki and Skelton, 1994), (Iwasaki and Skelton, 1995).

3.3.6 Nonlinear Programming Methods

In (Davison, 1965), Davison stabilizes the system (1.1) using SOF by minimizing the real part of the dominant eigenvalue of the closed-loop system. Such a performance index is non-differentiable, and the paper modifies it to a differentiable one. A similar approach was used with decentralized SOF in (Davison and Tripathi, 1978), for applications in power system control. Both of these papers use nonlinear programming methods in order to solve the problem and as such are not considered analytical.

3.3.7 Decision Methods

In 1975, a paper (Anderson et al., 1975) introduced decision methods to study the output feedback stabilization problem. By using a stability criterion, such as Routh-Hurwitz, the output feedback stabilizability problem can be reduced to a system of multivariable polynomial inequalities in k_{ij}, which are the ij-th component of the feedback matrix K. Decision methods permit one to establish, in a finite number of algebraic steps, the existence of real variables k_{ij} such that all polynomial inequalities are satisfied. Decision methods can be extended to eliminate not only the "existence" quantifier $\exists$, but also complex combinations of existence and "universal", $\forall$ quantifiers. This permits us to study not only nominal stabilizability but also robust stabilizability (Abdallah et al., 1996). Robust stability in this setting is with respect to parametric uncertainty in the coefficients of the transfer function. Decision methods are currently referred to as Quantifier Elimination or QE techniques (Basu et al., 1994; Tarski, 1951) and are briefly discussed next.

Given the set of polynomials with integer coefficients $P_i(X,Y), \;\; 1 \leq i \leq s$ where X represents a k dimensional vector of quantified real variables and Y represents an l dimensional vector of un-quantified real variables, let $X^{[i]}$ be a block of k_i quantified variables, Q_i be one of the quantifiers $\exists$ (there exists) or $\forall$ (for all), and let $\Phi(Y)$ be the quantified formula

$$\Phi(Y) = (Q_1 X^{[1]}, ..., Q_w X^{[w]}) F(P_1, ..., P_s), \tag{3.40}$$

where $F(P_1, ..., P_s)$ is a quantifier free *Boolean* formula, that is a formula containing the Boolean operators $\wedge$ (and) and $\vee$ (or), operating on *atomic predicates* of the form $P_i(Y, X^{[1]}, ..., X^{[w]}) \geq 0$ or $P_i(Y, X^{[1]}, ..., X^{[w]}) > 0$ or $P_i(Y, X^{[1]}, ..., X^{[w]}) = 0$. We can now state the general quantifier elimination problem

General Quantifier Elimination Problem: Find a quantifier-free Boolean formula $\Psi(Y)$ such that $\Phi(Y)$ is true if and only if $\Psi(Y)$ is true. ■

In control problems, the un-quantified variables are generally the compensator parameters, represented by the parameter vector $Y = q$, and the quantified variables are the plant parameters, represented by the plant parameter vector p, and the frequency variable ω. Uncertainty in plant parameters are characterized by quantified formulas of the type $\forall(p_i) \;\; [\underline{p_i} \leq p_i \leq \overline{p_i}]$ where $\underline{p_i}$ and $\overline{p_i}$ are rational numbers. *The quantifier-free formula $\Psi(q)$ then represents a characterization of the compensator design.*
An important special problem is the QE problem with no un-quantified variables (free variables), i.e.

$l = 0$. This problem is referred to as the *General Decision Problem*.
General Decision Problem: With no un-quantified variables, i.e. $l = 0$, determine if the quantified formula given in (3.40) is true or false. ■

The general decision problem may be applied to the problem of *existence* of compensators that meet given specifications, in which case an "existence" quantifier is applied to the compensator parameter q.

Algorithms for solving general QE problems were first given by Tarski (Tarski, 1951) and Seidenberg (Seidenberg, 1954), and are commonly called Seidenberg-Tarski decision procedures. Tarski showed that QE is solvable in a finite number of steps, but his algorithm and later modifications are exponential in the size of the problem. Researchers in Control Theory have been aware of Tarski's results and their applicability to Control problems since the 1970's but the tedious operations made the technique very limited (Anderson et al., 1975).

Recently, new algorithms have been developed for the QE problem and software packages have been introduced. A sample of these packages is the package QEPCAD *Quantifier Elimination by Partial Cylindrical Algebraic Decomposition* (Hong, 1990). This software has been used to solve different fixed-structure and output feedback problems in (Abdallah et al., 1996).

