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$\mathscr{H}_\infty$ filtering for discrete-time linear systems with bounded time-varying parameters

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ABSTRACT

In this paper, the problem of linear parameter varying (LPV) filter design for time-varying discrete-time polytopic systems with bounded rates of variation is investigated. The design conditions are obtained by means of a parameter-dependent Lyapunov function and extra variables for the filter design, expressed as bilinear matrix inequalities. An LPV filter, which minimizes an upper bound to the $\mathscr{H}_\infty$ performance of the estimation error, is obtained as the solution of an optimization problem. A convex model to represent the parameters and their variations as a polytope is proposed in order to provide less conservative design conditions. Robust filters for time-varying polytopic systems can be obtained as a particular case of the proposed method. Numerical examples illustrate the results.

1. Introduction

Technological advances have always pushed the control community to face more complex problems in several different framework. Concerning the linear filtering problem, which has extended from the earliest times [1], a large number of papers dealing with deterministic and stochastic scenarios can be seen in the literature. The search for more sophisticated structures has become decisive when dealing with signal recovering and estimation under time-varying or constant uncertainties, without mentioning that the design of optimal filters for precisely known models is well characterized nowadays.

In this context, the Lyapunov theory has been extensively applied as a tool to deal with the synthesis of filters that guarantee the stability of the estimation error dynamics meanwhile assuring a certain level of performance. For example, quadratic Lyapunov functions have been used to deal with time-invariant or arbitrarily time-varying systems as can be seen in [2–4] concerning the $\mathscr{H}_2$ and $\mathscr{H}_\infty$ robust filtering. Improvements of these results may be obtained by using parameter-dependent Lyapunov functions, as proposed in [5] for the time-invariant case and in [6] for the time-varying case with bounded rates of variation. Many works dealing with robust filtering have appeared in print lately.

Considering the case where the time-varying parameters, although may not be known *a priori*, can be measured online, gain scheduling techniques represent an interesting option for filtering or control of dynamic systems when contrasted with robust methods. Furthermore, as discussed in [7], gain-scheduling strategies extend the validity of the linearization approach of nonlinear systems to a range of operating points. As mentioned in [8], gain scheduling is an effective and economical method for nonlinear control design in

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practice. In the filtering framework, recent works include [9,10] where affine parameter varying filters, with limited rate of variation, are obtained [11] in the context of parameter-dependent filters by means of nonlinear fractional transformation and quadratic stability [12] concerned with LPV filtering for slowly varying systems and [13] where the LPV filtering for arbitrarily time-varying systems in polytopic domain is addressed.

Extending the powerful features of gain scheduling (well presented in [8]) to deal with the filtering problem is of great importance specially within the class of time-varying discrete-time systems. It is known from [14] that when stability analysis for time-varying discrete-time systems is at issue, robust stabilizability implies gain scheduling stabilizability, but the converse is not true. These facts in some sense motivate the results and effort of the present work.

This paper investigates the LPV filtering of time-varying systems with bounds on the rate of variation. A preliminary version of this paper appeared in [15] considering only the robust filter design, and applications in the context of networked robust filtering in [16]. The approach proposed in this paper complements and extends previous results appeared in the literature by presenting a systematic procedure for filtering design that can be applied in four different frameworks, namely LPV or robust filtering of time-varying systems with bounded or unbounded rates of variation. The Lyapunov theory is applied in order to obtain the design conditions of the filter. A parameter-dependent Lyapunov function is used to reduce the conservatism of the proposed method, resulting in a more general approach when compared to methods based on quadratic stability. All system matrices are assumed to be affected by time-varying parameters, which are supposed to lie inside a polytope. A more precise parameter variation modeling is applied to give a better description of the uncertainty domain and an $\mathscr{H}_\infty$ guaranteed cost is used as performance index. The $\mathscr{H}_\infty$ filtering limits the maximum possible variance of the error signal over all exogenous inputs with bounded variance [17], i.e. the $\mathscr{H}_\infty$ norm reflects the worst-case energy gain of the system and does not require statistical assumptions on the exogenous input (a situation in which the Kalman filtering cannot be employed [18]). Moreover, the $\mathscr{H}_\infty$ guaranteed cost provides robustness with respect to unmodeled uncertainties. The LPV filter is then obtained by the solution of an optimization problem that minimizes an upper bound to the $\mathscr{H}_\infty$ index of performance subject to a finite number of bilinear matrix inequality (BMI) constraints formulated only in terms of the vertices of a polytope. No grids in the parametric space are used. Extra variables introduced in the BMI conditions can be explored in the search for better $\mathscr{H}_\infty$ performance of the estimation error dynamic giving more flexibility to the design process. Robust filters for time-invariant and arbitrarily time-varying uncertain systems can be obtained as particular cases of the proposed method. Numerical examples illustrate the efficiency of the proposed results.

2. Problem statement and preliminary results

Consider the time-varying discrete-time system, for $k \geq 0$

$$\begin{aligned} x(k+1) &= A(\alpha(k))x(k) + B(\alpha(k))w(k) \\ z(k) &= C_1(\alpha(k))x(k) + D_1(\alpha(k))w(k) \\ y(k) &= C_2(\alpha(k))x(k) + D_2(\alpha(k))w(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state-space vector, $w(k) \in \mathbb{R}^m$ is the noise input belonging to $l_2[0,\infty)$, $z(k) \in \mathbb{R}^p$ is the signal to be estimated and $y(k) \in \mathbb{R}^q$ is the measured output. The time-varying vector of parameters $\alpha(k)$ belongs to the unit simplex (for all $k \geq 0$)

$$\mathscr{U}_N = \left\{ \delta \in \mathbb{R}^N : \sum_{i=1}^{N} \delta_i = 1,\ \delta_i \geq 0,\ i = 1, \ldots, N \right\}$$

and has bounded rates of variation of percentage $b \in [0,1]$. For instance, $b = 0.05$ indicates that the parameters are constrained to vary only 5% of their original values between two instants of time. The time-invariant case is modeled by $b = 0$ and arbitrarily fast variations by $b = 1$.

All matrices are real, with appropriate dimensions, belonging to the polytope[1]

$$\tilde{\mathscr{P}} \triangleq \left\{ \left[\begin{array}{c|c} A(\alpha) & B(\alpha) \\ \hline C_1(\alpha) & D_1(\alpha) \\ \hline C_2(\alpha) & D_2(\alpha) \end{array} \right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c} A_i & B_i \\ \hline C_{1i} & D_{1i} \\ \hline C_{2i} & D_{2i} \end{array} \right],\ \alpha \in \mathscr{U}_N \right\} \tag{2}$$

More specifically, the system matrices are given, for any time $k \geq 0$, by the convex combination of the well-defined vertices of the polytope $\mathscr{P}$.

A full order proper LPV filter is investigated here, being given by

$$\begin{aligned} x_f(k+1) &= A_f(\alpha)x_f(k) + B_f(\alpha)y(k), \quad x_f(0) = 0 \\ z_f(k) &= C_f(\alpha)x_f(k) + D_f(\alpha)y(k) \end{aligned} \tag{3}$$

where $x_f(t) \in \mathbb{R}^n$ is the filter state-space vector and $z_f(t) \in \mathbb{R}^p$ the estimated signal. All filter matrices are real, with appropriate dimensions, belonging to the polytope

$$\hat{\mathscr{P}} \triangleq \left\{ \left[\begin{array}{c|c} A_f(\alpha) & B_f(\alpha) \\ \hline C_f(\alpha) & D_f(\alpha) \end{array} \right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c} A_{fi} & B_{fi} \\ \hline C_{fi} & D_{fi} \end{array} \right],\ \alpha \in \mathscr{U}_N \right\} \tag{4}$$

The estimation error dynamics is given by

$$\begin{aligned} \varsigma(k+1) &= \hat{A}(\alpha)\varsigma(k) + \hat{B}(\alpha)w(k), \quad \varsigma(0) = 0 \\ e(k) &= \hat{C}(\alpha)\varsigma(k) + \hat{D}(\alpha)w(k) \end{aligned} \tag{5}$$

where $\varsigma(k) = [x(k)'\ \ x_f(k)']'$, $e(k) = z(k) - z_f(k)$ and

$$\hat{A}(\alpha) = \begin{bmatrix} A(\alpha) & \mathbf{0} \\ B_f(\alpha)C_2(\alpha) & A_f(\alpha) \end{bmatrix}, \quad \hat{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ B_f(\alpha)D_2(\alpha) \end{bmatrix}$$

$$\begin{aligned} \hat{C}(\alpha) &= [C_1(\alpha) - D_f(\alpha)C_2(\alpha) \quad -C_f(\alpha)], \\ \hat{D}(\alpha) &= [D_1(\alpha) - D_f(\alpha)D_2(\alpha)]. \end{aligned} \tag{6}$$

The filtering problem to be dealt with can be stated as follows:

Problem 1. Find matrices $A_{fi} \in \mathbb{R}^{n\times n}$, $B_{fi} \in \mathbb{R}^{n\times q}$, $C_{fi} \in \mathbb{R}^{p\times n}$ and $D_{fi} \in \mathbb{R}^{p\times q}$ $i = 1, \ldots, N$, of the filter (3), such that the estimation error system (5) is asymptotically stable, and

[1] The time dependence of $\alpha(k)$ will be omitted to lighten the notation.

