Finite-Time Stability of Nonlinear Networked-Control Systems

by

Silvia Mastellone

Laurea in Ingegneria Informatica,

Universita' Roma Tre, Rome Italy, March 2002

THESIS

Submitted in Partial Fulfillment of the
Requirements for the Degree of

Master of Science
Electrical Engineering

The University of New Mexico

Albuquerque, New Mexico

29 April, 2004

Dedication

Ai miei Mentori Peter Dorato, e Chaouki T. Abdallah,
Alla mia Famiglia,
A Peter

Acknowledgments

I dedicate all the devotion, time and sacrifice to realize this work to the people that make my life richer and the last year a great feat.

To my precious family: Mamma, for being the strongest and sweetest woman in my life, and for encouraging me to pursue my dreams and to *fly high*. Papa', from whom I learned how little things can make life great, and how to not be scared about most of the things in life. My sister Valeria, for showing me that all things in life, no matter how gloomy they appear to be, can be funny and worth a laugh.

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To my Peter for sharing with me the great adventure I started since I arrived in Albuquerque. For growing up with me through difficulties, enthusiasm and emotions. For our tears at the airport for every goodbye and every smile when meeting again. For being in my life and making it more than the wonderful dream I ever had.

For the next chapter of my life that I am just starting and whatever it brings along, I am extremely grateful to all these people who participated to make it possible, and I will do my best to make them proud.

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ABSTRACT OF THESIS

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Abstract

Finite-time stability of nonlinear networked control systems is studied in a stochastic and in a deterministic setting. Focusing on packet dropping, a deterministic model for networked control systems is realized by including the network dynamics in such model. This links the fields of study in control of networks and networked control systems.

Contents

List of Figures

List of Tables

List of Acronyms and Symbols

FT Finite-Time

FTS Finite-Time Stability

FTSS Finite-Time Stochastic Stability

EFTS Extended Finite-Time Stability

NCS Networked Control System

B-NCS Bounded Networked Control System

MB-NCS Model Based Networked Control System

EFTS Extended Finite-Time Stability

λ_{max} Maximum Eigenvalue

λ_{min} Minimum Eigenvalue

$\mathbb{N}$ Set of Natural Numbers

$\mathbb{R}$ Set of Real Numbers

$||x||_M = x^T M x$

$||x|| = \sqrt{x^T x}$

Chapter 1

Introduction

1.1 Introduction

In several recent works, the problem of networked control systems (NCS) has been posed and partially investigated. This new problem deals with the possibility of controlling a system remotely via a communication network and as such, instantaneous and perfect signals between controller and plant are not achievable (see Figure 1.2). This casts classical control problems into a setting that provides control solutions to remotely located systems such as: assembling space structures, exploring hazardous environment, executing tele-surgery, and many others.

Within this new setting we are able to overcome the necessity of collocated control and processes, thus overcoming many of the spatial restrictions. Networked control systems however, do not exist without new challenging sets of problems. In fact, the networks introduce delays of time-varying and possibly random nature, packet losses that degrade the performance of the system and possibly destabilize it, and limited bandwidth that compromises our otherwise achievable control objective. Most of classical control theory is based on the assumption that the controller, system, and

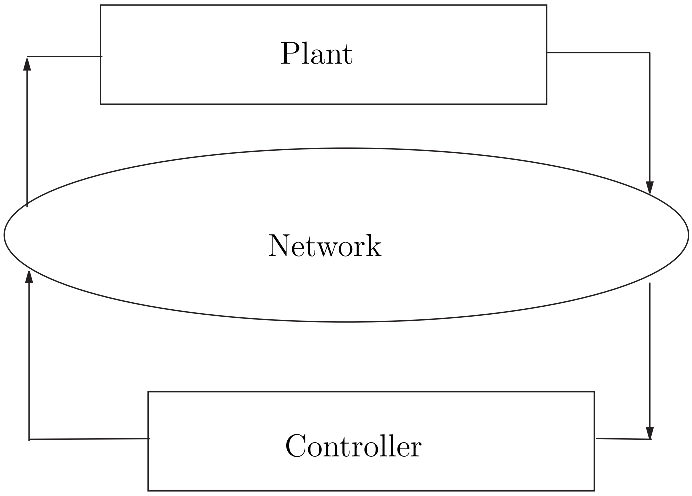

Figure 1.1: Networked Control System

sensors are collocated so the aforementioned problems were not apparent. A challenging aspect of the networked setting is that we need to compensate for the effects of the network in order to retain stability and performance of the system under study. Many models have been proposed to study the effects of the network, and in this thesis we aim to provide a novel model that links the effects of the network to the traditional control design.

Another novel aspect of this thesis is that, unlike the current trends that study the Lyapunov stability of networked systems, we use the concept of finite-time stability where specific bounds are desired on the performance of the system and the study is restricted to a finite interval of time. This issue appears in several problems where we are interested in the system's behavior only over a specific, finite-time interval. We also study how to control a system through a network which may subject the system to the loss of data.

1.2 Thesis Outline

The remainder of this thesis is divided as follows.

1.2.1 Chapter 2

Chapter two states the general problem and in particular describes the model used to control a nonlinear plant, assuming a model of the original plant available on the controller's side of the network. The state of the plant is sent through the network and is therefore subject to packet dropping. On the other side of the network, when a state is received it is used to update the model and the controller, or else the state provided by the model is used to update the controller. In both cases the controller is attempting to stabilize the closed-loop plant. The stability of the plant depends on the rate of packets lost, the accuracy of the model, and the initial conditions for the model and the plant. We also define in this chapter a specific class of networked control systems to study, and describe some of its properties.

1.2.2 Chapter 3

In this chapter we provide a model description of the networks used. For such models we describe how packets are dropped, and thus complete our model of the networked control system. In particular we describe two possible scenarios (stochastic and deterministic) of packet loss and complete the dynamics of the overall network.

1.2.3 Chapter 4

In chapter four finite-time stability of a general control systems is detailed. We focus in the first part on deterministic finite time stability, while in the second part

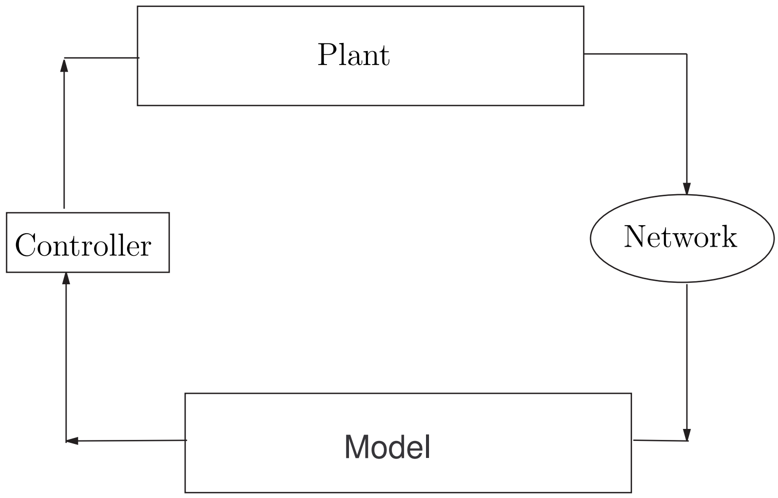

Figure 1.2: Model-Based Networked Control System

stochastic finite-time stability is studied.

1.2.4 Chapter 5

Exploiting the tools provided in the previous chapters we study in this chapter finite-time stochastic stability of the networked control system, in which the packet dropping is modelled as a random variable.

1.2.5 Chapter 6

In chapter six the deterministic model for the packet dropping is considered, and the deterministic finite-time stability is investigated for this case.

1.3 Contributions

One major contribution of this thesis lies in applying finite-time stability theory to a networked control system. Moreover, we provide a new link between two different areas of study, namely control of networks and networked control systems. Several efforts have been applied into the research of both areas [1]-[25], [34]-[38], and this thesis proposes a method to bridge their gap. Other contributions may also be found in extending model-based networked control systems into the nonlinear domain.

Chapter 2

Networked-Control-Systems

2.1 Introduction

In [1] a model for the networked control of linear time invariant systems was proposed. The network is modelled as a sampler placed between the plant and sensors on one side, and the controller on the other side of the network. Utilizing an approximate model of the process at the controller's side, the controller may be able to maintain stability while receiving only periodic updates of the actual state of the plant. Whenever a new update is received, the model is initialized with the new information. This idea was utilized in [2], where the system evolved in discrete-time, and state updates were either received or dropped at each sampling time due to the effects of the network. The characterization of such a dropout is achieved through the use of a Markov chain that takes on values of 0 or 1 depending on whether a sample was lost or received, respectively. Recently in [11], the model for a continuous-time plant and a network modelled with a fixed rate sampler was extended to bounded, yet random sample times driven by a Markov chain.

In this chapter, we extend the discrete-time result of [2] into a nonlinear setting, i.e. our plant and model used for state estimation are both nonlinear. We utilize the same model of packets being dropped according to either a stochastic model, or a deterministic one. In both cases we obtain results that guarantee finite-time stability in a stochastic or deterministic setting respectively.

The chapter is organized as follows: in Section 2.2, we reformulate the model-based networked control problem in the nonlinear discrete-time setting with generic packet dropout. We then describe in Section 2.3 a particular class of NCS and describe its properties. Finally Section 3.5 reports our conclusions.

2.2 Problem Formulation

In [2] a discrete-time model-based control with observation dropouts is proposed for linear discrete-time systems. Our objective in this chapter is to propose a similar framework in the case of nonlinear systems, and to study the finite-time stability of the resulting closed-loop system.

We consider the nonlinear discrete time plant described by the following

$$x_{k+1} = f(x_k) + g(x_k)u_k. \tag{2.1}$$

where $x_k \in \mathbb{R}^n$, and $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are two sufficiently smooth vector functions, and $u_k \in \mathbb{R}$ is a scalar input.

As depicted in Figure 2.1, discrete-time model-based control contains a plant and a model with the network residing between the sensors of the plant and the model and actuators.