Example 3.2 *As an illustration consider the static output feedback example of (Anderson et al., 1975) where the Tarski-Seidenberg theory was applied manually. We have the plant*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u; \\ y &= \begin{bmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{bmatrix} x \end{aligned} \tag{3.41}$$

with the static output feedback, $u = -Kx$ *where* $K = [q_1, q_2] = [v, w]$ *so that the closed-loop system matrix is*

$$A - BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -w & 13 + 5v - w & -v \end{bmatrix} \tag{3.42}$$

with a closed-loop characteristic polynomial $s^3 + vs^2 + (w - 5v - 13)s + w$. *The Liénard-Chipart criterion gives us the conditions*

$$v > 0, \quad w > 0, \quad -5v^2 - 13v + vw - w > 0 \tag{3.43}$$

for the polynomial to be stable, i.e. to have all roots with negative real part. The solution of the inequalities (3.43) can be stated as the quantifier elimination problem. We have stated it as

$$\exists w,\ (v > 0) \wedge (w > 0) \wedge (-5v^2 - 13v + vw - w > 0).$$

The QEPCAD code eliminated w *and gave us the unquantified equivalent formula* $v - 1 > 0 \wedge v > 0$ *in 0.034 seconds. Now we can choose* $v > 1$, *e.g* $v_0 = 2$, *and the inequalities (3.43) give us restrictions on* w, *namely for* v_0, *we obtain that* $w > 0 \wedge w - 46 > 0$ *so that one parameterization of stabilizing controllers is,* $v_0 = 2, w > 46$.

The basic limitation of QE methods is still the computational complexity of existing algorithms. Typically, these algorithms are doubly exponential in the number of blocks $X^{[i]}$ (Basu et al., 1994). Thus, only modestly-sized problems can be solved by these methods. Even so, some (robust) stabilization problems can only be solved using QE methods. In particular, QE methods have the distinct advantage over deterministic and random discretization methods in that results (when obtained) have no "holes" in the parameter space and require no probabilistic qualifications.

4 Pole Placement With Static Output Feedback

Here, it desired to select the gain K to place the poles (or the eigenvalues) in the closed-loop system (1.3) at desired locations. However, in a historical context a pole is said to be assignable (by output feedback) if K may be selected such that (1.3) has a pole arbitrarily close to a desired value. We say that (1.1) is (output) pole assignable if all the poles may be assigned given a desired symmetric (i.e. closed under complex conjugation) set of n poles. This problem is the most developed of all output feedback applications and recent results have provided necessary and sufficient conditions for the generic pole placement assignability.

4.1 Necessary Conditions

In (Herman and Martin, 1977) a necessary and sufficient condition for generic pole assignability with a **complex** gain matrix K was established as

$$mp \geq n, \tag{4.44}$$

however, simple counter-examples show that this is only necessary for the case of **real** K (Willemns and Hesselink, 1978). In (Giannakopoulos and Karcanias, 1985), the necessary condition was strengthened to (4.44) plus full rank of the so-called Plücker matrix. Reference (Kabamba and Longman, 1982) defined (1.1) as locally completely assignable (for a given K) if, for every desired set of small changes $\delta\sigma_i$ in the poles σ_i of $(A+BKC)$, there exists a δK such that $[A+B(K+\delta K)C]$ has poles at $(\sigma_i+\delta\sigma_i)$. A necessary and sufficient (but non-testable) condition for this to occur was given in terms of the independence of the closed-loop Markov parameter matrices.

4.2 Sufficient Conditions

In (Brasch and Pearson, 1970) it was shown that if (1.1) is minimal (i.e. controllable and observable), then almost any K will yield a cyclic $\bar{A} = (A + BKC)$, i.e. one such that $sI - A - BKC$ has only one non-unity invariant polynomial. Moreover, for almost any choice of a vector q, we make $\{\bar{A}, Bq\}$ controllable. Then, we can apply the scalar design formulas to obtain a gain matrix k such that $\det(sI - \bar{A} + Bqk)$ is the desired closed-loop polynomial. In (Davison, 1970; Davison, 1971; Davison and Chow, 1973), this approach was exploited to show that if (A, B, C) is minimal with B and C of full rank, then $\max(m, p)$ poles are assignable. Davison and Wang (Davison and Wang, 1975) and Kimura (Kimura, 1975; Kimura, 1978) showed that indeed, under these conditions, $\min(n, m+p-1)$ poles are assignable generically (i.e. for almost all A, B and C). This translates into the sufficient condition for generic pole assignability that