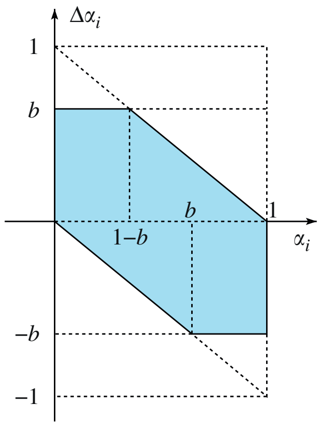

Fig. 1. Region on the plane $\Delta\alpha_i \times \alpha_i$ where $\Delta\alpha_i$ can assume values as a function of α_i (dark region).

an upper bound γ to the $\mathscr{H}_\infty$ estimation error performance is minimized, that is,

$$\sup_{\|w\|_2 \neq 0} \frac{\|e\|_2^2}{\|w\|_2^2} < \gamma^2, \quad w \in l_2[0,\infty) \tag{7}$$

In order to model the parameter variation when $-b < \Delta\alpha_i(k) < b$, $b \neq 0$, it must be taken into account that the feasible values of $\Delta\alpha_i(k)$ depend on the actual values of $\alpha_i(k)$, as show in Fig. 1 (darken area). Thus, any pair $(\alpha_i, \Delta\alpha_i)$ belongs to the polytope Λ_i, $i = 1, \ldots, N$ given by

$$\Lambda_i \triangleq \left\{ \delta \in \mathbb{R}^2 : \delta = \sum_{j=1}^{6} \lambda_j r_j, \quad \lambda \in \mathscr{U}_6 \right\},$$

$$\Lambda_i \triangleq \left\{ \delta \in \mathbb{R}^2 : \delta = \sum_{j=1}^{6} \lambda_j r_j, \quad \lambda \in \mathscr{U}_6 \right\},$$

$$[r_1 \cdots r_6] = \left[\begin{array}{c:c:c:c:c:c} 0 & 0 & 1-b & 1 & 1 & b \\ 0 & b & b & 0 & -b & -b \end{array} \right], \tag{8}$$

that is, Λ_i is the convex hull of vertices of the feasible area.

To construct the $(\alpha, \Delta\alpha)$-space, the Cartesian product of all Λ_i, $i = 1, \ldots, N$ must be considered, taking into account that the new vertices must satisfy $\alpha_1 + \cdots + \alpha_N = 1$ and $\Delta\alpha_1 + \cdots + \Delta\alpha_N = 0$. The resulting polytope, called Λ, is then given by

$$\Lambda \triangleq \left\{ \delta \in \mathbb{R}^{2N} : \delta = \sum_{i=1}^{M} \lambda_i s_i, \ \lambda \in \mathscr{U}_M \right\} \tag{9}$$

where $s_i \in \mathbb{R}^{2N}$ are given vectors. Thus, both α and $\Delta\alpha$ are embedded together in an augmented space, called λ-space, of dimension $2N$ and, as a consequence, the first step to search for a solution to any LMI/BMI depending on both α and $\Delta\alpha$ is to make a lifting to the λ-space. From (9) one has

$$(\alpha, \Delta\alpha)' = S\lambda, \quad S = [s_1, \ldots, s_M] \in \mathbb{R}^{2N \times M}, \quad \lambda \in \mathscr{U}_M \tag{10}$$

In the case of affine parameter-dependent matrices, that is

$$X(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k) X_i, \quad \alpha_i(k) = \sum_{j=1}^{M} \lambda_j S_{ij} \tag{11}$$

$$X(\alpha(k+1)) = \sum_{i=1}^{N} (\alpha_i(k) + \Delta\alpha_i(k)) X_i, \quad \Delta\alpha_i(k) = \sum_{j=1}^{M} \lambda_j S_{(i+N)j} \tag{12}$$

it follows that

$$\bar{X}(\lambda) = \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_j S_{ij} X_i = \sum_{j=1}^{M} \lambda_j \bar{X}_j$$

$$\tilde{X}(\lambda) = \sum_{i=1}^{N} \sum_{j=1}^{M} \lambda_j (S_{ij} + S_{(i+N)j}) X_i = \sum_{j=1}^{M} \lambda_j \tilde{X}_j \tag{13}$$

where[2]

$$\bar{X}_j = \sum_{i=1}^{N} S_{ij} X_i \tag{14}$$

$$\tilde{X}_j = \sum_{i=1}^{N} (S_{ij} + S_{(i+N)j}) X_i \tag{15}$$

Theorem 1 (*Stability analysis*). *For a given γ, if there exists bounded matrix sequences $\mathscr{P}(\alpha)' = \mathscr{P}(\alpha) > 0$, $G(\zeta)$, $H(\zeta)$, matrix F and full rank matrix T, with appropriate dimensions, such that (the term $(\star)$ indicates symmetric blocks in the matrix inequality)*

$$\begin{bmatrix} \mathscr{P}(\alpha_+) - F - F' & F\hat{A}(\alpha)' - F'TG(\zeta)'T^{-1} & F\hat{C}(\alpha)' - F'TH(\zeta)' \\ (\star) & \mathscr{L}_{22} & \mathscr{L}_{23} \\ (\star) & (\star) & \mathscr{L}_{33} \end{bmatrix} < 0 \tag{16}$$

$$\mathscr{L}_{22} = (T')^{-1} G(\zeta) T' F \hat{A}(\alpha)' + \hat{A}(\alpha) F' T G(\zeta)' T^{-1} - \mathscr{P}(\alpha) + \gamma^{-1} \hat{B}(\alpha) \hat{B}(\alpha)'$$

$$\mathscr{L}_{23} = (T')^{-1} G(\zeta) T' F \hat{C}(\alpha)' + \hat{A}(\alpha) F' T H(\zeta)' + \gamma^{-1} \hat{B}(\alpha) \hat{D}(\alpha)'$$

$$\mathscr{L}_{33} = H(\zeta) T' F \hat{C}(\alpha)' + \hat{C}(\alpha) F' T H(\zeta)' - \gamma \mathbf{I} + \gamma^{-1} \hat{D}(\alpha) \hat{D}(\alpha)'$$

for all $\alpha, \zeta \in \mathscr{U}_N$, where $\alpha_+ = \alpha(k+1)$, and bounded $\Delta\alpha$, then the error dynamics (5) is asymptotically stable with an upper bound γ to the $\mathscr{H}_\infty$ performance.

Proof. Firstly, multiply the inequality (16) to the left by $\mathscr{T}'$ and to the right by $\mathscr{T}$, with

$$\mathscr{T}' = \begin{bmatrix} \hat{A}(\alpha) & I & \mathbf{0} \\ \hat{C}(\alpha) & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

in order to obtain

$$\begin{bmatrix} \hat{A}(\alpha)\mathscr{P}(\alpha_+)\hat{A}(\alpha)' - \mathscr{P}(\alpha) + \gamma^{-1}\hat{B}(\alpha)\hat{B}(\alpha)' & \hat{A}(\alpha)'\mathscr{P}(\alpha_+)\hat{C}(\alpha)' + \gamma^{-1}\hat{B}(\alpha)\hat{D}(\alpha)' \\ (\star) & \hat{C}(\alpha)\mathscr{P}(\alpha_+)\hat{C}(\alpha)' + \gamma^{-1}\hat{D}(\alpha)\hat{D}(\alpha)' - \gamma\mathbf{I} \end{bmatrix} < 0$$

[2] The same conversion is applied to the system and filter matrices.

Secondly, by choosing $\nu(k)=\varsigma(k)'\mathscr{P}(\alpha)\varsigma(k)$ as a parameter-dependent Lyapunov function and considering the dual system (i.e. $\hat{A}=\hat{A}'$, $\hat{B}=\hat{C}'$, $\hat{C}=\hat{B}'$ and $\hat{D}=\hat{D}'$), it follows, after some algebraic manipulation, that the above inequality implies $\Delta\nu(k)<-\gamma^{-1}e(k)'e(k)+\gamma w(k)'w(k)$. Therefore, system (5) has an upper bound γ to the $\mathscr{H}_\infty$ performance, in accordance with the extension of the discrete-time version of the bounded real lemma to cope with time-varying parameters [6] and, from the Lyapunov theory [7], is asymptotically stable. □

It is important to stress that the additional variables F, $G(\zeta)$ and $H(\zeta)$ in (16) represent extra degree of freedom in the search for a feasible solution of Theorem 1. As pointed out in [19], these variables can be identified as Lagrangian multipliers and can be explored for design purpose. In this sense, different structures of matrices F, $G(\zeta)$ and $H(\zeta)$ can be used yielding different sufficient conditions for stability analysis. For instance, assuming polytopic structures, $G(\cdot)$ and $H(\cdot)$ can be parametrized in α or α_+, as used throughout this paper.

The nonlinear inequality conditions of Theorem 1 must be tested at all points of the simplex $\mathscr{U}_N$, i.e. at an infinite number of points. Hence, the main goal hereafter is to obtain finite-dimensional BMI conditions in terms of the vertices of the polytope $\mathscr{P}$ to solve Problem 1. Using Schur complement and change of variables, finite-dimensional BMIs assuring the existence of such filters are given in the next section.

3. Main results

By considering the particular structure

$$\mathscr{P}(\alpha)=\alpha_1P_1+\alpha_2P_2+\cdots+\alpha_NP_N,\quad \alpha\in\mathscr{U}_N \tag{17}$$

lifted to the λ-space, the following sufficient condition can be obtained.