The network is modelled as a two-value variable sequence θ_k, (assumed for now to be generic), where a measurement is dropped if $\theta_k = 0$, and a measurement is

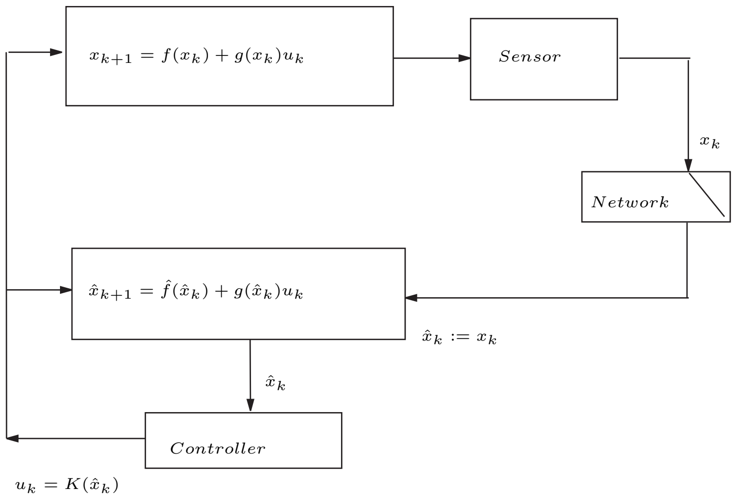

Figure 2.1: Model-Based NCS

received when $\theta_k = 1$. Due to our inability of receiving an update of the plant's state at each discrete instant of time, we use an inexact plant model on the controller side to provide us with a state estimate when packets are dropped. Such a model is given by

$$\hat{x}_{k+1} = \hat{f}(\hat{x}_k) + \hat{g}(\hat{x}_k)u_k. \tag{2.2}$$

in which $\hat{x}_k \in \mathbb{R}^n$, and $\hat{f}, \hat{g}$ are two smooth vector functions that map $\mathbb{R}^n$ into $\mathbb{R}^n$.

In order to carry out the analysis, we define the estimation error as $e_k = \hat{x}_k - x_k$, and augment the state vector x_k with e_k so that the closed-loop state vector is given by $z_k = \left(x_k^T; e_k^T\right)^T$, $z_k \in \mathbb{R}^{2n}$. The closed-loop system evolves according to

$$\begin{aligned} z_{k+1} &= \begin{pmatrix} f(x_k) \\ (f(x_k) - \hat{f}(x_k)) + (1-\theta_k)((\hat{f}(x_k) - \hat{f}(\hat{x}_k))) \end{pmatrix} \\ &+ \begin{pmatrix} g(x_k)K(\hat{x}_k) \\ (g(x_k) - \hat{g}(x_k))K(\hat{x}_k) + (1-\theta_k)(\hat{g}(x_k) - \hat{g}(\hat{x}_k))u_k \end{pmatrix}. \end{aligned} \tag{2.3}$$

In the above model $\theta_k \in \{0, 1\}$ is a sequence (or equivalently, $\varphi_k = \theta_k$ is a dropping sequence) that indicates the reception ($\theta_k = 1$) or the loss (θ_k=0) of the packet containing the state measurement x_k. We assume that at each step time k a state is sent across the network in one packet. If a packet is received, it is used as an initial condition for the next time step in the model, otherwise the previous state of the model is used. Note that $u_k = K(\hat{x})$ is a scalar state-feedback input. We then classify the NCS errors as follows:

(I). Model structure errors

$$e_{f1}(x_k) = f(x_k) - \hat{f}(x_k) \tag{2.4}$$

$$e_{g1}(x_k) = g(x_k) - \hat{g}(x_k). \tag{2.5}$$

These are the errors between the plant and the model evaluated at the plant's state, and are therefore dependent on the system's structure.

(II). State dependent errors

$$e_{f2}(x_k, \hat{x}_k) = \hat{f}(x_k) - \hat{f}(\hat{x}_k) \tag{2.6}$$

$$e_{g2}(x_k, \hat{x}_k) = \hat{g}(x_k) - \hat{g}(\hat{x}_k). \tag{2.7}$$

These represent the errors between the model evaluated at the plant's state and at its own state, i.e. the error introduced by the difference in the states.

(III). Structure and state dependent errors

$$e_{f3}(x_k, \hat{x}_k) = f(x_k) - \hat{f}(\hat{x}_k) \tag{2.8}$$

$$e_{g3}(x_k, \hat{x}_k) = g(x_k) - \hat{g}(\hat{x}_k), \tag{2.9}$$

which include both model structure and state dependent errors.

With the new notation, the system (2.3) becomes

$$z_{k+1} = \begin{pmatrix} f(x_k) + g(x_k)u_k \\ e_{f1}(x_k) + e_{g1}(x_k)u_k + (1-\theta_k)(e_{f2}(x_k,\hat{x}_k) + e_{g2}(x_k,\hat{x}_k)u_k) \end{pmatrix}$$

Based on the value of θ_k we have two possible situations:

1. for $\theta_k = 1$ the closed-loop system becomes

$$z_{k+1} = \begin{pmatrix} f(x_k) + g(x_k)u_k \\ e_{f1}(x_k) + e_{g1}(x_k)u_k \end{pmatrix} \tag{2.10}$$

2. for $\theta_k = 0$, we have

$$z_{k+1} = \begin{pmatrix} f(x_k) + g(x_k)u_k \\ e_{f3}(x_k,\hat{x}_k) + e_{g3}(x_k,\hat{x}_k)u_k \end{pmatrix} \tag{2.11}$$

For the remainder of this work we use the following compact form to represent the closed-loop system, and to highlight the fact that θ_k represents packet dropouts,

$$z_{k+1} = H_1(z_k) + H_2(z_k)(1-\theta_k),\ k \geq 0 \tag{2.12}$$

with

$$\begin{aligned} H_1(z_k) &= F_1(z_k) + G_1(z_k)u_k && (2.13) \\ H_2(z_k) &= F_2(z_k) + G_2(z_k)u_k. && (2.14) \end{aligned}$$

$$F_1(z_k) = \begin{pmatrix} f(x_k) \\ e_{f1}(x_k) \end{pmatrix} \tag{2.15}$$

$$F_2(z_k) = \begin{pmatrix} 0 \\ e_{f2}(x_k, \hat{x}_k) \end{pmatrix} \tag{2.16}$$

$$G_1(z_k) = \begin{pmatrix} g(x_k) \\ e_{g1}(x_k) \end{pmatrix} \tag{2.17}$$

$$G_2(z_k) = \begin{pmatrix} 0 \\ e_{g2}(x_k, \hat{x}_k) \end{pmatrix} \tag{2.18}$$

in which $H_i, F_i, G_i \in \mathbb{R}^{2n}$, $i = 1, 2$, are vector functions that map $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$.

Moreover, we assume that the control law $u_k = K(\hat{x}_k)$ stabilizes, in some sense, the plant in the case of full-state availability.

In the following we will refer to such networked-control-system (NCS) as a quadruple $(Plant, Model, Controller, Dropping\,Sequence\,\{\varphi_k\})$ and denote it as model-based networked control system $MB - NCS$.

2.3 Bounded Networked Control System

Next we define a particular class of NCS for which we characterize the accuracy of the model in representing the plant's dynamics, and describe how the model discrepancy affects the NCS structure.

Definition 2.1 ***Class*** C_{B-NCS} ***NCS***

A MB-NCS of the form (2.12), belongs to a class C_{B-NCS} *with the bounds* $(B_f, B_g, B_{efi}, B_{egi}; B_{h_i})$, $i = 1, 2$ *if for all* $k = 0, \ldots, N$, $N \in \mathbb{N}$ *and for all* $x_k \in S$, *where S is a given subset of $\mathbb{R}^n$, the system structure and error norms are bounded as follows*

$$
\begin{aligned}
&||f(x_k)|| \leq B_f \\
&||g(x_k)u(\hat{x}_k)|| \leq B_g(\hat{x}_k) \\
&||e_{f1}(x_k)|| \leq B_{ef1} \\
&||e_{f2}(x_k, \hat{x}_k)|| \leq B_{ef2}(\hat{x}_k) \\
&||e_{g1}(x_k)u(\hat{x}_k)|| \leq B_{eg1}(\hat{x}_k) \\
&||e_{g2}(x_k, \hat{x}_k)u(\hat{x}_k)|| \leq B_{eg2}(\hat{x}_k)
\end{aligned}
\tag{2.19}
$$

where B_f, B_{ef1} are constant bounds and $B_g(\hat{x}_k)$, $B_{ef2}(\hat{x}_k)$, $B_{eg1}(\hat{x}_k)$, $B_{eg2}(\hat{x}_k)$ are bounds that depend on the model state. Such NCS are called bounded model-based NCS (B-MB-NCS).

▲

The above definition describes the class of NCS, for which it is possible to define bounds on the plant and the NCS errors, and where such bounds depend only on the model's state.

Next we state a lemma that describes properties of class C_{B-NCS}. In particular the lemma describes how bounds on the norm of the B-MB-NCS errors imply bounds on the weighted norm of the NCS dynamics, i.e. on $||z_k||_M = z_k^T M(k) z_k$.