$$m + p \geq n + 1. \tag{4.45}$$

An alternate proof of this was offered in (Brockett and Byrnes, 1981; Schumacher, 1980) where the authors showed that the pole placement with SOF is equivalent to classical Schubert problem. Moreover, they showed that if

$$d(m,p) = \frac{1!2!\cdots(p-1)!(mp)!}{m!(m+1)!\cdots(m+p-1)!} \tag{4.46}$$

is odd, and whenever $\min\{m,p\} = 1$ or $\min\{m,p\} = 2$ and $\max\{m,p\} = 2^k - 1$, a real K exists to generically assigns the closed-loop poles.

Another sufficient condition for generic pole assignability was given in (Kimura, 1977) as

$$m + p + \beta > n + 1; \quad m > \beta; \quad p \geq \alpha \tag{4.47}$$

with α and β the controllability and observability indices respectively. If (1.1) is minimal with B of full rank and A_d is the desired closed-loop plant matrix, then another sufficient condition for pole assignability was given in (Vardulakis, 1975) as $(A - A_d)(I - C^\dagger C) = 0$, with superscript "†" again denoting the Moore-Penrose inverse. This may be interpreted as a condition that any differences between the actual and the desired plant matrices occur in the perpendicular of $\mathcal{N}(C)$ (with $\mathcal{N}(.)$ representing the Null space). More recently, Wang (Wang, 1994) has shown that $n < mp$ is sufficient for generic pole assignability. In fact, the result of Wang is described in the following theorem.

Theorem 4.1 *If $n < mp$, then the pole placement map*

$$\begin{aligned} \chi \quad : \quad & \mathbb{R}^{\mathrm{mp}} \longrightarrow \mathbb{R}^{\mathrm{n}} \\ & K \longmapsto \det(sI - A - BKC) \end{aligned}$$

is surjective (or onto) for a generic set of real matrices (A, B, C).

In this setting, a subset S of $\mathbb{R}^{\mathrm{k}}$ is generic if its complement is contained in the zero set of some nonzero polynomial $p(x_1, x_2, \cdots, x_k)$. A simpler proof of Wang's result appeared in (Leventides and Karcanias, 1995b; Rosenthal et al., 1994) where a linearization procedure was used to obtain the SOF controller.

Finally, a more recent result (Rosenthal and Wang, 1995; Leventides and Karcanias, 1995a) shows that generic pole assignability is possible with a compensator of MacMillan degree q as soon as

$$\begin{aligned} n &< q(m+p-1) + mp - \min\{r_m(p-1), r_p(m-1)\} \\ r_m &= q - m[q/m] \\ r_p &= q - p[q/p] \end{aligned}$$

i.e. r_m and r_p are the remainders of q divided by m and p respectively. More importantly, the authors provided an algorithmic procedure for obtaining the compensator when it exists. By letting $q = 0$, we obtain the condition $mp > n$ again.

4.3 Design Approaches and Limitations

It is worth discussing briefly the techniques used in some of the cited references. In (Davison, 1970; Davison, 1971; Davison and Chow, 1973; Davison and Wang, 1975), an explicit formula was given for K in terms of various matrices constructed from (A, B, C) and the desired poles. It amounts to an "Ackermann-type" formula for output feedback. In (Kimura, 1975; Kimura, 1977; Kimura, 1978) a different approach which relates closely to the eigenstructure assignment techniques in the next section was used. References (Brockett and Byrnes, 1981) and (Giannakopoulos and Karcanias, 1985) used the Grassman space (i.e. exterior algebra). In (Misra and Patel, 1989) an algorithm was given to assign the eigenvalues arbitrarily close to desired values for the case $m + p > n$. The Hessenberg form was used to solve two single-input problems. First, $p-1$ poles were placed, then $n-p+1$ poles were placed without disturbing the first poles assigned (c.f. (Srinathkumar, 1978)). A discussion on the relation between the pole-assignment problem and transmission zeros is also given. A related algorithm was given in (Miminis, 1985) to assign $\max(m, p)$ poles. If condition (4.45) fails to hold, then the techniques of this subsection generally allow the assignment of $m + p - 1 < n$ poles. There are no guarantees however, on the locations of the remaining closed-loop poles, which may often be unstable. A nice geometric framework involving lattices is provided in (Champetier and Magni, 1992; Magni and Champetier, 1988; Magni and Champetier, 1991). It is however difficult to translate that framework into computational techniques.