Theorem 2 ($\mathscr{H}_\infty$ *LPV Filtering*). *Given system* (1) *and matrix S as in* (10), *if there exist matrices Z, Y, R, $Q_i\in\mathbb{R}^{n\times n}$, $L_i\in\mathbb{R}^{n\times q}$, $J_i\in\mathbb{R}^{p\times n}$, $D_{fi}\in\mathbb{R}^{p\times q}$, G, $M_i=M_i'>0\in\mathbb{R}^{2n\times 2n}$, $H\in\mathbb{R}^{p\times 2n}$, $i=1,\dots,N$ and a scalar $\gamma>0$ such that, for matrices $\bar{Q}_i$, $\bar{L}_i$, $\bar{J}_i$, $\bar{D}_{fi}$, $\bar{M}_i$, $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_{1i}$, $\bar{C}_{2i}$, $\bar{D}_{1i}$ and $\bar{D}_{2i}$ given as in* (14) *and $\tilde{M}_i$ as in* (15)

$$\Xi_i\triangleq\begin{bmatrix}\mathscr{F}_{11} & \mathscr{F}_{12} & \hat{F}_{3i}-\hat{F}_1'H' & \mathbf{0}\\ (\star) & \mathscr{F}_{22} & G\hat{F}_{3i}+\hat{F}_{2i}'H' & \hat{F}_{4i}\\ (\star) & (\star) & H\hat{F}_{3i}+\hat{F}_{3i}'H'-\gamma\mathbf{I} & \mathscr{F}_{34}\\ (\star) & (\star) & (\star) & -\gamma\mathbf{I}\end{bmatrix}<0,\quad i=1,\dots,M \tag{18}$$

$$\mathscr{F}_{11}=\tilde{M}_i-\hat{F}_1-\hat{F}_1',\quad \mathscr{F}_{12}=\hat{F}_{2i}-\hat{F}_1'G'$$
$$\mathscr{F}_{22}=G\hat{F}_{2i}+\hat{F}_{2i}'G'-\bar{M}_i,\quad \mathscr{F}_{34}=\bar{D}_{1i}-\bar{D}_f\bar{D}_{2i}$$
$$\hat{F}_1=\begin{bmatrix}Z & Y'+R'\\ Z & Y'\end{bmatrix},\quad \hat{F}_{2i}=\begin{bmatrix}\bar{A}_i'Z & \bar{A}_i'Y'+\bar{C}_{2i}'\bar{L}_i'+\bar{Q}_i'\\ \bar{A}_i'Z & \bar{A}_i'Y'+\bar{C}_{2i}'\bar{L}_i'\end{bmatrix}$$
$$\hat{F}_{3i}=\begin{bmatrix}\bar{C}_{1i}'-\bar{C}_{2i}'\bar{D}_{fi}'-\bar{J}_i'\\ \bar{C}_{1i}'-\bar{C}_{2i}'\bar{D}_{fi}'\end{bmatrix},\quad \hat{F}_{4i}=\begin{bmatrix}Z'\bar{B}_i\\ Y\bar{B}_i+\bar{L}_i\bar{D}_{2i}\end{bmatrix}$$

$$\Xi_{ik}\triangleq\begin{bmatrix}\hat{\mathscr{F}}_{11} & \hat{\mathscr{F}}_{12} & \hat{F}_{3ik}-2\hat{F}_1'H' & \mathbf{0}\\ (\star) & \hat{\mathscr{F}}_{22} & G\hat{F}_{3ik}+\hat{F}_{2ik}'H' & \hat{F}_{4ik}\\ (\star) & (\star) & H\hat{F}_{3ik}+\hat{F}_{3ik}'H'-2\gamma\mathbf{I} & \hat{\mathscr{F}}_{34}\\ (\star) & (\star) & (\star) & -2\gamma\mathbf{I}\end{bmatrix}<0,\quad \begin{cases} i=1,\dots,M-1\\ k=i+1,\dots,M\end{cases} \tag{19}$$

$$\hat{\mathscr{F}}_{11}=\tilde{M}_i+\tilde{M}_k-2\hat{F}_1-2\hat{F}_1',\quad \hat{\mathscr{F}}_{12}=\hat{F}_{2ik}-2\hat{F}_1'G'$$
$$\hat{\mathscr{F}}_{22}=G\hat{F}_{2ik}+\hat{F}_{2ik}'G'-\bar{M}_i-\bar{M}_k$$
$$\hat{\mathscr{F}}_{34}=\bar{D}_{1i}+\bar{D}_{1k}-\bar{D}_{fi}\bar{D}_{2k}-\bar{D}_{fk}\bar{D}_{2i}$$

$$\hat{F}_{2ik}=\begin{bmatrix}(\bar{A}_i'+\bar{A}_k')Z & (\bar{A}_i'+\bar{A}_k')Y'+\bar{C}_{2i}'\bar{L}_k'+\bar{C}_{2k}'\bar{L}_i'+\bar{Q}_i'+\bar{Q}_k'\\ (\bar{A}_i'+\bar{A}_k')Z & (\bar{A}_i'+\bar{A}_k')Y'+\bar{C}_{2i}'\bar{L}_k'+\bar{C}_{2k}'\bar{L}_i'\end{bmatrix}$$
$$\hat{F}_{3ik}=\begin{bmatrix}\bar{C}_{1i}'+\bar{C}_{1k}'-\bar{C}_{2i}'\bar{D}_{fk}'-\bar{C}_{2k}'\bar{D}_{fi}'-\bar{J}_i'-\bar{J}_k'\\ \bar{C}_{1i}'+\bar{C}_{1k}'-\bar{C}_{2i}'\bar{D}_{fk}'-\bar{C}_{2k}'\bar{D}_{fi}'\end{bmatrix}$$
$$\hat{F}_{4ik}=\begin{bmatrix}Z'(\bar{B}_i+\bar{B}_k)\\ Y(\bar{B}_i+\bar{B}_k)+\bar{L}_i\bar{D}_{2k}+\bar{L}_k\bar{D}_{2i}\end{bmatrix}$$

then there exists a robust filter in the form of (3), *ensuring the asymptotic stability of the estimation error dynamic* (5) *and an $\mathscr{H}_\infty$ guaranteed cost γ, for all $(\alpha,\Delta\alpha)\in\Lambda$ with vertices given by*

$$A_{fi}=\hat{V}^{-1}Q_i(UZ)^{-1},\quad B_{fi}=\hat{V}^{-1}L_i,\quad C_{fi}=J_i(UZ)^{-1},\quad D_{fi} \tag{20}$$

where $U\in\mathbb{R}^{n\times n}$ and $\hat{V}\in\mathbb{R}^{n\times n}$ are matrices arbitrarily chosen such that $R=\hat{V}UZ$.

Proof. Applying the following operation:

$$\Xi(\lambda)=\sum_{i=1}^{N}\lambda_i^2\Xi_i+\sum_{i=1}^{N-1}\sum_{k=i+1}^{N}\lambda_i\lambda_k\Xi_{ik} \tag{21}$$

to the BMIs (18) and (19) one gets

$$\Xi(\lambda)=\begin{bmatrix}\mathscr{F}_{11} & \mathscr{F}_{12} & \mathscr{F}_{13} & \mathbf{0}\\ (\star) & \mathscr{F}_{22} & \mathscr{F}_{23} & \hat{F}_4\\ (\star) & (\star) & \mathscr{F}_{33} & \mathscr{F}_{34}\\ (\star) & (\star) & (\star) & -\gamma\mathbf{I}\end{bmatrix}<0 \tag{22}$$

$$\mathscr{F}_{11}=\tilde{M}(\lambda)-\hat{F}_1-\hat{F}_1',\quad \mathscr{F}_{12}=\hat{F}_2(\lambda)-\hat{F}_1'G',\quad \mathscr{F}_{13}=\hat{F}_3(\lambda)-\hat{F}_1'H'$$
$$\mathscr{F}_{22}=G\hat{F}_2(\lambda)+\hat{F}_2(\lambda)'G'-\bar{M}(\lambda),\quad \mathscr{F}_{23}=G\hat{F}_3(\lambda)+\hat{F}_2(\lambda)'H'$$
$$\mathscr{F}_{33}=H\hat{F}_3(\lambda)+\hat{F}_3(\lambda)'H'-\gamma\mathbf{I},\quad \mathscr{F}_{34}=\bar{D}_1(\lambda)-\bar{D}_f(\lambda)\bar{D}_2(\lambda)$$

where

$$\hat{F}_2(\lambda)=\begin{bmatrix}\bar{A}(\lambda)'Z & \bar{A}(\lambda)'Y'+\bar{C}_2(\lambda)'\bar{L}(\lambda)'+\bar{Q}(\lambda)'\\ \bar{A}(\lambda)'Z & \bar{A}(\lambda)'Y'+\bar{C}_2(\lambda)'\bar{L}(\lambda)'\end{bmatrix}$$
$$\hat{F}_3(\lambda)'=[\bar{C}_1(\lambda)-\bar{D}_f(\lambda)\bar{C}_2(\lambda)-\bar{J}(\lambda)\ \ \bar{C}_1(\lambda)-\bar{D}_f(\lambda)\bar{C}_2(\lambda)]$$
$$\hat{F}_4(\lambda)'=[\bar{B}(\lambda)'Z\ \ \bar{B}(\lambda)'Y'+\bar{D}_2(\lambda)'\bar{L}(\lambda)']$$