Lemma 2.1 *Consider the NCS (2.12) and $M(k) > 0$, $(2n \times 2n)$ time-varying real-valued matrix,*

$$
M(k) \;=\; \left[\begin{array}{c|c} m_1(k) & m_2(k) \\ \hline m_3(k) & m_4(k) \end{array}\right],\; m_i(k) \in \mathbb{R}^{n\times n},\; m_2(k)^T = m_3(k) \tag{2.20}
$$

Also assume the system belongs to class C_{B-NCS}. Then the following bounds hold on the norm of the NCS dynamics weighted by $M(k)$ for

$i, j = \{1, 2\}$, $j \neq i$, $k = 0, \ldots, N$, $N \in \mathbb{N}$ *and for all* $x_k \in S$, *where* $S \subset \mathbb{R}^n$,

$$\begin{aligned} &H_i^T M(k+1) H_j \leq B_{H_{i,j}}(\hat{x}_k) \\ &H_i^T M(k+1) H_i \leq B_{H_i}(\hat{x}_k) \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} &F_i^T(z_k) M(k) F_i(z_k) \leq B_{Fi}(\hat{x}_k) \\ &G_i^T(z_k) M(k) G_i(z_k) \leq B_{Gi}(\hat{x}_k) \\ &|F_i^T(z_k) M(k) G_j(z_k)| u_k \leq B_{FiGj}(\hat{x}_k) u_k \end{aligned} \tag{2.22}$$

where the bounds on the vector functions are related to the bounds on the errors as follows:

$$\begin{aligned} B_{H_1}(\hat{x}_k) \quad &= (B_f + B_g(\hat{x}_k))\lambda_{max}(m_1(k+1)) + (B_{ef1} + B_{eg1} \\ &(\hat{x}_k))(||m_3(k+1)|| + ||m_2(k+1)||)(B_f + B_g(\hat{x}_k)) \\ &+(B_{ef1} + B_{eg1}(\hat{x}_k))\lambda_{max}(m_4(k+1)) \end{aligned} \tag{2.23}$$

$$B_{H_{1,2}}(\hat{x}_k) = (B_{ef2}^T(\hat{x}_k) + B_{eg2}^T(\hat{x}_k))\lambda_{max}(m_4(k+1)) \tag{2.24}$$

$$\begin{aligned} B_{H_2}(\hat{x}_k) \quad &= (B_{ef1} + B_{eg1}(\hat{x}_k))(||m_4(k+1)||)(B_{ef2} + B_{eg2}(\hat{x}_k)) \\ &(B_f + B_g(\hat{x}_k))(||m_4(k+1)||)(B_{ef2} + B_{eg2}(\hat{x}_k)) \end{aligned} \tag{2.25}$$

and

$$\begin{aligned}
B_{F_1}(\hat{x}_k) &= B_f^T(\hat{x}_k)\lambda_{max}(m_1(k+1)) + B_{ef1}(||m_3(k+1)|| + \\
&||m_3(k+1)||)B_f(\hat{x}_k) + B_{ef1}\lambda_{max}\{m_4(k+1)\} \\
B_{G_1}(\hat{x}_k) &= B_g(\hat{x}_k)\lambda_{max}\{m_1(k+1)\} + B_{eg1}^T(\hat{x}_k)(||m_3(k+1)|| + \\
&||m_2(k+1)||)B_g(\hat{x}_k) + B_{eg1}(\hat{x}_k)\lambda_{max}\{m_3(k+1)\} \\
B_{FG_2}(\hat{x}_k) &= B_{ef2}^T(\hat{x}_k)||m_4(k+1)||B_{eg2}(\hat{x}_k) \\
B_{F_2}(\hat{x}_k) &= B_{ef2}(\hat{x}_k)\lambda_{max}\{m_4(k+1)\} \\
B_{G_2}(\hat{x}_k) &= B_{eg2}(\hat{x}_k)\lambda_{max}\{m_4(k+1)\} \\
B_{F_1G_2}(\hat{x}_k) &= B_f^T(\hat{x}_k)||m_2(k+1)||B_{eg2}(\hat{x}_k) + B_{ef1}^T(\hat{x}_k)||m_4(k+1)||B_{eg2}(\hat{x}_k) \\
B_{F_2G_1}(\hat{x}_k) &= B_{ef2}^T(\hat{x}_k)||m_3(k+1)||B_g(\hat{x}_k) + B_{ef2}^T(\hat{x}_k)||m_4(k+1)||B_{eg1}(\hat{x}_k)
\end{aligned} \tag{2.26}$$

Proof. We will only prove the statement for $H_1^T(z_k)M(k+1)H_1(z_k)$ in (2.21), as the proof for all other inequalities is similar. Consider the vector function

$$H_1^T(z_k)M(k+1)H_1(z_k),\ z_k \in \mathbb{R}^{2n} \tag{2.27}$$

Expanding (2.27) using the NCS errors bounds and the fact that $x^T Mx \leq \lambda_{max}\{M\}$ and $x^T My \leq ||x^T||||M||||y||$, we obtain

$$\begin{aligned}
&H_1^T(z_k)M(k+1)H_1(z_k) \leq \\
&(f(x_k) + g(x_k)u(\hat{x}_k))^T(f(x_k) + g(x_k)u(\hat{x}_k))\lambda_{max}\{m_1(k+1)\} + \\
&(e_{f1}(x_k) + e_{g1}(x_k)u(\hat{x}_k))^T||m_2(k+1) + m_3(k+1)||(f(x_k) + g(x_k)u(\hat{x}_k)) + \\
&(e_{f1}(x_k) + e_{g1}(x_k)u(\hat{x}_k))^T(e_{f1}(x_k) + e_{g1}(x_k)u(\hat{x}_k))\lambda_{max}\{m_4(k+1)\} \leq \\
&(B_f + B_g(\hat{x}_k))\lambda_{max}(m_1(k+1)) + (B_{ef1} + B_{eg1} \\
&(\hat{x}_k))(||m_3(k+1)|| + ||m_2(k+1)||)(B_f + B_g(\hat{x}_k)) \\
&+(B_{ef1} + B_{eg1}(\hat{x}_k))\lambda_{max}(m_4(k+1)) = B_{H_1}(\hat{x}_k)
\end{aligned} \tag{2.28}$$

which completes the proof.

■

The above lemma states that if in a NCS the norms of the plant and of the NCS errors are bounded by constants, or by a model's state bound in the finite interval of time $[0, N]$, see (2.19), then there exists a bound on the weighted norm of the NCS dynamics in the interval of time $[0, N]$, and moreover this bound depends on the errors bounds. Assuming the NCS is such that the above bounds on the errors hold, then it is possible to bound the weighted norm defined by the matrix M of the B-MB-NCS dynamics. In particular bounds defined on the vector function H_1 do not depend on packet dropping, whereas those on H_2 do, see equations (2.21). Also, going into more details, the H_i can be decomposed into a part F_i, $i = 1, 2$, independent of the input, and one dependent on the input, G_i, $i = 1, 2$ both of which may be bounded, see (2.22).

Lemma 2.2 *Consider the NCS (2.12) and $M(k) > 0$ matrix, and denote $\| x \|_M = x^T M x$, then for all $x_k \in S \subset \mathbb{R}^n, \forall k = 0, \ldots, N$*

$$||z_k||_M \geq \lambda_{min}\{M\} B_z(\hat{x}) \tag{2.29}$$

Also assume the system belongs to class C_{B-NCS} then for all $x_k \in S \subset \mathbb{R}^n$ and Euclidian norm $||.||$

$$||x_k|| \geq B_f + B_g(\hat{x}_k) = B_x(\hat{x}) \tag{2.30}$$

$$||e_k|| \geq B_{ef1} + B_{eg1}(\hat{x}_k) + B_{ef2}(\hat{x}_k) + B_{eg2}(\hat{x}_k) = B_e(\hat{x}) \tag{2.31}$$

$$||z_k|| \geq B_x(\hat{x}) + B_e(\hat{x}) = B_z(\hat{x}) \tag{2.32}$$

Proof.

For the first part of the lemma observe that

$$||z_k||_M \geq \lambda_{min}\{M\}||z_k|| = \lambda_{min}\{M\}(||x_k|| + ||e_k||) \geq \\ \lambda_{min}\{M\}(B_x(\hat{x}) + B_e(\hat{x})) = \lambda_{min}\{M\}B_z(\hat{x}) \tag{2.33}$$

The second part trivially follows from the system definition.

■

2.4 Conclusion

Nonlinear networked control systems have been introduced by extending the model based approach proposed in [1] to a nonlinear setting, and focusing on the packets dropping aspect.

The model for packet dropping remains for the time being unspecified, but will be defined in the next chapter by including the network dynamics.

Also a class of such NCS, namely B-MB-NCS with bounded errors between the model and the plant has been introduced and its properties have been explored.

Chapter 3

Network Control and Models for Packets Dropout

3.1 Introduction

Communication networks and their complex dynamics have been studied by several researchers, see, for example, [34, 35, 36]. Due to the Internet growth in size and complexity, and with the advent of industrial networks, an understanding of the organization and efficiency of communication networks has become necessary.

As communication between two systems takes place across a network, several problems arise such as delays and loss of information due to limited bandwidth and congestion. Considering the bandwidth as a fixed resource, in order to avoid the loss of information and delays, an efficient use of such resources is required. Congestion control represents an important aspect of the problem. As an example, in [34] the network is modelled as a dynamical system and the congestion control problem is reformulated as an optimization problem. Two main aspects of congestion control are highlighted; first the characterization of the equilibrium conditions from the point of view of fairness, efficiency in resource usage, the dependence on network parameters, etc. Second the stability of the postulated equilibria is studied in terms of performance metrics such as speed of convergence, capacity tracking, etc. In the present chapter we define and model a simplified network. In chapter 2 we described a model for the NCS with the dropping sequence θ_k considered generic. Here, using the network model we aim to model the dropping sequence using either random or deterministic packet dropouts.

The chapter is organized as follows: in Section 3.2, we describe a simplified version of a communication network and model its dynamics. Then in Section 3.3 we present two stochastic models for the packet dropout. In the first model the packet dropout is modelled as an homogeneous Markov chain, while in the second an independent Markov chain, i.e. a process of i.i.d. random variables, is used. Then in Section 3.4 a deterministic model for packet dropout is proposed. Finally in Section 3.5, we present our conclusions.

3.2 A Simple Model for Communication Networks

The problem studied in NCS is a stability problem, with the added complication that the plant is being controlled across a network. On the other hand, congestion control studies the problem of multiple users sharing a common resource on the network. Congestion control can also be interpreted as a stability problem, see [34].

In NCS, delays and packet drops are viewed as network effects affecting the capability to control and therefore the stability of the plant. Those effects are studied without paying any attention to their causes. In a congestion control framework, delays and packet drops need to be reduced or eventually eliminated by acting on their causes such as congestion, sources rates, and so on. In both arenas we note an interplay between networks and control. Our goal is to merge those two areas of study, i.e. to define a simplified network model and, focusing on packets dropping, to model the loss of packets directly in terms of their cause, i.e. the network dynamics such as sources rates and channel capacity. In order to explore the causes of packet

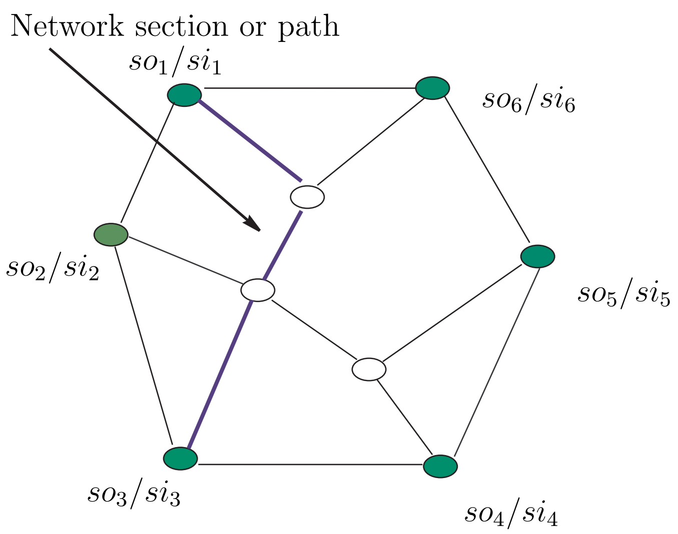

Figure 3.1: Undirected or bidirectional Network, nodes are sources and sinks.

dropping we start by defining the network setting in which the packet drops take place and exploring some of their properties.