In the following, we present yet another set of the so-called necessary and sufficient (but non-testable) conditions for pole placement using output feedback. For notational ease assume that B has full column rank and that C has full row rank. We suppose $p \geq m$; the other case is handled in a similar way.

The open-loop input-coupling, output-coupling, and transfer relations are revealed in matrix-fraction description (MFD) form by

$$\begin{aligned}(sI-A)^{-1}B &= N_1(s)D^{-1}(s) && (4.48)\\ C(sI-A)^{-1} &= F^{-1}(s)G_1(s) && (4.49)\\ H(s) &= CN_1(s)D^{-1}(s)=N(s)D^{-1}(s) && (4.50)\\ H(s) &= F^{-1}(s)G_1(s)B=F^{-1}(s)G(s) && (4.51)\end{aligned}$$

with (4.48, 4.50) normalized right MFDs (e.g. right coprime, $D(s)$ column-reduced and column-degree ordered), and (4.49, 4.51) normalized left MFDs (e.g. left coprime, $F(s)$ row-reduced and row-degree ordered). The next result was shown in (Syrmos et al., 1992).

Theorem 4.2 Let $[N(s), D(s)]$ be a normalized right MFD for $H(s)$. There exists a feedback K that assigns the invariant polynomials if and only if the equation

$$\begin{bmatrix} -Y_m & X_m \\ F(s) & G(s) \end{bmatrix}\begin{bmatrix} -N(s) & X_p \\ D(s) & Y_p \end{bmatrix}=\begin{bmatrix} R_m(s) & 0 \\ 0 & R_p(s) \end{bmatrix} \tag{4.52}$$

is satisfied for some $R_m(s)$ and $R_p(s)$, both having the desired closed–loop nonunit invariant polynomials. The solution must satisfy the conditions:

1. $[G(s), F(s)]$ left coprime, $F(s)$ row reduced and row-degree ordered.
2. X_m, Y_m, X_p, Y_p constant matrices with X_m and X_p nonsingular.

Then the required output feedback is given by $K=-X_m^{-1}Y_m=-Y_pX_p^{-1}$ ■

Note 4.1 Equation (4.52) is equivalent to

$$\begin{aligned}Y_mN(s)+X_mD(s) &= R_m(s);\quad F(s)N(s)=G(s)D(s)\\ F(s)X_p+G(s)Y_p &= R_p(s);\quad Y_mX_p=X_mY_p.\end{aligned}$$

Therefore, the condition is in terms of coupled Diophantine equations, which should be contrasted with the coupled LMI equations in the previous section.

Note that a similar derivation may be obtained using matrices in $\mathcal{S}$ rather than polynomial matrices. Finally, note that a common limitation to these pole placement approaches (except for (Rosenthal and Wang, 1995)), is that unless all poles can be placed in the stability region, no guarantee exists for the stability of the closed-loop system.

5 Eigenstructure Assignment With Static Output Feedback

First, we review eigenstructure assignment by state-variable feedback $u(t)=-Fx(t)+v(t)$ (i.e. $C=I$). While the pole-placement problem for multivariable systems is fairly complicated, Moore (Moore, 1976) showed that the problem of assigning both eigenvalues and eigenvectors has a straightforward solution. Given a symmetric set of desired closed-loop poles $\{\mu_i\}$, $i=1,\dots q$, vectors $\{v_i\}$ and $\{u_i\}$ are found such that

$$[\mu_iI-A\quad B]\begin{bmatrix} v_i \\ u_i \end{bmatrix}=0 \tag{5.53}$$

Then a feedback gain F defined by

$$F\,[v_1\cdots v_q]=[u_1\cdots u_q] \tag{5.54}$$

results in the closed-loop structure

$$[\mu_i I - (A - BF)]\, v_i = 0 \tag{5.55}$$

so that the v_i are assigned as the closed-loop eigenvectors for eigenvalues μ_i.