Then, define the partitioned matrices [20]

$$F=\begin{bmatrix}X' & U'\\ \hat{U}' & \hat{X}'\end{bmatrix},\quad F^{-1}=\begin{bmatrix}Y & \hat{V}\\ V & \hat{Y}\end{bmatrix},\quad T=\begin{bmatrix}X^{-1} & Y'\\ \mathbf{0} & \hat{V}'\end{bmatrix}$$

together with the following variable transformation

$$\begin{bmatrix} \bar{Q}(\lambda) & \bar{L}(\lambda) \\ \bar{J}(\lambda) & \bar{D}_f(\lambda) \end{bmatrix} = \begin{bmatrix} \hat{V} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \bar{A}_f(\lambda) & \bar{B}_f(\lambda) \\ \bar{C}_f(\lambda) & \bar{D}_f(\lambda) \end{bmatrix} \begin{bmatrix} UZ & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad R = \hat{V}UZ \tag{23}$$

where $Z = X^{-1}$. Using the above change of variable, multiply inequality (22) to the left by $\hat{S}'$ and to the right by $\hat{S}$ with

$$\hat{S} = \begin{bmatrix} \mathscr{S} & \mathbf{0} \\ \mathbf{0} & \mathscr{I} \end{bmatrix}, \quad \mathscr{S} = \begin{bmatrix} T^{-1} & \mathbf{0} \\ \mathbf{0} & T^{-1} \end{bmatrix}, \quad \mathscr{I} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

yielding the following inequality

$$\begin{bmatrix} \hat{\mathscr{P}}(\lambda) - F - F' & F\hat{A}(\lambda)' - F'TG'T^{-1} & F\hat{C}(\lambda)' - F'TH' & \mathbf{0} \\ (\star) & \hat{\mathscr{L}}_{22} & \hat{\mathscr{L}}_{23} & \hat{B}(\lambda) \\ (\star) & (\star) & \hat{\mathscr{L}}_{33} & \hat{D}(\lambda) \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0 \tag{24}$$

$$\hat{\mathscr{L}}_{22} = (T')^{-1}GT'F\hat{\bar{A}}(\lambda)' + \hat{\bar{A}}(\lambda)F'TG'T^{-1} - \hat{\mathscr{P}}(\lambda)$$

$$\hat{\mathscr{L}}_{23} = (T')^{-1}GT'F\hat{\bar{C}}(\lambda)' + \hat{\bar{A}}(\lambda)F'TH'$$

$$\hat{\mathscr{L}}_{33} = HT'F\hat{\bar{C}}(\lambda)' + \hat{\bar{C}}(\lambda)F'TH' - \gamma\mathbf{I}$$

where $\mathscr{P}(\lambda) = (T')^{-1}M(\lambda)T^{-1}$ and the matrices $\hat{A}(\cdot)$, $\hat{B}(\cdot)$, $\hat{C}(\cdot)$ and $\hat{D}(\cdot)$ have the same structure of (6), but in the λ-space. Finally, considering the lift of the BMI to the λ-space and applying Schur complement, inequality (24) reduces to (16) of Theorem 1 with $G(\zeta) = G$ and $H(\zeta) = H$. The filter matrices are obtained by the change of variables (23). □

Corollary 1. *The minimum γ attainable by the conditions of Theorem 2 is given by the optimization problem*

$$\min \gamma \quad \text{s.t. } (18)\text{–}(19) \tag{25}$$

Theorem 2 is presented in terms of BMI constraints due to the use of extra variables F, G and H. The advantages of this approach come from the fact that such variables can be used in the search for better performance of the closed-loop system. For instance, a lower $\mathscr{H}_\infty$ guaranteed cost may be obtained exploring the new variables G and H. Nevertheless, by choosing $G = \mathbf{0}$ and $H = \mathbf{0}$ the conditions of Theorem 2 reduce to LMIs, and, in this case, Corollary 1 becomes a convex optimization problem that can be handled by semi-definite programming (SDP) algorithms.

In order to solve Corollary 1 within the BMI framework, many methods appeared so far in the literature could be applied, as the two following algorithms. The first one is sometimes called alternating SDP method [21] and consists of fixing some variables and searching for others in such a way that at each step a convex optimization problem is solved. The second one is called path-following method [22] and consists of linearizing the BMIs. Although in both cases there is no guarantee of convergence, these methods are easy to implement and provide good results. In this paper, the first approach is used and the algorithm is as follows.

Algorithm 1. Let $G = \mathbf{0}$ and $H = \mathbf{0}$. Let ε and k_{max} be given. Set $k = 1$ and iterate:

(1) Fix the variables H and G, minimize w.r.t. γ_k, Z, Y, R, Q_i, L_i, J_i, D_{fi} and M_i. Get the new values of Z, Y, R, Q_i, L_i, J_i and D_{fi}.
(2) Fix the variables Z, Y, R, Q_i, L_i, J_i and D_{fi}, minimize w.r.t. γ_k, H, G and M_i. Get the new values of H and G.
(3) If $|\gamma_k - \gamma_{k-1}| < \varepsilon$, then stop (no significant changes).
(4) Set $k = k+1$ and go to step 1 if $k \le k_{max}$. Otherwise stop.

In order to reduce the number of BMIs and the computational time required to solve optimization problem (25), the conditions of Theorem 2 were obtained with $G(\zeta) = G$ and $H(\zeta) = H$. If $G(\zeta)$ and $H(\zeta)$ were parametrized in terms of α, a more sophisticated procedure, as the one proposed in [23], should be applied.

If $b = 0$, Problem 1 corresponds to the filtering problem of time-invariant uncertain systems. In this case, Theorem 2 provides sufficient conditions to design filters for uncertain discrete-time systems in polytopic domains. In the case $b = 1$, i.e. the parameters may vary arbitrarily inside the unit simplex $\mathscr{U}_N$, the conditions of Theorem 2 encompass the ones provided in [24, Theorem 2] leading to less conservative results when contrasted to LPV filters designed through quadratic Lyapunov functions.

3.1. Robust filtering

For the robust case, consider $P(\alpha)$ as in (17) and the particular structures

$$G(\zeta) = G(\alpha) = \sum_{i=1}^{N} \alpha_i G_i, \quad H(\zeta) = H(\alpha) = \sum_{i=1}^{N} \alpha_i H_i, \quad \alpha \in \mathscr{U}_N$$

lifted to the λ-space, yielding the following result.

Theorem 3 ($\mathscr{H}_\infty$ Robust Filtering). *Given system* (1) *and matrix S as in* (10), *if there exist matrices Z, Y, R, $Q \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times q}$, $J \in \mathbb{R}^{p\times n}$, $D_f \in \mathbb{R}^{p\times q}$, G_i, $M_i = M_i' > 0 \in \mathbb{R}^{2n\times 2n}$, $H_i \in \mathbb{R}^{p\times 2n}$, $i = 1,\dots,N$ and a scalar $\gamma > 0$ such that, for matrices $\bar{Q}$, $\bar{L}$, $\bar{J}$, $\bar{D}_f$, $\bar{G}_i$, $\bar{H}_i$, $\bar{M}_i$, $\bar{A}_i$, $\bar{B}_i$, $\bar{C}_{1i}$, $\bar{C}_{2i}$, $\bar{D}_{1i}$ and $\bar{D}_{2i}$ given as in* (14) *and $\tilde{M}_i$ as in* (15)

$$\Xi_i \triangleq \begin{bmatrix} \mathscr{F}_{11} & \mathscr{F}_{12} & \hat{F}_{3i} - \hat{F}_1'\bar{H}_i' & \mathbf{0} \\ (\star) & \mathscr{F}_{22} & \bar{G}_i\hat{F}_{3i} + \hat{F}_{2i}'\bar{H}_i' & \hat{F}_{4i} \\ (\star) & (\star) & \bar{H}_i\hat{F}_{3i} + \hat{F}_{3i}'\bar{H}_i' - \gamma\mathbf{I} & \mathscr{F}_{34} \\ (\star) & (\star) & (\star) & -\gamma\mathbf{I} \end{bmatrix} < 0, \quad i = 1,\dots,M \tag{26}$$

$$\mathscr{F}_{11} = \tilde{M}_i - \hat{F}_1 - \hat{F}_1', \quad \mathscr{F}_{12} = \hat{F}_{2i} - \hat{F}_1'\bar{G}_i'$$

$$\mathscr{F}_{22} = \bar{G}_i\hat{F}_{2i} + \hat{F}_{2i}'\bar{G}_i' - \bar{M}_i, \quad \mathscr{F}_{34} = \bar{D}_{1i} - \bar{D}_f\bar{D}_{2i}$$

$$\hat{F}_1 = \begin{bmatrix} Z & Y' + R' \\ Z & Y' \end{bmatrix}, \quad \hat{F}_{2i} = \begin{bmatrix} \bar{A}_i'Z & \bar{A}_i'Y' + \bar{C}_{2i}'\bar{L}' + \bar{Q}' \\ \bar{A}_i'Z & \bar{A}_i'Y' + \bar{C}_{2i}'\bar{L}' \end{bmatrix}$$

$$\hat{F}_{3i} = \begin{bmatrix} \bar{C}_{1i}' - \bar{C}_{2i}'\bar{D}_f' - \bar{J}' \\ \bar{C}_{1i}' - \bar{C}_{2i}'\bar{D}_f' \end{bmatrix}, \quad \hat{F}_{4i} = \begin{bmatrix} Z'\bar{B}_i \\ Y\bar{B}_i + \bar{L}\bar{D}_{2i} \end{bmatrix}$$