Among several models and descriptions of communication networks provided in the literature, we choose to redefine the network in a simpler framework.

Definition 3.1 ***Network***

A network is a couple (L, S)*, where* L *is a set of* n_L *links, and* S *is a set of* n_S *nodes that can potentially perform as sources or sinks of traffic. Each link is a transmission medium whose capacity, also referred to as bandwidth or data rate, is measured in packets per second, where a packet is the information carrier. Each source* so_i *has an associated rate* $r_i(k)$ *that is a function of time, and denotes the* *number of packets per second sent to the sink* si_i*.*

▲

In Figure 3.1 a communication network is depicted: in particular the network may be bidirectional or undirected, and the nodes can be either sources or sinks, i.e. they can send or receive packets.

Definition 3.2 ***Network's Section***

Consider a network (L, S)*, a couple* (L_s, S_s)*, in which* $L_s \subset L$*,* $S_s \subset S$ *is called a network section of the network* (L, S)*. Moreover* L_s *is called a path and is a set of* n_l *links, and* S_s *is a set of* n_s *nodes that access the path.*

▲

Consider a network (L, S) composed of n_L links l_j, $j = 1, \ldots, n_L$ in which n_S sources access sending information to n_L sinks. Each source so_i, $i = 1, \ldots, n_s$ sends information to the sink si_i encoded in packets trough the network with time rate $r_i(k)$, $k = 0, 1, 2, \ldots,$. Also each link l_i has an associated a fixed bandwidth capacity C_i and at each time k we have a corresponding left over capacity $c_i(k)$, $0 \leq c_i(k) \leq C_i$ that represents the amount of packets per second it can support. Let each link l_i be used by n_s sources, each sending at a rate $r_j(k)$ and therefore the global rate at the $i-th$ link is $G_i(k) = \sum_j^{n_s} r_j(k)$. We are interested in the section network (L_{si}, S_{si}) that is being used by the system. Where L_{si} is the path, or set of $n_{l,i}$ links, associated with each source-sink (so_i, si_i). We define the following possible state for the network.

Definition 3.3 ***Congested Link***

A link l_i is congested at time k if the amount of packets sent through it exceed its leftover capacity, i.e.

$$G_i(k) > c_i(k),\ fork \in \mathbb{N} \tag{3.1}$$

▲

Definition 3.4 ***Congested Network Section***

A network section (L_s, S_s) is congested at time k if at least one link is congested

▲

If a link is congested then it starts dropping packets. After a certain period of time, the congestion disappears as a consequence of the sources reducing their rate so that $c_i(k) \geq G_i(k)$. A sink does not receive packets as a consequence of congestion in one of the links in the path associated with it.

3.3 Stochastic Model for Packet Dropout

The packet dropout is caused by network congestion, so it is mainly related to the network dynamics. We assume the network model is not exactly known, but only its target values of performance,(such as stochastic limits on the packet drops) are known.

In order to explore possible stochastic models for packet dropping, we briefly recall some Markov chains concepts.

3.3.1 Markov Chains

Since we are interested in modelling random packet dropouts, we observe first that a packet dropout is a stochastic process. Among random processes, Markov chains are relatively simple because the random variable is discrete and so is time. More importantly, Markov chains (and for that matter Markov processes in general) have the basic property that their future evolution is determine by their current state and does not depend on their past. We would like to use this property, with some additional restrictions, to represent the fact that dropping a packet at each time k does not depend on whether or not there was a prior packet drop. We next proceed with some standard definitions [30]-[33].

Definition 3.5 *(Markov Chain)*

Consider the probability space $(\Omega,\ F, P)$*, in which* Ω *is the sample space,* F *is a* σ*-algebra of subsets of* Ω*, and* P *the probability measure defined on* F*. Let* $\{\theta_k\}_{k\in\mathbb{N}}$ *be a sequence of random variables that take values on* S *then, the sequence is a homogeneous Markov chain with state space* S*, transition probability matrix* $\mathbb{P} = (p(i,j))$ *, if for every* $k \in \mathbb{N}$

$$P\{\theta_{k+1} = j|\theta_k = i, \theta_{k-1} = i_{k-1}, \ldots, \theta_0 = i_0\} = P\{\theta_{k+1} = j|\theta_k = i\} = p(i,j) \quad (3.2)$$

for all $(j, i, i_{k-1}, \ldots, i_0) \in S^{k+2}$ *and* $\forall k$*, where* $p(i,j)$ *is the transition probability from state* i *to state* j*.*

▲

In other words, the state of the Markov chain depends only on the previous state, and not on the whole history of the chain.

The first identity in equation (3.2), which is also called "Markov property", defines the "memory" or "order" of the chain. In the case of equation (3.2), the order equals one since the transition probabilities are entirely determined by the preceding state. The second identity in (3.2) is called the homogeneity condition. It assures that the transition probabilities do not vary with the time k, i.e. they are stationary.

Let $\mathbb{P} = [p(i,j)]_{i,j\in S}$ denote the transition probability matrix of a Markov chain $\{\theta_n\}$. To complete the construction of a Markov chain we need to specify an initial distribution. Let us denote by D_S the set of discrete distributions on S,

$$D_S = \{\mathbf{P} = (P_i)_{i\in S} : P_i \geq 0, \sum_{i\in S} P_i = 1\} \tag{3.3}$$

We call $\mathbf{P}_0 = (P_{0i})_{i\in S} \in D_S$ the initial distribution of the chain $\{\theta_k\}$ if $P\{\theta_0 = i\} = P_{0i}$ for all states $i \in S$.

Definition 3.6 *(Independent chain)*

Let $\mathbf{P} = (P_1, \ldots, P_m) \in D_S$ and define an m-state Markov chain with transition matrix $\mathbb{P} = (p_{i,j})_{i,j\in S}$ given by $p_{i,j} = P_j$, $i \in S$, and arbitrary initial distribution $\mathbf{P}_0$. Then for all $k \in \mathbb{N}$

$$P\{\theta_{k+1} = j|\theta_k = i, \theta_{k-1} = i_{k-1}, \ldots, \theta_0 = i_0\} = P\{\theta_{k+1} = j\} =: P_j \tag{3.4}$$

We call this the "independent chain" with respect to $\mathbf{P}$.

▲

Corollary 1 *In an independent chain the sequence of states is a sequence of independent random variables.*

Independent chains have no memory and they are also called zero-order Markov chains.

3.3.2 Packets Dropout Models

In this first part of packets dropout modelling, we recall the independent Markov chain model proposed in [2] and extend it to the case of dependent Markov chains. Both resulting models are stochastic and do not directly involve the network dynamics.

In the first place we consider the case in which partial information is available about the network, and therefore we can reduce the level of uncertainty. Assume that the loss of packets is not completely unknown, but depends for example, on congestion in the network that occur with a known frequency. Also assume the time needed to eliminate the congestion is known. In this case if a packet is dropped at time k as a consequence of network congestion, it is likely to be dropped at the next time $k+1$. Therefore packets dropping can be modelled as a dependent-elements stochastic process and $\{\varphi_k\}$ becomes a two-state Markov chain defined as follows: $\{\varphi_k\} \in S = \{0, 1\}$, $\varphi_k = 1 - \theta_k$

$$\begin{aligned}
\mathbb{P}\left(\theta_{k+1} = 0 | \theta_k = 0\right) &= p_{00} \\
\mathbb{P}\left(\theta_{k+1} = 0 | \theta_k = 1\right) &= p_{01} \\
\mathbb{P}\left(\theta_{k+1} = 1 | \theta_k = 0\right) &= p_{10} \\
\mathbb{P}\left(\theta_{k+1} = 1 | \theta_k = 1\right) &= p_{11}
\end{aligned}$$

and the transition probability matrix is

$$\mathcal{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \tag{3.5}$$

Also assuming that the loss of packets depends on the network congestion and that as soon as congestion takes place, the network starts dropping packets for a certain time, we have that the probability of dropping a packet at time k is larger if a packet has been dropped at the previous time $k-1$, i.e. $p_{00} > p_{01}$, $p_{11} > p_{10}$.

For the second model, we assume that the event of congestion is completely random and happens with a certain probability q. Also assume the time for a link to eliminate congestion is not known, therefore the event of dropping a packet is random with probability q. We can then model the packet dropping through the independent Markov chain $\varphi_k = (1 - \theta_k)$ (independent sequence of i.i.d. random variable) with a binary phase space $S = \{0, 1\}$, governed by the following transition probabilities

$$\begin{aligned} \mathbb{P}(\theta_{k+1} = 0) = \mathbb{P}(\theta_{k+1} = 0|\theta_k = 0) &= \mathbb{P}(\theta_{k+1} = 0|\theta_k = 1) = p \\ \mathbb{P}(\theta_{k+1} = 1) = \mathbb{P}(\theta_{k+1} = 1|\theta_k = 0) &= \mathbb{P}(\theta_{k+1} = 1|\theta_k = 1) = 1 - p = q \end{aligned} \tag{3.6}$$

that is, at each time k the probability of getting or not getting a packet is independent whether or not a packet was received before. Then the state transition probability matrix is given by

$$\mathcal{P} = \begin{pmatrix} p & p \\ 1-p & 1-p \end{pmatrix} \tag{3.7}$$

which is not block diagonal, and hence the chain is said to be irreducible(for a definition of irreducible Markov chain and associated properties see [32]), i.e. the probability of either state occurring at time k is never zero. Now consider the complement of the process θ_k, $\varphi_k = (1 - \theta_k)$, which is also a sequence of two-state i.i.d. random variable, representing the packets received. We then have the following statistics of φ_k

$$\mu_\varphi = \mathbb{E}[\varphi] = \sum_{j=0}^{1} j p_j = q \tag{3.8}$$

$$\mu_{\varphi^2} = \mathbb{E}\left[\varphi^2\right] = \sum_{j=0}^{1} j^2 p_j^2 = q \tag{3.9}$$

In our study we only consider the stochastic packets dropping as a random process of i.i.d. random variables, i.e. as an independent two-state Markov chain.