5.1 Necessary Conditions

There is a certain freedom in the choice of the v_i, but for a real F to exist they must satisfy

1. $v_i \in (\mu_i I - A)^{-1} R(B)$
2. $v_i = v_j^*$ when $\mu_i = mu_j^*$, (where "*" means complex conjugation)
3. $\{v_i\}$ is a linearly independent set.

The integer q may be taken equal to n, but any uncontrollable poles must be included in $\{mu_i\}$, with the associated v_i satisfying $w_i^T v_i \neq 0$, where w_i is the left eigenvector associated with μ_i. Note that we may write (5.53) as the generalized Lyapunov equation

$$VJ - AV = -BU \tag{5.56}$$

with $V = [v_1 \cdots v_q]$, $U = [u_1 \cdots u_q]$, $J = \mathrm{diag}(\mu_i)$. Then (5.54) reduces to $FV = U$. Turning to the case of output feedback (1.2), Reference (Bengtsson and Lindahl, 1974) assumes that a state-variable feedback F which places both eigenvalues and eigenvectors has been selected by some procedure. Then, a method is given to find an output feedback K that preserves some of the poles of $(A - BF)$ in (1.3). Although eigenvector assignment was not specifically addressed, the technique involves in fact preserving the eigenvectors v_i associated with the modes $\{\mu_i, i = 1, ..., q\}$. Indeed, although $KC = -F$, may have no solution K, the reduced equation $KCV = -FV$ may have a solution, so that (5.55) becomes $[\mu_i I - (A - BKC)]\, v_i = 0$. In (Srinathkumar, 1978), the technique of (Moore, 1976) was extended to output feedback, essentially by replacing (5.54) with $KCV = U$. ¿From that work, it is clear that $\max(m, p)$ poles are assignable by this method. The algorithm given assigns $p-1$ poles, and an additional (interesting but fairly complicated) procedure was given to assign a total of $\min(n, m+p-1)$ poles generically. The case of constrained output feedback (i.e. where some of the entries of K are set to zero) was covered in (Calvo-Ramon, 1986).

5.2 Sufficient Conditions

A major breakthrough occurred in (Kwon and Youn, 1987) where some techniques of (Kimura, 1977) were extended to show that, in some cases, $m+p$ poles may be assigned. This is a better result than those associated with (4.45). It was obtained by considering the closed-loop right and left eigenstructure. A design example demonstrates the assignment of $m+p$ poles. However, it is not clear in the paper what is actually going on in terms of system structure. A somewhat streamlined description of the main result is as follows. Let the desired closed-loop structure be described by the (possibly non-simple) Jordan matrix J. If there exist a direct sum decomposition $J = J_1 \oplus J_2$ and matrices V_1, W_2, U, and Z such that

$$\begin{aligned} V_1 J_1 - AV_1 &= -BU && (5.57)\\ J_2 W_2^T - W_2^T A &= -Z^T C && (5.58)\\ W_2^T V_1 &= 0 && (5.59) \end{aligned}$$

then $K = U(CV_1)^{-1}$ makes J the Jordan matrix of $(A + BKC)$. Moreover, the right eigenvectors corresponding to the poles in J_1 are the columns of V_1, and the left eigenvectors corresponding to the poles in J_2 are the columns of W_2. It should now be noted that p poles may be placed by using equation of (5.57) (c.f. (5.56)), and possibly m by using the dual relation, equation of (5.58). The construction of the required matrices in (5.57)-(5.59) may be confronted by using the right Null space of $[\mu_i I - A \;\; B]$ and the left Null space of

$$\begin{bmatrix} \mu_i I - A \\ C \end{bmatrix}$$

with $\{\mu_i\}$ the desired poles (Kwon and Youn, 1987). Unfortunately, the proposed solution algorithm is derived from only a sufficient condition, and relies on selecting some vectors to guarantee various conditions, so that some artistic ability and intuition is needed, along with a bit of luck, to apply the technique. In the case where $p + m > n$ a computationally efficient algorithm is proposed in (Syrmos and Lewis, 1993) for the solution of the coupled Sylvester equations (5.57)-(5.59). The fixed-order compensator problem was also studied using two Coupled Sylvester equations in (Syrmos and Syrmos, 1992).