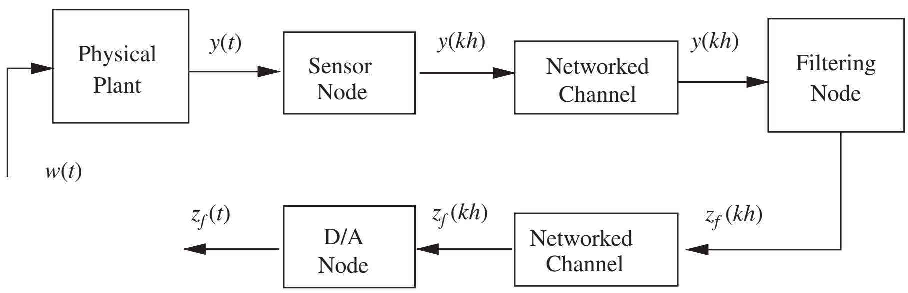

Fig. 2. Networked filtering model.

$$\Xi_{ik} \triangleq \begin{bmatrix} \hat{\mathscr{F}}_{11} & \hat{\mathscr{F}}_{12} & \hat{F}_{3i}+\hat{F}_{3k}-\hat{F}_1'(\bar{H}_i'+\bar{H}_k') & \mathbf{0} \\ (\star) & \hat{\mathscr{F}}_{22} & \bar{G}_i\hat{F}_{3k}+\bar{G}_k\hat{F}_{3i}+\hat{F}_{2i}'\bar{H}_k'+\hat{F}_{2k}'\bar{H}_i' & \hat{F}_{4i}+\hat{F}_{4k} \\ (\star) & (\star) & \bar{H}_i\hat{F}_{3k}+\bar{H}_k\hat{F}_{3i}+\hat{F}_{3i}'\bar{H}_k'+\hat{F}_{3k}'\bar{H}_i'-2\gamma\mathbf{I} & \hat{\mathscr{F}}_{34} \\ (\star) & (\star) & (\star) & -2\gamma\mathbf{I} \end{bmatrix} < 0 \quad (27)$$

$i = 1,\ldots,M-1, \;\; k = i+1,\ldots,M$

$$\hat{\mathscr{F}}_{11} = \tilde{M}_i + \tilde{M}_k - 2\hat{F}_1 - 2\hat{F}_1', \quad \hat{\mathscr{F}}_{12} = \hat{F}_{2i} + \hat{F}_{2k} - \hat{F}_1'(\bar{G}_i' + \bar{G}_k')$$

$$\hat{\mathscr{F}}_{22} = \bar{G}_i\hat{F}_{2k} + \bar{G}_k\hat{F}_{2i} + \hat{F}_{2i}'\bar{G}_k' + \hat{F}_{2k}'\bar{G}_i' - \bar{M}_i - \bar{M}_k$$

$$\hat{\mathscr{F}}_{34} = \bar{D}_{1i} + \bar{D}_{1k} - \bar{D}_f(\bar{D}_{2i} + \bar{D}_{2k})$$

then there exists a robust filter in the form of (3), *ensuring the asymptotic stability of the estimation error dynamic* (5) *and an* $\mathscr{H}_\infty$ *guaranteed cost* γ, *for all* $(\alpha, \Delta\alpha) \in \Lambda$ *with vertices given as in* (20).

Proof. Similar to the proof of Theorem 2. □

The remarks done for Theorem 2 also hold for Theorem 3. Additionally, for $b=0$, $G(\alpha)=\mathbf{0}$ and $H(\alpha)=\mathbf{0}$, the conditions of Theorem 3 reduce to the $\mathscr{H}_\infty$ extension of the results in [5, Theorem 5.1].

3.2. *Practical appeal and possible extensions*

The filter design method presented in this section can be applied to all types of dynamical process that can be written as (1). It encompasses the cases of time-invariant ($b=0$), bounded time-varying ($0<b<1$) and arbitrarily time-varying ($b=1$) systems. Consequently, it can be used in many different practical situations, including systems that exchange information through a communication channel, commonly known as networked control systems (NCSs). The usefulness and importance of NCS architectures is largely due to advances in digital control and computer interfaced structures. Drawbacks associated with NCS are discussed in [25–27]. In the filtering framework, the problem of estimating a signal of a precisely known continuous-time system, sampled by a zero order hold with a time-varying sampling period, through an NCS can be faced by the proposed technique. By using the Cayley–Hamilton theorem or the Taylor series expansion, the time-varying sampled-data matrices can be rewritten as in (2) and Theorem 2 can be applied to provide the filter matrices. More specifically, consider a time-invariant continuous-time system sampled by a zero order hold with a period h. The structure of the filtering model is illustrated in Fig. 2. Assuming that h may change its value at run-time due to different reasons, as bandwidth allocation and scheduling decisions, let the actual value of h at each instant k (i.e. h_k) lie inside a finite discrete set as specified below

$$h_k \in \{h_{min},\ldots,h_{max}\}, \quad h_k = \kappa \cdot g, \;\; \kappa \in \mathbb{N} \quad (28)$$

The parameter g is known as the processor/network clock granularity, [28]. The clock granularity is related with the processor frequency and $k \in \mathbb{N}$ is a function of time that specifies how many times g the sampling period h will be at instant k.

To represent the set of all possible sampled-data system matrices due to uncertain sampling rates, a polytopic model may be considered. In this case, the system matrices, for any time $k \geq 0$, are described as a convex combination of well-defined vertices, which are given by the arrangements of the extreme values of (28) with the help of the Cayley–Hamilton theorem or Taylor series expansion [29]. The sampled system is then rewritten as (1) and Theorem 2 (or Theorem 3) can be used to provide a networked filter such that the estimation error is asymptotically stable under time-varying sampling rates. This problem is of great interest specially when dealing with scheduling or dynamic bandwidth allocation for bandwidth reduction [30].

Other improvements of Theorems 2 and 3 can be obtained by exploring the structure of the Lyapunov matrix $\mathscr{P}(\alpha)$ and the extra variables F, $G(\zeta)$ and $H(\zeta)$ of Theorem 1. As can be seen in (17), the Lyapunov matrix used in Theorem 2 is affine in α. More sophisticated structures may lead to better results, for example, the polynomially parameter-dependent Lyapunov (PPDL) functions used in [31] can be explored for $b<1$. The case $b=1$ (arbitrarily parameter variation) seems to be more involved. Whether or not PPDL functions with higher degree will help to improve the performance when compared to affine functions for synthesis purpose with $b=1$ is still an open question. Nevertheless, parameter-dependent Lyapunov matrices that depend on more than one instant of time, as the path-dependent Lyapunov function proposed in [32,33], can provide better results for $b=1$ when contrasted to the affine Lyapunov matrix.

Changes in the structure of matrices F, $G(\cdot)$ and $H(\cdot)$ appeared in (16) may also lead to better results, following for instance the lines given in [34,35]. A result for arbitrarily time-varying systems, obtained with the path-dependent Lyapunov matrix[3]

$$\mathscr{P}(\alpha) = P(\alpha, \alpha_+) = \sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_i \alpha_{+_j} P_{ij}, \quad \alpha, \alpha_+ \in \mathscr{U}_N \tag{29}$$

and the particular choices

$$G(\zeta) = G(\rho) = \sum_{i=1}^{N} \rho_i G_i, \quad H(\zeta) = H(\rho) = \sum_{i=1}^{N} \rho_i H_i, \quad \rho \in \mathscr{U}_N$$

with $\rho \in \mathscr{U}_N$ is presented in the next theorem. Note that, since $b = 1$ (arbitrarily fast rates of variation), there is no need to lift the matrices to the λ-space.