3.4 Deterministic Model for Packet Dropout

Next we move to the second part of this work, in which we model the packet dropout by considering the network dynamics. In particular we are interested in the network section that includes the path that a packet is going to follow. This path is composed of a number of n_l links, and with each link is associated an actual traffic, depending on the number and rate of sources that are accessing the path, and on the link physical capacity see figure (5.4). We want to study how the loss of packets affects

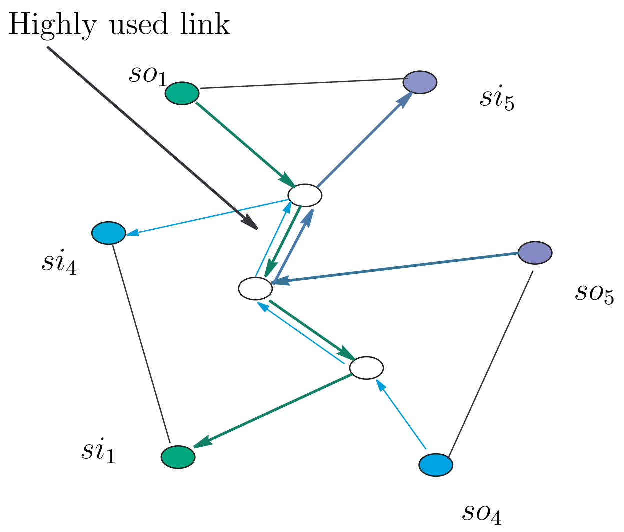

Figure 3.2: Sources and their paths trough the network links.

the stability of the overall system by including the network dynamics in the model. In particular this will allow us to explicitly relate the stability of the system to the capacity of the links involved in the path used by the system, and to the rate of the sources that are accessing such a path. This relation gives us the possibility of eventually designing for the stability of the system by controlling the rate of the sources accessing the path.

3.4.1 Congestion Control Model

Let (L, S) be a network in which each source s_i has an associated rate $r_i(k)$ that is a function of time at which it sends packets trough a set $L_i \subset L$ of links. So through every link l_j a total rate that is the sum of all the rates of n_s sources is given by:

$$R_j(k) = \sum_{i=1}^{n_s} r_i(k) \tag{3.10}$$

Moreover, each link will have a capacity function proportional to the total rate that will indicate the level of occupation of the link

$$G_j(k) \quad = \quad K_l R_j(k),\ j = 1, \ldots, n_l \tag{3.11}$$

A link has a limiting capacity beyond which it will drop packets. In particular there is a critical level of leftover capacity $c_i(k)$ above which the link will accomodate packets, and below which it will start dropping them. The packet drop will be modelled by the binary value variable θ_k, as discussed earlier. Consider the indicator function defined as follows

$$I_{c_j(k) \geq G_j(k)}(G_j(k)) = \begin{cases} 1 & , G_j(k) \leq c_j(k) \\ 0 & , G_j(k) > c_j(k) \end{cases} \tag{3.12}$$

then we have at every instant of time k

$$\theta_k = \prod_{j=1}^{l_r} I_{c_i \geq G_j(k)}(G_j(k)) \tag{3.13}$$

which may also be described as:

$$\theta_k = \prod_{j=1}^{n_l} \left[\frac{sign(c_i - G_j(k)) + 1}{2} \right] \tag{3.14}$$

where the function $sign : \mathbb{R} \rightarrow \{-1, 1\}$ is defined as follows

$$sign(a) = \begin{cases} 1 & a \geq 0 \\ -1 & a < 0 \end{cases} \tag{3.15}$$

The complementary variable $\varphi_k = 1 - \theta_k$ can then be obtained as follows

$$\varphi_k = \left[1 - \prod_{j=1}^{n_l} \left[\frac{sign(c_j(k) - G_j(k)) + 1}{2} \right] \right] \tag{3.16}$$

With the provided framework we are now able to study the stability of the following

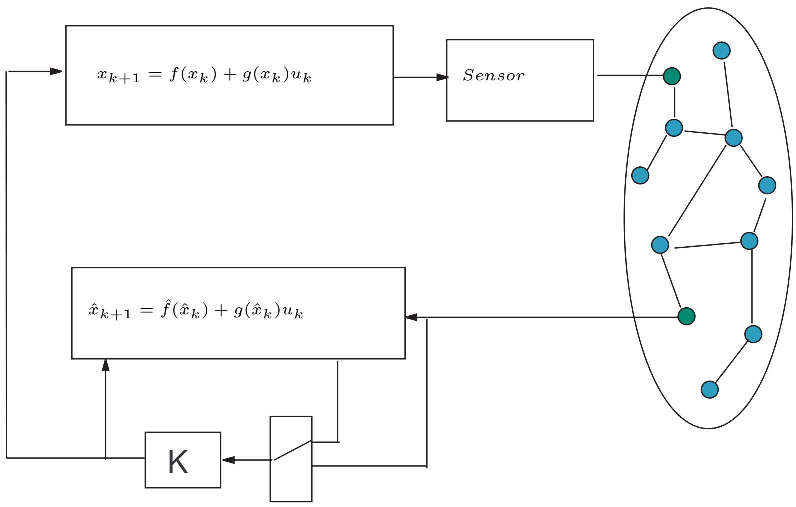

Figure 3.3: Model-Based NCS

dynamical nonlinear time varying system (Figure 3.3):

$$
\begin{aligned}
z_{k+1} &= (F_1(z_k) + G_1(z_k)u_k) + (F_2(z_k) + G_2(z_k)u_k)\varphi_k \\
&= (F_1(z_k) + G_1(z_k)u_k) + (F_2(z_k) + G_2(z_k)u_k) \\
&\quad \left[1 - \prod_{j=1}^{n_l}\left[\frac{sign(c_j(k) - G_j(k)) + 1}{2}\right]\right] \\
&= (F_1(z_k) + G_1(z_k)u_k) + (F_2(z_k) + G_2(z_k)u_k) \\
&\quad \left[1 - \prod_{j=1}^{n_l}\left[\frac{sign(c_i(k) - K\sum_{j=1}^{n_s} r_i(k)) + 1}{2}\right]\right]
\end{aligned} \tag{3.17}
$$

where $G_j(k)$ is given by (3.11), and where $r_i(k)$ are the known sequence of rates for sources accessing the path.

This model of NCS is a discrete-time, time-varying dynamical system that incorporates the system state z_k, and the network dynamics $c_i(k), r_j(k)$. The network is therefore an integral part of the overall system, therefore achieving our chapter goal as depicted in Figure (3.3).

3.5 Conclusion

In this chapter we discussed the interplay between networks and control that occurs in the areas of NCS and congestion control. Recalling some stochastic models for packet dropout based on independent Markov chains, we extended such models to dependent Markov chains. We also described a model for the network involved in our study. Based on the network's model we also proposed stochastic and deterministic mechanisms for packets dropping.

Chapter 4

Finite-Time Stability

4.1 Introduction

In this chapter we extend some of the existing results in finite-time stability to the design of discrete-time stochastic systems. In many practical problems it is of interest to investigate the stability of a system over a finite interval of time, since it might be crucial to stay within given bounds over a finite time. Classical control theory does not directly address this requirement because it focuses mainly on the asymptotic behavior of the system (over an infinite time interval), and does not usually specify bounds on the trajectories. On the other hand, finite-time stability (or short-time stability [39]) plays an important role in the study of the transient behaviors of systems and in some way answers the question proposed in [50], on how is “asymptotic” defined.

It is important to underline how the two stability concepts are disconnected, i.e. neither one of them implies nor excludes the other. In fact a system can be finite-time stable, i.e. a state starting within a “specified” bound α does not exceed a “specified” bound β in a specified time interval $[0, N]$, but may become unstable after the specified interval of time. On the other hand, the state trajectory might exceed the given bound over a certain time interval, but asymptotically go to zero. Asymptotic stability is specified with respect to arbitrary bounds, i.e. a trajectory starting within a bound $\delta(\epsilon)$ stays in an "arbitrary" ϵ and eventually converges to the origin, while finite-time stability is always defined with respect to pre-specified bounds α and β. In Figure 6.2 the two stability concepts are contrasted.

At first the concept of finite-time stability emerged under the name of "practical stability" [49], in which specific bounds on the state were given. For finite-time stability the interval of operation is assumed finite. The finite-time stability analysis problem has been discussed for linear systems [39, 41], and nonlinear systems [44]. A stochastic version of finite-time stability has been developed in [42] for analysis and in [47, 48] for optimal control design. Deterministic finite-time stability theory has been recently applied to several control problems in linear systems [40].

After discussing the deterministic case in Section 4.2, we move to stochastic finite-time stability in Section 4.3. In particular, we introduce in Section 4.3.1 useful bounds, then in Section 4.3.2 we use those bounds to state sufficient conditions for a stochastic system to be finite-time stable. Section 4.3.3 compares and discusses the results in the previous sections. We then proceed in Section 4.3.4 to extend the analysis techniques to designing controllers. Finally in Section 4.3.5, we propose an optimal feedback law for finite-time stability of a dynamical stochastic system.