5.3 Design Approaches and Limitations

Although a given number of poles is generically assignable by the above approaches, nothing is known of the remaining closed-loop poles, which may be unstable. In (Fletcher and Ho, 1986) a technique was given for approximate pole assignment which gives some idea of the location of all of the closed-loop poles. Eigenstructure assignment with output feedback was treated for some special cases in (Fletcher and Magni, 1987; Magni, 1987). Note that the condition expressed in terms of (5.57)-(5.59) is sufficient only. A necessary and sufficient condition for eigenstructure assignment using output feedback was also given in (Kwon and Youn, 1987); however, it was not used as the basis of any design algorithm. Yet another necessary and sufficient condition was given in (Syrmos and Lewis, 1994) in terms of a Bilinear Sylvester equation. However, it was not used as the basis of any algorithm either.

6 LQ Regulator With Static Output Feedback

It is desired here to select K to minimize, subject to the constraint (1.3), the performance index

$$J = \int_0^\infty (x^T Q x + u^T R u) dt \tag{6.60}$$

with $Q \geq 0$ and $R > 0$, while stabilizing the closed-loop system. In (Johnson and Athans, 1970; Levine and Athans, 1970; Levine et al., 1971; Moerder and Calise, 1985), necessary conditions for optimality were given as

$$0 = A_c^T S + S A_c + Q - C^T K^T R K C \tag{6.61}$$
$$0 = A_c P + P A_c^T + X \tag{6.62}$$
$$0 = -RKCPC^T - B^T S P C^T, \tag{6.63}$$

with $X = x(0)x(0)^T$ and $A_c = A + BKC$. Generally, optimal control with reduced information results in such coupled nonlinear matrix equations. If it is desired to eliminate the dependence of (6.61)-(6.63) on the specific initial conditions, then expected values may be taken of the performance index (6.60) so that $X = \mathcal{E}\{x(0)x(0)^T\}$ in (6.62). It is generally assumed that $x(0)$ is uniformly distributed on the unit sphere so that $X = I$ (Levine and Athans, 1970). The tracking problem with output feedback is considered in (Bernstein and Haddad, 1987).

Conditions for the existence and global uniqueness of solutions to (6.61)-(6.63) such that P and S are positive definite and (1.3) is stable are not known. It has been shown (Ermer and Vandelinde, 1973) that in the discrete case there exists a gain that minimizes (6.60) locally and also stabilizes the system if $Q \geq 0$, $R > 0$, rank$(C) = p$, $X > 0$, and (A, B, C) is output stabilizable; that is, there exists a K such that A_c is stable. However, there may be more than one local minimum, so that solution of (6.61)-(6.63) may not yield the global minimum. Similar sufficient conditions were given in (Moerder and Calise, 1985).

Necessary and sufficient conditions for the existence of a solution to the suboptimal LQ problem with output feedback in terms of LMIs are given in (Iwasaki et al., 1994). H^2 optimal control with output feedback is treated in (Peres et al., 1993) using the techniques of section 3.3.4. However, the above approaches suffer from the same drawbacks discussed in section 3.3.

6.1 Design Approaches and Limitations

Algorithms for the solution of (6.61)-(6.63) and their discrete counterparts were proposed in (Choi and Sirisena, 1974; Kreisselmeier, 1975; Levine and Athans, 1970; Moerder and Calise, 1985; O'Reilly, 1978; Söderström, 1978; Toivonen, 1985). These algorithms are all iterative in nature. Convergent iterative algorithms for the continuous case were finally presented in 1985 (Moerder and Calise, 1985; Toivonen, 1985). The algorithm in (Moerder and Calise, 1985) requires repetitive solution of (6.61) and (6.62) for fixed values of K so that they are considered as two Lyapunov (i.e. linear matrix) equations, and the form $K = R^{-1}B^T SPC^T(CPC^T)^{-1}$ as a candidate for the next choice for K. Compare this expression with that in Section 2.3.2 when $L = 0$. Note however, that it guarantees only a local minimum. Unfortunately, iterative algorithms such as these require the selection of an initial stabilizing gain. A direct procedure for finding such a K is unknown as discussed in section 3.

Finally, (Davison et al., 1973) considers nonlinear programming methods in order to design a SOF controller which minimizes (6.60) in power system applications.