Theorem 4 (Path-dependent approach). *Given system* (1), *if there exist matrices* Z, Y, R, $Q_i \in \mathbb{R}^{n\times n}$, $L_i \in \mathbb{R}^{n\times q}$, $J_i \in \mathbb{R}^{p\times n}$, $D_{fi} \in \mathbb{R}^{p\times q}$, G_i, $M_{ij} = M_{ij}' > 0 \in \mathbb{R}^{2n\times 2n}$, $H_i \in \mathbb{R}^{p\times 2n}$ $i,j = 1,\ldots,N$ *and a scalar* $\gamma > 0$ *such that*

$$\Xi_{ij\ell} \triangleq \begin{bmatrix} M_{j\ell} - \hat{F}_1 - \hat{F}_1' & \hat{F}_{2i} - \hat{F}_1' G_j' & \hat{F}_{3i} - \hat{F}_1' H_j' & \mathbf{0} \\ (\star) & G_j \hat{F}_{2i} + \hat{F}_{2i}' G_j' - M_{ij} & G_j \hat{F}_{3i} + \hat{F}_{2i}' H_j' & \hat{F}_{4i} \\ (\star) & (\star) & H_j \hat{F}_{3i} + \hat{F}_{3i}' H_j' - \gamma \mathbf{I} & D_{1i} - D_{fi} D_{2i} \\ (\star) & (\star) & (\star) & -\gamma \mathbf{I} \end{bmatrix} < 0 \tag{30}$$

$$i = 1,\ldots,N, \quad j = 1,\ldots,N, \quad \ell = 1,\ldots,N$$

$$\Xi_{ikj\ell} \triangleq \begin{bmatrix} 2M_{j\ell} - 2\hat{F}_1 - 2\hat{F}_1' & \hat{F}_{2ik} - 2\hat{F}_1' G_j' & \hat{F}_{3ik} - 2\hat{F}_1' H_j' & \mathbf{0} \\ (\star) & \hat{\mathscr{F}}_{22} & G_j \hat{F}_{3ik} + \hat{F}_{2ik}' H_j' & \hat{F}_{4ik} \\ (\star) & (\star) & H_j \hat{F}_{3ik} + \hat{F}_{3ik}' H_j' - 2\gamma \mathbf{I} & \hat{\mathscr{F}}_{34} \\ (\star) & (\star) & (\star) & -2\gamma \mathbf{I} \end{bmatrix} < 0 \tag{31}$$

$$i = 1,\ldots,N-1, \quad k = i+1,\ldots,N, \quad j = 1,\ldots,N, \quad \ell = 1,\ldots,N$$

$$\hat{\mathscr{F}}_{22} = G_j \hat{F}_{2ik} + \hat{F}_{2ik}' G_j' - M_{ij} - M_{kj}$$

$$\hat{\mathscr{F}}_{34} = D_{1i} + D_{1k} - D_{fj} D_{2k} - D_{fk} D_{2i}$$

where $\hat{F}_1$, $\hat{F}_{2i}$, $\hat{F}_{3i}$, $\hat{F}_{4i}$, $\hat{F}_{2ik}$, $\hat{F}_{3ik}$ *and* $\hat{F}_{4ik}$ *have the same structure of the ones from Theorem* 2 *but in the* α *domain, then there exists an LPV filter in the form of* (3), *ensuring the asymptotic stability of the estimation error dynamic* (5) *and an* $\mathscr{H}_\infty$ *guaranteed cost* γ, *for all* $\alpha \in \mathscr{U}_N$ *with arbitrary rates of variation and vertices given as in* (20).

Proof. Similar to the proof of Theorem 2 except that now there is no lift to the λ-space and the operation (21) becomes

$$\Xi(\alpha,\rho,\eta) = \sum_{\ell=1}^{N} \eta_\ell \left\{ \sum_{j=1}^{N} \rho_j \left\{ \sum_{i=1}^{N} \alpha_i^2 \Xi_{ij\ell} + \sum_{i=1}^{N-1}\sum_{k=i+1}^{N} \alpha_i \alpha_k \Xi_{ikj\ell} \right\} \right\} \qquad \square$$

Note that the Lyapunov matrix (29) would imply on three instants of time $\alpha(k)$, $\alpha(k+1)$ and $\alpha(k+2)$ in Theorem 1. Since these values are completely independent when $b = 1$, they are represented respectively by α, ρ and η (all of them belonging to unit simplexes, for all $k \geq 0$), yielding matrix $\Xi(\alpha,\rho,\eta)$ in Theorem 4. The robust version of Theorem 4 can be obtained in a similar way of Theorem 3.

[3] The Lyapunov matrix can also be generalized for any number of instants ahead following the lines given in [33], at the price of a quick increase on the computational effort.

4. Numerical experiments

All the experiments have been performed in a PC equipped with Athlon 64 X2 6000+ (3.0 GHz), 2 GB RAM (800 MHz), using the SDP solver SeDuMi [36] interfaced by the parser YALMIP [37]. The numerical complexity is estimated in terms of the computational times given in seconds. Particularly to the iterative procedure given in Algorithm 1, the time of the i-th iteration is the total time cumulated up to this iteration.

Example I. Consider the following time-varying discrete-time system borrowed from [6]

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0 & -0.5 \\ 1 & 1+\theta(k) \end{bmatrix} x(k) + \begin{bmatrix} -6 & 0 \\ 1 & 0 \end{bmatrix} w(k) \\ z(k) &= [1 \;\; 0] x(k) \\ y(k) &= [-100 \;\; 10] x(k) + [0 \;\; 1] w(k) \end{aligned} \tag{32}$$

where $\underline{\theta} \leq \theta(k) \leq \overline{\theta}$ and $|\Delta\theta(k)| \leq \delta$. The equivalent polytopic representation of system (32) is obtained by the change of variables $\theta(k) = \alpha_1(k)\underline{\theta} + \alpha_2(k)\overline{\theta}$ and $|\Delta\alpha_1(k)| = |\Delta\alpha_2(k)| \leq \delta/|\overline{\theta} - \underline{\theta}| = b$. With respect to the ranges of the time-varying parameters, the case to be investigated is $\overline{\theta} = -\underline{\theta} = 0.4$ and $0 \leq \delta \leq 0.8$ (corresponding to $0 \leq b \leq 1$).

The first task is to synthesize robust filters using Algorithm 1 with Theorem 3 and the approaches from [6, Lemma 4] (Lyapunov matrix affine in $\theta(k)$) and [6, Theorem 2] (Lyapunov matrix quadratic in $\theta(k)$). Algorithm 1 is performed twice, considering the maximum number of iterations as $k_{max} = 1$ and $k_{max} = 5$. Fig. 3 shows the minimum γ achieved with strictly proper filters ($D_f = \mathbf{0}$). Note that with only one iteration, where in fact the conditions of Theorem 3 reduce to LMIs, the proposed

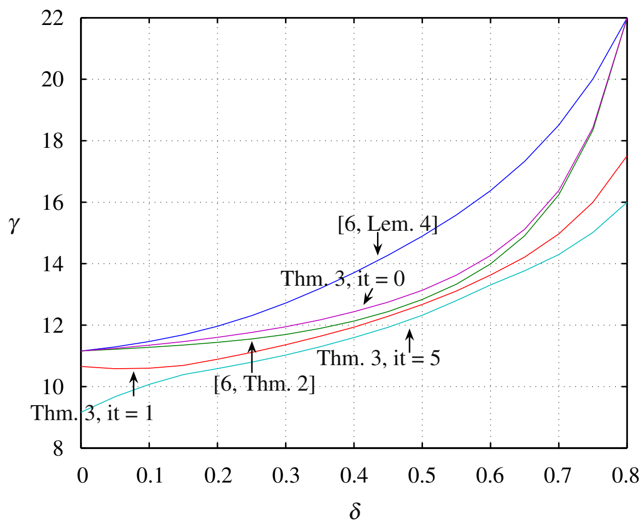

Fig. 3. $\mathscr{H}_\infty$ upper bound attained by using strictly proper robust filters in the design problem of Example I.

Table 1
$\mathscr{H}_\infty$ guaranteed costs and computational times obtained in the design problem of Example I for time-invariant ($\delta = 0$) and arbitrarily fast ($\delta = 0.8$) parameters.

Method	Filter	δ	γ	Time
[6, Lem. 4]	Robust	0	11.16	0.44
[6, Thm. 2]		0	11.16	0.44
[34, Thm. 2]$_{it=5}$		0	9.30	1.45
Theorem $3_{it=0}$		0	11.16	0.38
Theorem $3_{it=1}$		0	10.65	0.78
Theorem $3_{it=5}$		0	9.16	3.55
[6, Lem. 4]	Robust	0.8	21.99	0.57
[6, Thm. 2]		0.8	21.99	0.58
Theorem $3_{it=0}$		0.8	21.99	0.46
Theorem $3_{it=1}$		0.8	17.72	0.95
Theorem $3_{it=5}$		0.8	16.04	5.15
Theorem $4_{it=0}$		0.8	17.59	0.22
Theorem $4_{it=1}$		0.8	15.68	0.44
Theorem $4_{it=5}$		0.8	14.52	2.18
[38, Thm. 3]	LPV	0.8	8.49	0.42
Theorem $2_{it=0}$		0.8	8.49	0.33
Theorem $2_{it=1}$		0.8	8.49	0.48
Theorem $4_{it=0}$		0.8	8.49	0.24
Theorem $4_{it=1}$		0.8	8.49	0.33

approach based on affine parameter-dependent Lyapunov matrix outperforms the best method of [6] that is based on a Lyapunov matrix quadratic in $\theta(k)$. The zero iteration case ($it = 0$) shown in the figure was obtained without introducing the extra variables $G(\cdot)$ and $H(\cdot)$. Smaller guaranteed costs can be obtained through the iterative procedure given in Algorithm 1 at the price of a higher computational effort.

The second part of the experiment concerns a more detailed comparison between the proposed design conditions and the methods from the literature for the specific cases $\delta = 0$ (time-invariant parameter) and $\delta = 0.8$ (arbitrarily fast). In the case $\delta = 0$ the nonconvex procedure from [34, Theorem 2] is also included in the comparisons. For $\delta = 0.8$, the LPV filter design conditions proposed in the paper are compared to [38, Theorem 3]. The results are shown in Table 1, where $it = 0$ means without the extra variables $G(\cdot)$ and $H(\cdot)$. In the robust filtering case, the proposed conditions provide the best $\mathscr{H}_\infty$ guaranteed costs with five iterations at the price of slightly higher computational efforts. In the LPV filtering case the proposed conditions presented the same $\mathscr{H}_\infty$ guaranteed costs than [38, Theorem 3] for the case $\delta = 0.8$. Note that, differently from [38, Theorem 3], the proposed conditions could still synthesize LPV filters for the range $0 < \delta < 0.8$.