4.2 Deterministic Finite-Time Stability

We focus on discrete-time dynamical systems described by

$$x_{k+1} = f(x_k),\ x \in \mathbb{R}^n,\ x(0) = x_0 \tag{4.1}$$

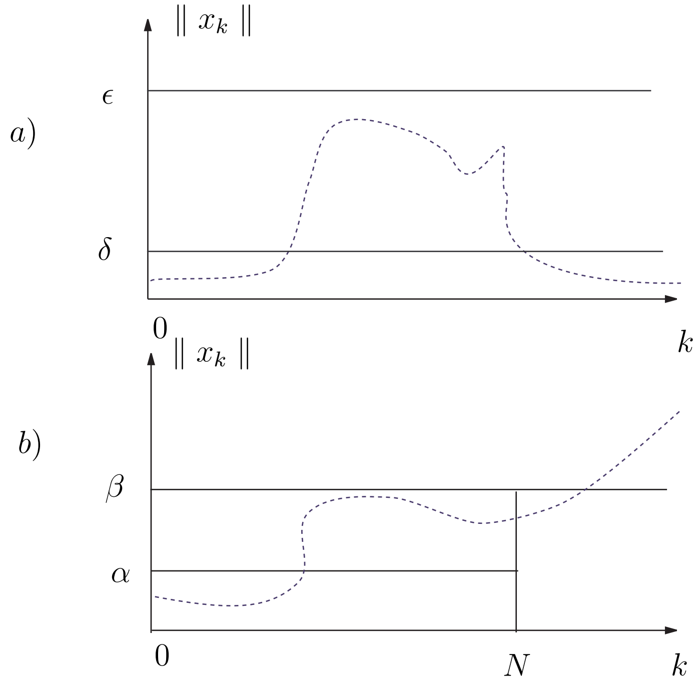

Figure 4.1: Asymptotic stability $a)$ versus finite-time stability $b)$.

Where $x \in \mathbb{R}^n$ is the system state, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth vector function. We are interested in studying the state trajectory of the system in a finite time interval, in other words we want to guarantee that specific bounds on the state are maintained in this finite time interval.

Definition 4.1 ***Finite-Time Stability***

The system (4.1) is finite-time stable (FTS) with respect to $(\alpha, \beta, N, ||.||)$ *with* $\alpha < \beta$ *if every trajectory* x_k *starting in* $||x_0|| \leq \alpha$ *satisfies the bound* $\| x_k \| < \beta$ *for all* $k = 1, \ldots, N$.

▲

We consider three classes of systems described in Figure (4.2): a) systems for which the state trajectories always increase in magnitude, b) systems for which states always decrease in magnitude, and c) systems whose state trajectories behavior's is mixed. The first step consists of exploring the state trajectories using a discrete

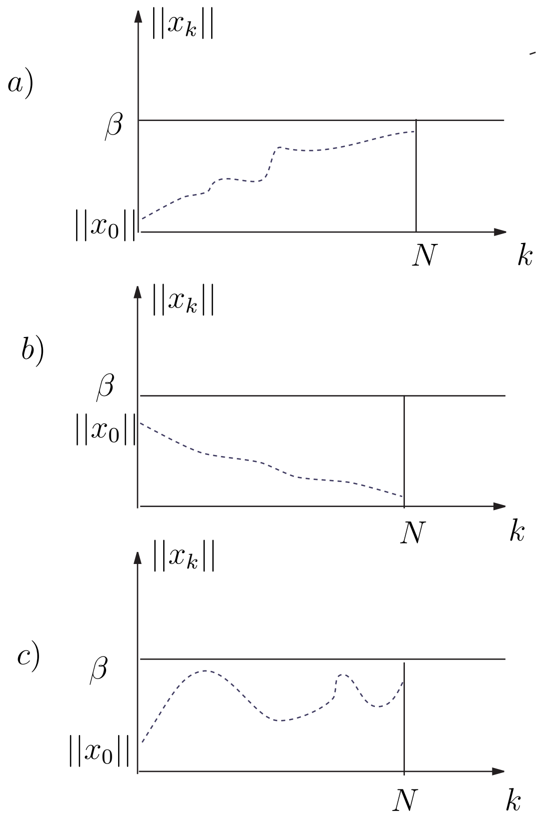

Figure 4.2: $a)$ Increasing dynamics. $b)$ Decreasing dynamics. $c)$ Mixed dynamics.

version of the Bellman-Gronwall inequality [46]. If the state trajectory is always increasing (in the norm) during the time interval of interest, then it is enough to verify that the state at the last time of the interval does not exceed the bound. In the case where the trajectory is always decreasing and it starts inside the bound, then FTS is guaranteed since $\alpha < \beta$. In the case of a mixed behavior, it is necessary to explore if the trajectory is suitably bounded at each time step.

In the next theorem we formulate conditions for finite-time stability of system (4.1).

Theorem 4.1 *The system (4.1) is finite-time stable with respect to* $(\alpha, \beta, N, ||.||)$, $\alpha \leq \beta$, *if for a continuous function* $V(x_k, k) = V_k \geq 0$ *such that for some* $\delta_1 > 0$ $\delta_1||x_k||^2 \leq V_k \leq \delta_2||x_k||^2$, $\gamma = \delta_1\beta$, $\gamma_0 = \delta_2\alpha$, *and* $S_\beta = \{x_k : ||x_k|| \leq \beta\}$ *one of the following three conditions occur:*

- **Case 1:** $\rho_k \geq 0$

$$\Delta V_k \leq \rho_k V_k, \, \forall k = 0, \ldots, N, \, \forall x_k \in S_\beta \tag{4.2}$$

$$\frac{\gamma}{\gamma_0} \geq \prod_{i=0}^{N-1}(1+\rho_i) \tag{4.3}$$

- **Case 2:** $0 \geq \rho_k \geq -1$

$$\Delta V_k \leq \rho_k V_k, \, \forall k = 0, \ldots, N, \, \forall x_k \in S_\beta \tag{4.4}$$

- **Case 3:** $\rho_k \geq -1$

$$\Delta V_k \leq \rho_k V_k, \, \forall k = 0, \ldots, N, \, x_k \in S_beta \tag{4.5}$$

$$\frac{\gamma}{\gamma_0} \geq \sup_k \prod_{i=0}^{k-1}(1+\rho_i) \tag{4.6}$$

Proof.

The proof of each of the three cases is provided separately:

- **Case 1:** Consider condition (4.2) with $\rho_k \geq 0$

$$\Delta V_k = V_{k+1} - V_k \leq \rho_k V_k, \tag{4.7}$$

from which it follows that

$$V_{k+1} - (1+\rho_k)V_k \leq 0\,,\ \forall k = 0, \ldots, N \tag{4.8}$$

Since $\rho_k \geq 0$, iterating the difference inequality and considering the upper bound on $V_0 \leq \gamma_0$, we obtain

$$V_N \leq V_0 \prod_{i=0}^{N-1}(1+\rho_i) \leq \gamma_0 \prod_{i=0}^{N-1}(1+\rho_i). \tag{4.9}$$

Finally, using the condition in (4.3), it follows that $V_N \leq \gamma$, and since the function is at most always increasing with a rate $\rho_i \geq 0$ it follows that the bound is never exceeded $\forall k = 0, \ldots, N$.

- **Case 2:** Now let us evaluate (4.2) for $0 \geq \rho_k \geq -1$

$$\Delta V_k = V_{k+1} - V_k \leq \rho_k V_k,\ \forall k = 0, \ldots, N \tag{4.10}$$

from which it follows that

$$V_{k+1} - (1+\rho_k)V_k \leq 0,\ \forall k = 0, \ldots, N \tag{4.11}$$

From the condition $0 \geq \rho_k \geq -1$, it follows that the function V_k is decreasing, so the finite-time stability condition is trivially satisfied since the upper bound of the initial state α is below the required state bound β.

- **Case 3:** Finally for $\rho_k \geq -1$ we have

$$\Delta V_k = V_{k+1} - V_k \leq \rho_k V_k,\ \forall k = 0, \ldots, N. \tag{4.12}$$

from which it follows

$$V_{k+1} - (1 + \rho_k)V_k \leq 0,\ \forall k = 0, \ldots, N \tag{4.13}$$

Because now $\rho_k \geq -1$, it is no longer possible to simply iterate the difference inequality for the time interval $k = 0, \ldots, N$. It is thus necessary that all intermediate terms satisfy the inequality, then iterating the partial difference inequalities and considering the upper bound on $V_0 \leq \gamma_0$ we get

$$\begin{aligned} V_k &\leq V_0 \prod_{i=0}^{k-1}(1 + \rho_i) \\ &\leq \gamma_0 \prod_{i=0}^{k-1}(1 + \rho_i),\ \forall k = 0, \ldots, N \end{aligned} \tag{4.14}$$

finally, using the condition in (4.6) it follows that $V_k \leq \gamma$, $\forall k = 0, \ldots, N$, which then guarantees the system is finite-time stable with respect to the specified parameters.

■

4.2.1 Extended Finite-Time Stability

We introduce next to introduce a novel concept, which has not been discussed in earlier works. In particular we consider the case in which the state norm may exceed the bound β, but only for a finite number of consecutive steps, after which it needs to contract again below the bound β. The rationale for this is to consider for the deterministic case an equivalent concept to the stochastic one, where the possibility of exceeding the bound for some time is allowed. The proposed extension fits many real situations such as the example of driving a car in a tunnel, where we do not want to hit the tunnel walls, but in case the car is robust enough, we may hit the walls for short periods of time. Another example, may be to consider hot object we need to grab, which even if the temperature is high we can touch it for short time. Therefore we allow a tolerance time within which we can support the object, but after which we need to release it and eventually grab it again. We formalize such a concept with the following definition.

Definition 4.2 ***Extended Finite-Time Stable*** *The nonlinear discrete-time system (4.1) is EFTS with respect to* $(\alpha, \beta; N, N_o)$, *if one of the following holds*

(I.) for some $k \in [0, N]$ *either*

$$\{||x_k|| < \beta : k \in [0, N] | \, ||x_0|| \leq \alpha\} \tag{4.15}$$

or

(II.)

$$\{\forall j \in [0, N] : ||x_j|| > \beta, \Rightarrow \min_{j+1 \leq i \leq j+N_o+1} ||x_i|| \leq \beta\}, \; N_o < N \tag{4.16}$$

where N_o *is the number of consecutive steps the system state is allowed to exceed the FT bound.*

▲

Definition 4.3 ***Attracted System***

A discrete-time system of the form (4.1) is an attracted system with respect to $(\alpha_1, \beta, \alpha_2, N, N_r)$, $\alpha_1 \leq \beta_z \leq \alpha_2$ *if it is FTS with respect to* (α_1, β, N) *and contracting with respect to* (α_2, β, N_r), *i.e.*

$$||x_0|| \leq \alpha_1 \Rightarrow ||x_k|| \leq \beta, \; k = [0, N]$$

$$\alpha_2 \geq ||x_0|| \geq \beta \Rightarrow ||x_k|| \leq \beta,\ k = [N_r, N]$$

▲

Theorem 4.2 *Consider a system (4.1) and assume it is attracted with respect to* $(\alpha_1, \beta, \alpha_2, N, N_r)$ *where the region* $[-\beta, \beta]$ *is a global region of attraction for the state. Also assume* N_r *is the number of steps needed for the state to contract into the ball of radius* β *from a distance* α_2*. Then the system is EFTS with respect to* $(\alpha_1, \beta, N, N_r + 1)$

Proof. In the case of $||x_0|| \leq \alpha_1$ the assumption that the system is contractive implies that $||x_k|| \leq \beta$, $k = [0, N]$ from which FTS follows and therefore EFTS.