6.2 Inverse Problem

It was shown in (Gu, 1990a) that for square open-loop transfer functions, a necessary and sufficient condition for the existence of an output feedback that will stabilize the closed-loop system and minimize (6.60) is given as follows;

Theorem 6.1 Let the system (1.1) have no transmission zeros on the jw axis. Then, there exists a matrix K such that $u = Ky$ will stabilize the closed-loop system and minimize (6.60) for some $Q \geq 0$ and $R > 0$, with $\{\sqrt{Q}, A\}$ observable, if and only if $\det(CB) \neq 0$ and the open-loop function $C(sI - A)^{-1}B$ is minimum phase. ■

Note that this necessary and sufficient condition is the same as that required to make the closed-loop system $K(sI - A - BK)^{-1}B$ Strictly-Positive-Real (SPR).

7 How Hard is SOF?

The question many researchers have begun asking is whether it is worth spending any more time looking for an analytical solution to the SOF problem. In fact, many of them have pointed out that algorithmic and numerical solutions may be called upon to solve the problem in many interesting cases. The hope is then that someone can come up with an algorithm that can solve most of the SOF problems encountered in practice. In this section we review results from Computational Complexity theory to suggest that such hope may not be realistic, at least for moderate and large size problems.

Recently, many control problems have been shown to be $\mathcal{NP}$-complete (or $\mathcal{NP}$-Hard), (Blondel and Tsitsiklis, 1995; Poljak and Rohn, 1993; Nemirovskii, 1993; Coxson, 1993; Toker and Özbay, 1995). For the SOF problem, the exponential-time Tarski-Seidenberg elimination method (Tarski, 1951) can theoretically be used to determine whether or not a solution to the multivariable polynomial inequalities (obtained from the Routh-Hurwitz test) exists (Anderson et al., 1975). This answers the question of the *decidability* of the problem, but it does not address the more practical problem of whether or not efficient (i.e., polynomial-time) methods exist for solving the problem.

Until recently, it was felt that decidable problems are practically solved and thus not very interesting. The introduction of Computational complexity theory has since changed this misconception. Computational complexity theory is often used to establish the tractability or intractability of computational problems, and is concerned with the determination of the intrinsic computational difficulty of these problems (Garey and Johnson, 1979). One important concept in this theory is that of a *polynomial-time algorithm*. In practice, such an algorithm can be feasibly implemented on a real computer. This is in contrast to an *exponential-time algorithm*, which is only feasible if the problem being solved is extremely small.

The complexity class $\mathcal{P}$ consists of all decision problems that can be decided in polynomial-time, using a Turing machine model of computation. The simplicity of the Turing machine model appears to make it of little practical value; however, the Church-Turing Thesis holds that the class of problems solvable on a Turing machine in polynomial time is robust across all other reasonable models of computation (including the computers we use).

The complexity class $\mathcal{NP}$ consists of all decision problems that can be decided algorithmically in *nondeterministic* polynomial-time. An algorithm is nondeterministic if it is able to choose or guess a sequence of choices that will lead to a solution, without having to systematically explore all possibilities. This model of computation is not realizable, but it is of theoretical importance since it is strongly believed that $\mathcal{P} \neq \mathcal{NP}$. In other words, these two complexity classes form an important boundary between the tractable and intractable problems. A problem is said to be $\mathcal{NP}$-hard if it is as hard as any problem in $\mathcal{NP}$. Thus, if $\mathcal{P} \neq \mathcal{NP}$, the $\mathcal{NP}$-hard problems can only admit deterministic solutions that take an unreasonable (i.e., exponential) amount of time, and they require (unattainable) nondeterminism in order to achieve reasonable (i.e., polynomial) running times.

The central idea used to demonstrate $\mathcal{NP}$-hardness evolves around the $\mathcal{NP}$-complete problems. A problem is said to be $\mathcal{NP}$-complete if every decision problem in $\mathcal{NP}$ is polynomial-time reducible to it. This means that the $\mathcal{NP}$-complete problems are as hard as any decision problem in $\mathcal{NP}$. Given two decision problems Π_1 and Π_2, Π_1 is said to be polynomial-time reducible to Π_2 (written as $\Pi_1 \leq_p \Pi_2$), if there exists a polynomial time algorithm R which transforms every input x for Π_1 into an equivalent input $R(x)$ for Π_2. By equivalent we mean that the answer produced by Π_2 on input $R(x)$ is always the same as the answer Π_1 produces on input x. Thus, any algorithm which solves Π_2 in polynomial time can be used to solve Π_1 on input x in polynomial time by simply computing $R(x)$, and then running Π_2.