Example II. Consider a time-varying system with state-space matrices given by

$$A = \begin{bmatrix} 0.265 - 0.165\theta(k) & 0.45(1+\theta(k)) \\ 0.5(1-\theta(k)) & 0.265 - 0.215\theta(k) \end{bmatrix}$$

$$B = \begin{bmatrix} 1.5 - 0.5\theta(k) \\ 0.1 \end{bmatrix}, \quad C_2' = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Table 2
$\mathscr{H}_\infty$ guaranteed costs and computational times obtained in the design problem of Example II.

Method	Filter	γ	Improvement (%)	Time
$T4_{it=1}$	Robust	19.41	–	0.89
$T4_{it=2}$		9.10	53.10	2.03
$T4_{it=3}$		7.55	61.07	3.17
$T4_{it=4}$		6.82	64.85	4.21
$T4_{it=5}$		6.56	66.18	5.30
$T4_{it=6}$		6.26	67.75	6.41
$T4_{it=1}$	LPV	1.22	–	0.85
$T4_{it=2}$		1.22	0.00	1.96

The computational time (in seconds) is the cumulated time as the number of BMI iterations evolves.

where $D_2 = 1$, $C_1 = \mathbf{I}_2$, $D_1 = \mathbf{0}_2$ and $-1 \le \theta(k) \le 1$ is an arbitrarily fast time-varying parameter ($\Delta\theta(k) = 2$). The polytopic representation of the system is obtained as in Example I. The aim is to synthesize robust and LPV $\mathscr{H}_\infty$ filters using the conditions proposed in the paper and the ones from [6] and [38]. For the LPV case, only Theorems 2 and 4 were able to provide a feasible solution. In the robust case, all methods failed except the robust version of Theorem 4. The results can be seen in Table 2. The robust filter matrices after one iteration are given by

$$A_f = \begin{bmatrix} 0.524 & 1.584 \\ -0.041 & -0.093 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.785 \\ 0.812 \end{bmatrix}$$

$$C_f = \begin{bmatrix} 0.007 & 0 \\ 0.524 & 1.797 \end{bmatrix}, \quad D_f = \begin{bmatrix} 0.993 \\ -0.703 \end{bmatrix}$$

and after six iterations, with an improvement of approximately 67%, by

$$A_f = \begin{bmatrix} 0.251 & 0.367 \\ -0.020 & -0.045 \end{bmatrix}, \quad B_f = \begin{bmatrix} -6.314 \\ 4.104 \end{bmatrix}$$

$$C_f = \begin{bmatrix} 0 & -0.013 \\ 0.170 & 0.560 \end{bmatrix}, \quad D_f = \begin{bmatrix} 1.001 \\ -0.385 \end{bmatrix}$$

As expected, the $\mathscr{H}_\infty$ guaranteed cost associated to the LPV filter was better but no improvement was obtained with the BMI iterations. This example illustrates the fact that there may exist systems where robust filters can only be designed by using path-dependent Lyapunov matrices, which encompass the methods based on Lyapunov matrices depending (affinely, quadratically or polynomially) on parameters only at the current instant of time k.

Fig. 4 shows the results for the noise input generated by the Matlab command $w(k) = 0.3 * \texttt{randn}$, for $0 \le k \le 50$, and zero initial condition. After six iterations, the first state of the error vector had an improvement of 2.29% and the second state of 40.05%.

Example III. This example, borrowed from [29], consists of a simplified model of an armature voltage-controlled DC servo motor, consisting of a stationary field and a rotating armature and load. All effects of the field are neglected. The aim is to design an $\mathscr{H}_\infty$ robust filter to estimate the armature current given the speed of the

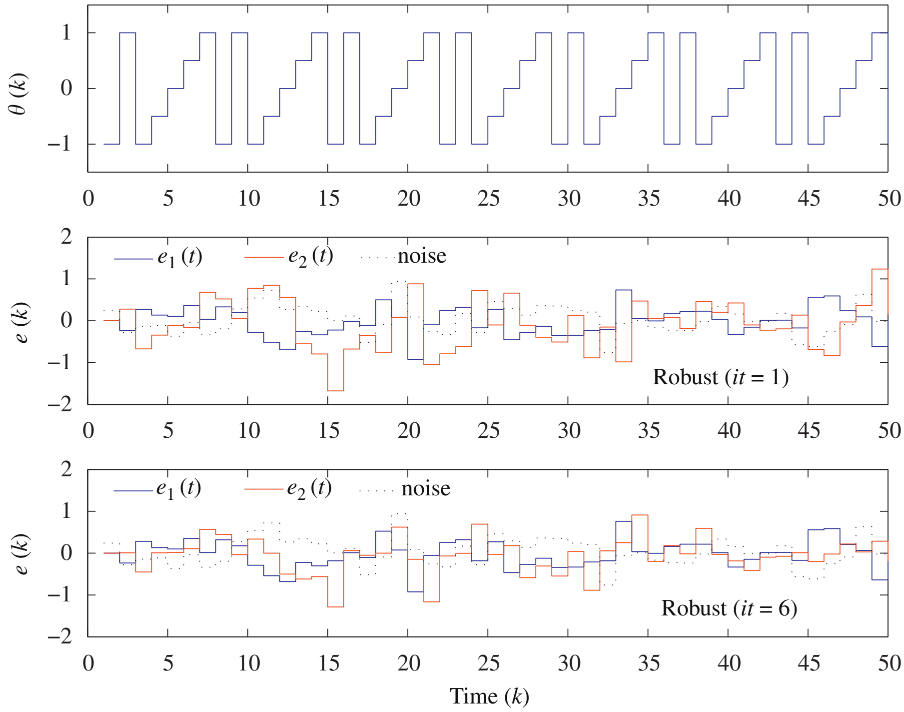

Fig. 4. Time-domain analysis. The first graph illustrates the parameter variation in time while the others show the estimation errors for two robust filters designed in Example II.

shaft. All information is sent through a communication network. The behavior of the DC servo motor shown in Fig. 5 can be described by

$$\begin{bmatrix} \ddot{\theta} \\ \dot{\rho}_a \end{bmatrix} = \begin{bmatrix} -\frac{b_v}{J} & \frac{K_T}{J} \\ -\frac{K_\theta}{L_a} & -\frac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \rho_a \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \omega$$

$$y = [1 \quad 0] \begin{bmatrix} \dot{\theta} \\ \rho_a \end{bmatrix} \tag{33}$$

where e_a is the externally applied armature voltage, ρ_a is the armature current, R_a the resistance of the armature winding, L_a the armature winding inductance, e_m the back-emf voltage induced by the rotating armature winding ($e_m = K_\theta \dot{\theta}, K_\theta > 0$), b_v the viscous damping due to bearing friction, J the moment of inertia of the armature and load and θ the shaft position. Further, the torque generated by the motor is given by $T = K_T i_a$. For $J = 0.01\,\text{kgm}^2/\text{s}^2$, $b_v = 0.1\,\text{Nms}$, $K_T = K_\theta = 0.01\,\text{Nm/Amp}$, $R_a = 1\,\Omega$ and $L_a = 0.5\,\text{H}$, system (33) can be rewritten in the form (1) with the following sampled-data matrices, presented as a function of h_k,

$$A_s = \begin{bmatrix} \exp(-10h_k) - 0.0003\exp(-2h_k) & 0.125(\exp(-2h_k) - \exp(-10h_k)) \\ 0.002(\exp(-10h_k) - \exp(-2h_k)) & -0.0003\exp(-10h_k) + \exp(-2h_k) \end{bmatrix}$$

$$B_s = \begin{bmatrix} 0.025\exp(-10h_k) - 0.125\exp(-2h_k) + 0.099 \\ 0.0000626\exp(-10h_k) - 0.99\exp(-2h_k) + 0.99 \end{bmatrix}$$

$$C_{1s} = [0 \quad 1], \quad C_{2s} = [1 \quad 0], \quad D_{1s} = 0, \quad D_{2s} = 0 \tag{34}$$

The sampling rate is allowed to vary within the interval $h_k \in [0.001\ 0.099]$. The system is then expressed by polytope (2) with four vertices ($N = 4$), obtained by

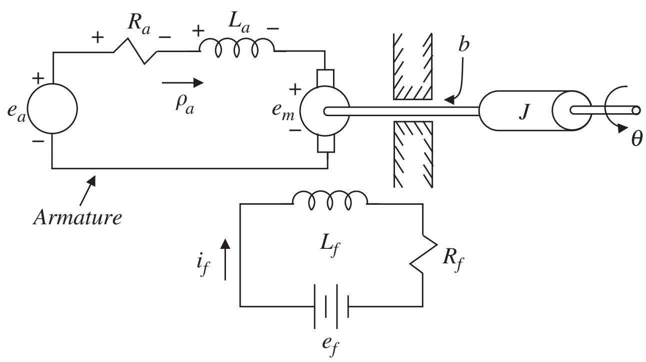

Fig. 5. DC servo motor as presented in [29].

evaluating $\exp(-10h_k)$ and $\exp(-2h_k)$ at the extreme values of h_k, where the parameters α_i are related to h_k and $b = 1$. Theorem 3 provided a robust filter after one iteration with $\mathscr{H}_\infty$ upper bound $\gamma = 1.1519$

$$A_f = \begin{bmatrix} 9.453 & 76.445 \\ -1.162 & -9.396 \end{bmatrix}, \quad B_f = \begin{bmatrix} -1917.224 \\ 253.188 \end{bmatrix}$$

$$C_f = [0.021 \ \ 0.172], \quad D_f = [5.101]$$

5. Conclusion

The $\mathscr{H}_\infty$ LPV filtering for uncertain discrete-time systems with bounded time-varying parameters has been addressed in this paper, where all system matrices are considered to be affected by time-varying parameters.