In the case of $\alpha_2 \geq ||x_0|| \geq \beta$ we have $||x_k|| \leq \beta$, $k = [N_r, N]$ which implies $||x_{N_r}|| \leq \beta$ and therefore $min_{0 \leq j \leq N_r+1} ||x_j|| \leq \beta$ which means EFTS with respect to $(\alpha_1, \beta, N, N_r + 1)$.

■

4.3 Stochastic Finite-Time Stability

Next, we want to describe how finite-time stability, which was originally defined for deterministic systems may be extended to stochastic systems. Consider a discrete time, stochastic dynamical system in which the state is a Markov process in $\mathbb{R}^n$

$$x_{k+1} = f(x_k, \theta_k),\ x \in \mathbb{R}^n,\ x(0) = x_0 \tag{4.17}$$

where $x \in \mathbb{R}^n$ is the system state, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a vector function, and θ_k a stationary independent random sequence. In stochastic dynamical systems it is meaningful to consider the probability for the trajectory not to exceed a given bound over a finite time interval. Such a probability is called "Inclusion Probability", as described in [47].

Definition 4.4 ***Inclusion Probability***

Consider the dynamical stochastic system (4.17), the associated inclusion probability with respect to $(\alpha, \beta, N, ||.||)$ *is defined as follows:*

$$P_{in}(x_k; \alpha, \beta, N) =$$
$$P\{||x(k)|| < \beta : k \in [0, N]; ||x(0)|| \leq \alpha\}$$

▲

We also define the "Exit Probability" as the probability for the supremum over all states norms in the given time interval to exceed a given bound.

Definition 4.5 ***Exit Probability***

Consider the dynamical stochastic system (4.17), the associated exit probability with respect to $(\alpha, \beta, N, ||.||)$ *is defined as follows:*

$$P_{ex}(x_k; \alpha, \beta, N) =$$
$$P\{\sup_{N \geq k \geq 0} ||x(k)|| > \beta; ||x(0)|| \leq \alpha\}$$

▲

Note that $P_{ex}(x_k; \alpha, \beta, N) = 1 - P_{in}(x_k; \alpha, \beta, N)$. Therefore in this context we define stochastic finite-time stability for the stochastic system (4.36) according to the following:

Definition 4.6 ***Finite Time Stochastic Stability***$(FTSS)$

The dynamical system (4.36) is finite-time stochastically stable (FTSS) with respect to $(\alpha, \beta, N, \lambda, ||.||)$

$$P_{in}(x_k; \alpha, \beta, N) \geq (1 - \lambda) \tag{4.18}$$

or equivalently

$$P_{ex}(x_k; \alpha, \beta, N) < \lambda \tag{4.19}$$

▲

We can also relate FTSS to inclusion and exit probabilities of a continuous smooth function $V_k \geq 0$ associated with the dynamical system such that $\delta_1 \parallel x_k \parallel^2 \leq V_k \leq \delta_1 \parallel x_k \parallel^2$, $\forall k = 0, \ldots, N$ and $\gamma = \delta_1 \beta$, $\gamma_0 = \delta_2 \alpha$. In particular the inclusion probability associated with $V(x_k, k)$ is defined as

$$P_{in}(V_k; \gamma_0, \gamma, N) =$$
$$P\{V(x_k, k) < \gamma : k \in [0, N]; V(x_0, 0) \leq \gamma_0\}$$

and consequently the associated exit probability will be defined as follows:

$$P_{ex}(V_k; \gamma_0, \gamma, N) =$$
$$P\{\sup_{N \geq k \geq 0} V(x_k, k) > \gamma; V(x_0, 0) \leq \gamma_0\}$$

We will show how the study of finite-time stochastic stability (FTSS) can be indirectly approached by studying the exit and inclusion probabilities associated with a function $V(x_k, k)$.

4.3.1 Bounds on Exit Probability

In order to analyze, and to eventually design, for the finite-time stability of a process, we provide in this section upper bounds on the exit probability of the process (4.17) and on the associated function V_k. These upper bounds will allow us to indirectly study the FTSS of the system.

Our first theorem is based on the following principle in the deterministic case described in [42]: consider the upper bound on the increments of V_k, $\Delta V_k = V_{k+1} - V_k \le \phi_k$ in $S_m = \{x_k : V(x_k, k) < m(k), m(k) > 0\}$, where ϕ_k is a non-negative constant. Then, a state trajectory stays in the set S_m for at least a time $N = \frac{(m-X(x_0))}{\phi_k}$. This can be seen by considering the condition $\Delta V_k \le \phi_k$, taking the summation on both sides and choosing $\phi = max_k \phi_k$

$$\sum_{k=0}^{N-1} V(x_{k+1}, k+1) - V(x_k, k) \le \sum_{k=0}^{N-1} \phi \tag{4.20}$$

which implies

$$V(x_N, N) \le V(x_0, 0) + N\phi \tag{4.21}$$

then the smallest value of the interval length N that will guarantee that the trajectory stays in Q_m is $N = \frac{(m-X(x_0))}{\phi}$. Following the above principle we present next a stochastic finite-time stability theorem that is a slight extension of the one in [43].

Theorem 4.3 *Consider a discrete-time Markov process x_k, $k = 0, 1, \ldots,$ and the continuous function $V(x_k, k) \ge 0$, and define the open set $S_\gamma = \{x_k : V(x_k, k) \le \gamma\}$. Let the first exit time for $V(x_k, k) = V_k$. If the following conditions are satisfied*

$$\begin{aligned} &\mathbb{E}_{x_k}[V(x_{k+1}, k+1)] \le \infty \qquad \forall x_k \in S_\gamma, \\ &\mathbb{E}_{x_k}[V(x_{k+1}, k+1) - V(x_k, k)] \le \phi_{k+1} \qquad \forall x_k \in S_\gamma,\ \phi_k \ge 0 \end{aligned} \tag{4.22}$$

Then for the initial condition $x(0) = x_0$ *we have*

$$P_{ex}(V(x_k, k); \gamma_0, \gamma, N) \leq \frac{[V(x_0, 0) + \Phi_N]}{\gamma} \tag{4.23}$$

where $\Phi_N = \sum_{i=1}^{N} \phi_i$

Proof. The proof of the above theorem can be found in [43].

■

The last theorem gives an upper bound for the exit probability of V_k. This upper bound depends on the initial conditions through V_0, on the desired bound through γ, and on the time interval and state dynamics indirectly through Φ_N. Next, we aim to directly bound the exit probability of the state dynamics of (4.17).

Theorem 4.4 *Consider the dynamical Markov process (4.17) and its exit probability with respect to* $(\alpha, \beta, N, ||.||)$

$$P_{ex}(x_k; \alpha, \beta, N) = P\{\sup_{N \geq k \geq 0} ||x_k|| > \beta; ||x_0|| \leq \alpha\}$$

we have the following upper bound on $P_{ex}(x_k; \alpha, \beta, N)$

$$P_{ex}(x_k; \alpha, \beta, N) \leq E\left[\frac{\sup_{N \geq k \geq 0} ||x_k||}{\beta}; ||x_0|| < \alpha\right]$$
$$\leq \mathbb{E}\left[\frac{\sup_{N \geq k \geq 0} V(x_k)}{\gamma}; ||V(x_0)|| < \gamma_0\right]$$

Proof. The proof easily follows from Chebychev inequality [31] $P\{|X - \mu_X| > \epsilon\} \leq \frac{\mathbb{E}[|X - \mu_X|]}{\epsilon}$. In the following, I is the indicator function such that $I = I_{\{\sup_{N \geq j \geq 0} ||x_j|| > \beta\}}$.

Then

$$P_{ex}(x_k;\alpha,\beta,N) = P\{\sup_{N\geq k\geq 0}||x_k|| > \beta; ||x_0|| \leq \alpha\}$$
$$= \mathbb{E}\left[I(\sup_{N\geq j\geq 0}||x_j||); ||x_0|| \leq \alpha\right]$$
$$\leq \mathbb{E}\left[I(\sup_{N\geq j\geq 0}||x_j||)\frac{\sup_{N\geq k\geq 0}||x_k||}{\beta}; ||x_0|| \leq \alpha\right]$$
$$\leq \mathbb{E}\left[\frac{\sup_{N\geq k\geq 0}||x_k||}{\beta}; ||x_0|| < \alpha\right]$$

■

Again the bound on $P_{ex}(x_k;\alpha,\beta,N)$ is directly related to the bounds on the state α, β, the state dynamics, and the time interval N.

4.3.2 Stochastic Finite-Time Stability Analysis

In the previous section we showed how the exit probability relative to the state dynamics x_k and to the associated function $V(x_k,k)$ may be bounded and how the bound depends on the parameters describing the finite-time stability objective. In this section we use the described bounds to provide sufficient conditions for the system (4.17) to be finite-time stochastically stable.

Theorem 4.5 *Consider the dynamical system (4.17) and a function $V(x_k,k)$ such that for a given δ_1 we have $\delta_1||x_k||^2 \leq V(x_k,k) \leq \delta_2||x_k||^2$, and $\gamma = \beta\delta_1$. Then the system is finite-time stochastically stable with respect to $(\alpha,\beta,N,||.||,\lambda)$, if any of the following three conditions is satisfied*

(i)

$$\mathbb{E}_{x_k}\left[V(x_{k+1},k+1) - V(x_k,k)\right] \leq \phi_{k+1},\ \forall x_k \in S_\gamma \tag{4.24}$$

$$\frac{[\alpha\delta_2 + \Phi_N]}{\beta\delta_1} \leq \lambda \tag{4.25}$$
$$\Phi_N = \sum_{k=1}^{N} \phi_k,\ \phi_k \geq 0$$

(ii)

$$\mathbb{E}\left[\frac{\sup_{N\geq k\geq 0} ||x_k||}{\beta}; ||x_0|| \leq \alpha\right] \leq \lambda \tag{4.26}$$

(iii)

$$P\{\Delta V_k \leq \rho_k V_k\} \geq (1-\lambda) \tag{4.27}$$

$$\frac{\beta}{\alpha} \geq \sup_k \prod_{i=0}^{k-1}(1+\rho_i) \tag{4.28}$$
$$\forall x_k \in S_\beta,\ \rho_k \geq -1,\ \forall k = 0, \ldots, N$$

Proof.