In order to show that a particular (control) decision problem Π_2 is $\mathcal{NP}$-complete, one starts with a problem Π_1 in $\mathcal{NP}$-complete , and attempts to show that $\Pi_1 \leq_p \Pi_2$. This shows that Π_2 is $\mathcal{NP}$-hard. To complete the proof that Π_2 is $\mathcal{NP}$-complete, it must be demonstrated that a candidate solution can be verified in polynomial time. In control theory, researchers have followed this "reduction" method to study the computational difficulty of some decidable problems.

In the language of computational complexity theory, the SOF problem is formulated as follows:

Static Output Feedback

Instance: A LTI plant of the form $\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t)$, under the influence of static output feedback of the form $u(t) = Ky(t) + v(t)$.

Question: Does there exist a real gain matrix K which guarantee the closed-loop stability of the LTI plant? ■

A problem closely related to Static Output Feedback was studied in Stable matrix in unit interval family

Instance: A positive integer n, a partition of $I = \{(i,j) \mid 1 \leq i,j \leq n\}$ into disjoint sets I_1 and I_2, rational numbers a^*_{ij} for $(i,j) \in I_1$.

Question: Does the set $\mathcal{A}$ of $n \times n$ matrices defined by

$$\mathcal{A} = \{A = a_{ij} \mid a_{ij} = a^*_{ij} \text{ for } (i,j) \in I_1, a_{ij} \in [-1,1] \text{ for } (i,j) \in I_2\}$$

contain at least one stable matrix? ■

Notice, however, that some constraints are placed on all of the elements of A–each element is either a fixed rational number, or a rational number in the interval $[-1, 1]$. In (Blondel and Tsitsiklis, 1995), it was shown that Stable matrix in unit interval family $\in \mathcal{NP}$-Hard and used the result to show that the SOF problem when the entries of K are constrained to lie in some intervals is $\mathcal{NP}$-Hard. In (Toker and Özbay, 1995) the problem of Stable matrix remains $\mathcal{NP}$-Hard even if no bounds is placed on the variations of a_{ij}. Blondel and Tsitsiklis conjectured that the computational complexity of their problems remains the same even in the absence of constraints on K (Blondel and Tsitsiklis, 1995). To date however, no such result is available.

In effect, the introduction of computational complexity methods into the study of the static output problem is suggesting that general algorithms (such as those obtained from the Decision methods) are almost doomed to failure. Computational complexity however, does not necessarily lead to the conclusion that every (or even most) SOF problem is computationally intractable. On the contrary, and due to the genericity results discussed in Section 3, the SOF problems may be solved for many specific problems. The complexity methods do however suggest that every effort should be applied to exploit the particular structure of a given SOF problem, thus restricting the class of systems and reducing the computational cost.

8 Conclusion

It is clear from the studies cited here that the problem of static output feedback is still open. Various unconnected necessary conditions, sufficient conditions, and ad hoc solution techniques abound. Except for the generic pole assignment problem, where an algorithm exists (Leventides and Karcanias, 1995b; Rosenthal and Wang, 1995), and the QE software, not much exists in terms of an organized design. Unfortunately, the generic pole assignment problem is too restrictive and the decision methods are computationally inefficient. The result is total confusion for all but the expert in mathematical system theory, and the failure to use analytical output-feedback design in many applications. The so-called necessary and sufficient conditions are not efficiently testable, and as such only succeed in transforming the problem into another unsolved problem or into a numerical search problem with no guarantee of convergence to a solution. A common thread throughout these methods however, is the fact that the problem is equivalent to obtaining the solution of a coupled set of matrix (Lyapunov, Riccati, LMI, Bezout, etc) equations. The recent indications that the output feedback problem may be $\mathcal{NP}$-Hard implies that moderately large problems are computationally intractable. Exploitation of the special structure of particular problems seems to be the only promising approach to follow.

Acknowledgments: V. Syrmos acknowledges the financial support of the National Science Foundation under contract NCR-9210408, by the Advanced Research Programs Agency under contract MDA-972-93-1-0032 and by the University of Hawaii Research Council Funds. C.T. Abdallah acknowledges the financial support of Boeing Computer Services under Contract W-300445, and fruitful discussions with G. Heileman and P. Goldberg. K. Grigoriadis acknowledges the financial support of a University of Houston PEER Award.

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