With a convex description of the parameter time variation, a less conservative design condition was obtained. Extra variables were used to derive BMI conditions that may be explored in the search for a better $\mathscr{H}_\infty$ performance. The filter design is accomplished by means of an optimization problem, formulated only in terms of the vertices of the polytope. The proposed approach provides improvements and advantages when compared to other methods from the literature, as illustrated by examples.

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References

[1] T. Kailath, A view of three decades of linear filtering theory, IEEE Trans. Inform. Theory 20 (2) (March 1974) 146–181.
[2] J.C. Geromel, J. Bernussou, G. Garcia, M.C. de Oliveira, $\mathscr{H}_2$ and $\mathscr{H}_\infty$ robust filtering for discrete-time linear systems, SIAM J. Control Optim. 38 (5) (May 2000) 1353–1368.
[3] M.C. de Oliveira, J.C. Geromel, $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filtering design subject to implementation uncertainty, SIAM J. Control Optim. 44 (2) (2005) 515–530.
[4] J.C. Geromel, R.A. Borges, Joint optimal design of digital filters and state-space realizations, IEEE Trans. Circuits & Syst. II: Exp. Briefs 53 (12) (December 2006) 1353–1357.
[5] J.C. Geromel, M.C. de Oliveira, J. Bernussou, Robust filtering of discrete-time linear systems with parameter dependent Lyapunov functions, SIAM J. Control Optim. 41 (3) (2002) 700–711.
[6] C.E. de Souza, K.A. Barbosa, A. Trofino, Robust $\mathscr{H}_\infty$ filtering for discrete-time linear systems with uncertain time-varying parameters, IEEE Trans. Signal Proces. 54 (6) (June 2006) 2110–2118.
[7] H.K. Khalil, Nonlinear Systems, third ed., Prentice-Hall, Upper Saddle River, NJ, 2002.
[8] W.J. Rugh, J.S. Shamma, Research on gain scheduling, Automatica 36 (10) (October 2000) 1401–1425.
[9] G.I. Bara, J. Daafouz, F. Kratz, Advanced gain scheduling techniques for the design of parameter-dependent observers, in: Proceedings of 40th IEEE Conference on Decision and Control, Orlando, FL, USA, December 2001, pp. 3892–3897.
[10] M. Sato, Filter design for LPV systems using biquadratic Lyapunov functions, in: Proceedings of 2004 American Control Conference, Boston, MA, USA, July 2004, pp. 1368–1373.
[11] N.T. Hoang, H.D. Tuan, P. Apkarian, S. Hosoe, Gain-scheduled filtering for time-varying discrete systems, IEEE Trans. Signal Proces. 52 (9) (September 2004) 2464–2476.
[12] H. Gao, J. Lam, P. Shi, C. Wang, Parameter-dependent filter design with guaranteed $\mathscr{H}_\infty$ performance, IEE P. Contr. Theor. Appl. 152 (5) (September 2005) 531–537.
[13] R.A. Borges, V.F. Montagner, R.C.L.F. Oliveira, P.L.D. Peres, P.-A. Bliman, Parameter-dependent $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filter design for linear systems with arbitrarily time-varying parameters in polytopic domains, Signal Process. 88 (7) (July 2008) 1801–1816.
[14] F. Blanchini, S. Miani, C. Savorgnan, Stability results for linear parameter varying and switching systems, Automatica 43 (10) (October 2007) 1817–1823.
[15] R.A. Borges, R.C.L.F. Oliveira, C.T. Abdallah, P.L.D. Peres, $\mathscr{H}_\infty$ filtering of time-varying systems with bounded rates of variation, in: Proceedings of 47th IEEE Conference on Decision and Control, Cancun, Mexico, December 2008, pp. 1678–1683.
[16] R.A. Borges, R.C.L.F. Oliveira, C.T. Abdallah, P.L.D. Peres, $\mathscr{H}_\infty$ filtering of networked systems with time-varying sampling rates, in: Proceedings of 2009 American Control Conference, St. Louis, MO, USA, June 2009, pp. 3372–3377.
[17] M.A. Rotea, D. Williamson, Optimal realization of finite wordlength digital filters and controllers, IEEE Trans. Circuits Syst. I 42 (2) (February 1995) 61–72.
[18] J.M. Velni, K.M. Grigoriadis, Delay-dependent $\mathscr{H}_\infty$ filtering for time-delayed LPV systems, Syst. Contr. Lett. 57 (4) (April 2008) 290–299.
[19] M.C. de Oliveira, R.E. Skelton, Stability tests for constrained linear systems, in: S.O. Reza Moheimani (Ed.), Perspectives in Robust Control, Ser. Lecture Notes in Control and Information Science, vol. 268, Springer, New York, 2001, pp. 241–257.
[20] M. Chilali, P. Gahinet, $\mathscr{H}_\infty$ design with pole placement constraints: an LMI approach, IEEE Trans. Automat. Contr. 41 (3) (March 1996) 358–367.
[21] M. Fukuda, M. Kojima, Branch-and-cut algorithms for the bilinear matrix inequality eigenvalue problem, Comput. Optim. Appl. 19 (1) (2001) 79–105.
[22] A. Hassibi, J. How, S. Boyd, A path-following method for solving BMI problems in control, in: Proceedings of 1999 American Control Conferene, San Diego, CA, USA, June 1999, pp. 1385–1389.
[23] D.C.W. Ramos, P.L.D. Peres, A less conservative LMI condition for the robust stability of discrete-time uncertain systems, Syst. Contr. Lett. 43 (5) (August 2001) 371–378.
[24] R.A. Borges, P.L.D. Peres, $\mathscr{H}_\infty$ LPV filtering for linear systems with arbitrarily time-varying parameters in polytopic domains, in: Proceedings of 45th IEEE Conference on Decision and Control, San Diego, CA, USA, December 2006, pp. 1692–1697.
[25] W. Zhang, M.S. Branicky, S.M. Phillips, Stability of networked control systems, IEEE Control Syst. Mag. 21 (1) (February 2001) 84–99.
[26] F.-L. Lian, J.R. Moyne, D.M. Tilbury, Performance evaluation of control networks: Ethernet, ControlNet and DeviceNet, IEEE Control Syst. Mag. 21 (1) (February 2001) 66–83.
[27] G.C. Walsh, H. Ye, Scheduling of networked control systems, IEEE Control Syst. Mag. 21 (1) (February 2001) 57–65.
[28] M. Velasco, P. Marti, R. Villa, J.M. Fuertes, Stability of networked control systems with bounded sampling rates and time delays, in: 31st Annual Conference of IEEE Industrial Electronics Society, Raleigh, NC, USA, November 2005, pp. 2417–2422.
[29] P.J. Antsaklis, A.N. Michel, Linear Systems, Birkhäuser, Boston, 2006.
[30] L.A. Montestruque, P. Antsaklis, Stability of model-based networked control systems with time-varying transmission times, IEEE Trans. Automat. Contr. 49 (9) (September 2004) 1562–1571.
[31] R.C.L.F. Oliveira, P.L.D. Peres, Parameter-dependent LMIs in robust analysis: characterization of homogeneous polynomially parameter-dependent solutions via LMI relaxations, IEEE Trans. Automat. Contr. 52 (7) (July 2007) 1334–1340.
[32] J.-W. Lee, G.E. Dullerud, Uniform stabilization of discrete-time switched and Markovian jump linear systems, Automatica 42 (2) (February 2006) 205–218.
[33] J.-W. Lee, On uniform stabilization of discrete-time linear parameter-varying control systems, IEEE Trans. Automat. Contr. 51 (10) (October 2006) 1714–1721.
[34] Z.S. Duan, J.X. Zhang, C.S. Zhang, E. Mosca, Robust $\mathscr{H}_2$ and $\mathscr{H}_\infty$ filtering for uncertain linear systems, Automatica 42 (11) (November 2006) 1919–1926.
[35] H. Gao, X. Meng, T. Chen, A new design of robust H_2 filters for uncertain systems, Syst. Contr. Lett. 57 (7) (July 2008) 585–593.
[36] J.F. Sturm, Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones, Optim. Method Softw. 11–12 (1999) 625–653 ⟨http://sedumi.mcmaster.ca/⟩.
[37] J. Löfberg, YALMIP: a toolbox for modeling and optimization in MATLAB, in: Proceedings of 2004 IEEE International Symposium on Computer Aided Control System Design, Taipei, Taiwan, September 2004, pp. 284–289, ⟨http://control.ee.ethz.ch/~joloef/yalmip.php⟩.
[38] S. Zhou, J. Lam, A. Xue, $\mathscr{H}_\infty$ filtering of discrete-time fuzzy systems via basis-dependent Lyapunov function approach, Fuzzy Sets Syst. 158 (2) (January 2007) 180–193.