In order to prove the above statements we will explore $(i)-(iii)$ and verify that each of these conditions imply finite-time stability for the system. Finite-time stability easily follows from (i) considering that for $\delta_1||x_k||^2 \leq V(x_k,k) \leq \delta_2||x_k||^2$, $\forall k = 0, \ldots, N$ and $\gamma_0 = \delta_2\alpha$, $\gamma = \delta_1\beta$ we have

$$P_{ex}(x_k; \alpha, \beta, N) \leq P_{ex}(V_k; \gamma_0, \gamma, N) \tag{4.29}$$

and therefore from theorem 4.3 and (i)

$$P_{ex}(x_k; \alpha, \beta, N) \leq \lambda \tag{4.30}$$

Now recalling that $P_{ex}(x_k;\alpha,\beta,N) + P_{in}(x_k;\alpha,\beta,N) = 1$ we deduce the finite-time stability for the system (4.17) with respect to $(\alpha,\beta,N,||.||,\lambda)$ i.e.

$$P_{in}(x_k;\alpha,\beta,N) \geq (1-\lambda) \tag{4.31}$$

For point (ii), since from theorem 4.4 the first term in (4.25) is an upper bound on $P_{ex}(x_k;\alpha,\beta,N)$ with the same principle as before, we obtain immediately

$$P_{ex}(x_k;\alpha,\beta,N) \leq \lambda \tag{4.32}$$

and therefore

$$P_{in}(x_k;\alpha,\beta,N) \geq (1-\lambda) \tag{4.33}$$

Finally to prove (iii) let us consider the following for $\rho_k \geq -1$

$$\begin{aligned} P\{\Delta V_k \leq \rho_k V_k\} = P\{V_{k+1} - (1+\rho_k)V_k \leq 0\} \\ \forall k = 0,\ldots,N \end{aligned}$$

then iterating the partial difference inequalities and considering the upper bound on $V_0 \leq \gamma_0$ we obtain

$$\begin{aligned} P\{\Delta V_k \leq \rho_k V_k\} \leq P\{V_k \leq \gamma_0 \prod_{i=0}^{k-1}(1+\rho_i)\} \\ \forall k = 0,\ldots,N \end{aligned}$$

then using the condition (4.29) from (iii) it follows that

$$\begin{aligned} P\{\Delta V_k \leq \rho_k V_k\} \leq P\{V_k \leq \gamma\} \\ \forall k = 0,\ldots,N \end{aligned}$$

and moreover

$$(1-\lambda) \leq P\{\Delta V_k \leq \rho_k V_k\} \leq P\{V_k \leq \gamma\} \quad \forall k = 0, \ldots, N$$

that implies finite time stability with respect to $(\alpha, \beta, N, ||.||, \lambda)$.

■

4.3.3 Relations of FTS Conditions

In this section we compare the above results for FTS analysis and study how they may be related. First we study how the two upper bounds presented in section 4.3.1 are related. In particular let us consider (recall theorem 4.3) the following

$$P_{ex}(V_k; \gamma_0, \gamma, N) \leq \frac{[V_0 + \Phi_N]}{\gamma} \tag{4.34}$$

where $\Phi_N = \sum_{i=1}^{N} \phi_i$ and from theorem 4.1

$$P_{ex}(x_k; \alpha, \beta, N) \leq \mathbb{E}\left[\frac{\sup_{N \geq k \geq 0} ||x_k||}{\beta}; ||x_0|| < \alpha\right]$$
$$\leq \mathbb{E}\left[\frac{\sup_{N \geq k \geq 0} V(x_k, k)}{\gamma}; ||V(x_0, 0)|| < \gamma_0\right]$$

then using the fact that $\delta_1 ||x_k||^2 \leq V(x_k) \leq \delta_2 ||x_k||^2$ and $\gamma = \delta_1 \beta$ we have

$$P_{ex}(x_k; \alpha, \beta, N) \leq P_{ex}(V_k; \gamma_0, \gamma, N) \tag{4.35}$$

and moreover, by Chebychev inequality

$$P_{ex}(V_k; \gamma_0, \gamma, N) \leq \mathbb{E}\left[\frac{\sup_{N \geq k \geq 0} V_k}{\gamma}; ||V_0|| < \gamma_0\right]$$

from the last two inequalities we can conclude that $P_{ex}(V_k; \gamma_0, \gamma, N)$ is a less conservative bound on $P_{ex}(x_k; \alpha, \beta, N)$ than the one in (4.36). Finally, we compare the two bounds in (4.34) and (4.36). In particular we observe that in (4.34),the bound on P_e depends on the initial condition $V(0)$, the bound on V, γ, and on bounds on its increments ϕ_k. In (4.36) we are actually considering the expected value of supremum over all V_k in the studied interval. In principle the second bound on exit probability is less conservative and does not require the evaluation of the increment at each step, but on the other hand it is not easy to directly calculate the value of the supremum of V_k.

Now let us consider part (iii) of theorem 4.5 from which we have for $k = 0, \ldots, N$

$$
\begin{aligned}
& P\{\Delta V_k \leq \rho_k V_k\} \leq P\{V_k \leq \gamma_0 \prod_{i=0}^{k-1}(1+\rho_i)\} \\
= \;& 1 - P\{\sup_{N \geq k \geq 0} V_k \geq \gamma_0 \prod_{i=0}^{k-1}(1+\rho_i)\} \\
\leq \;& 1 - P\{\sup_{N \geq k \geq 0} V_k > \gamma\},\; k = 0, \ldots, N
\end{aligned}
$$

We observe how the last term, the inclusion probability, is the complement of the exit probability for V_k, and therefore by theorem 4.5 $P_{in} \geq (1 - \lambda)$.

Since the three parts of the theorem 4.5 are comparable, from now on we will just focus on the first part (i), since it is more general and does not directly require the knowledge of the states of the system.

4.3.4 Finite-Time Stochastic Stability Design

The previous theorems focused on analysis but may be extended to designing controllers that stochastically stabilize a system over a finite time. Consider the system

$$x_{k+1} = f(x_k, \theta_k) + g(x_k)u_k \tag{4.36}$$

Where $x \in \mathbb{R}^n$ is the system state, u_k is a one-dimensional control input, f and g are vector functions defined over $\mathbb{R}^n$ and θ_k is an independent stationary random sequence with mean μ_θ and variance σ_θ. In particular, we consider systems in which the random sequence θ_k appears linearly in the system i.e. $f(x_k, \theta_k) = f(x_k)\theta_k$. In order to simplify notation, we will use the following notation $g(x_k) = g_{x_k}$ and $f(x_k) = f_{x_k}$.

We aim to design a state-feedback control law $u_k = u(x_k)$, such that the closed-loop system is FTSS with respect to the parameters $(\alpha, \beta, N, ||.||, \lambda)$. The proposed design technique is based on part (i) of theorem 4.5. From now on, we also restrict our study to the choice of $V_k = x_k^T x_k$.

Theorem 4.6 *Let us consider the system given in (4.36). Consider the FTSS condition (4.24), and let us choose* $\Phi_N = \gamma\lambda - \gamma_0$ *and* $\phi_k = \frac{\gamma\lambda-\gamma_0}{N}$, $\forall k = 1, \ldots, N$. *Then, the system is stabilizable over a finite time with respect to* $(\alpha, \beta, N, ||.||, \lambda)$ *if for the function* $V(x_k) = x_k^T x_k$ *there exists an input law* $u(x_k)$ *such that*

$$\begin{aligned} & g_{x_k}^T g_{x_k} = (f_{x_k}^T g_{x_k} + g_{x_k}^T f_{x_k}) = 0 \\ \Rightarrow\ & (f_{x_k}^T f_{x_k} - x_k^T x_k) < \phi_k, \\ & \forall k = 1, \ldots, N, \forall x_k \in S_\gamma \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} \mathbb{E}\left[\Delta V(x_k)\right] &= ((\sigma_\theta f_{x_k}^T f_{x_k} - x_k^T x_k) + g_{x_k}^T g_{x_k} u_k^2 \\ &+ \mu_\theta (f_{x_k}^T g_{x_k} + g_{x_k}^T f_{x_k}) u_k) \\ &\le \phi_k; \quad \forall k = 1, \ldots, N \end{aligned} \tag{4.38}$$

$$\mu_\theta^2 (f_{x_k}^T g_{x_k} + g_{x_k}^T f_{x_k})^2 \ge 4 g_{x_k}^T g_{x_k} (\sigma_\theta f_{x_k}^T f_{x_k} - x_k^T x_k - \phi_k)$$

The set of possible control laws is given by

$$u_k : u_1 \leq u_k \leq u_2, \ for \ g^T_{x_k} g_{x_k} \neq 0, \ and \, (f^T_{x_k} g_{x_k} + g^T_{x_k} f_{x_k}) \neq 0;$$
$$u_k = 0, \ for \ g^T_{x_k} g_{x_k} = (g^T_{x_k} f_{x_k} + f^T_{x_k} g_{x_k}) = 0$$

Where

$$u_{1,2} = \frac{-\mu_\theta A_1 \pm \sqrt{\mu^2_\theta (A^2_1) - 4 g^T_{x_k} g_{x_k} B_1}}{2 g^T_{x_k} g_{x_k}}$$

in which $A_1 = (f^T_{x_k} g_{x_k} + g^T_{x_k} f_{x_k}), \ B_1 = (\sigma_\theta f^T_{x_k} f_{x_k} - x^T_k x_k - \phi_k),$

Proof.

Consider condition (4.38). Because of the choice of ϕ_k we have

$$\mathbb{E}_{x_k} [V_{k+1} - V_k] \leq \frac{\gamma\lambda - \gamma_0}{N}, \forall k = 1, \ldots, N \tag{4.39}$$

and also

$$\Phi_N = \sum_{k=0}^{N} \frac{\gamma\lambda - \gamma_0}{N} = \gamma\lambda - \gamma_0, \tag{4.40}$$

from theorem 4.3 the above conditions imply

$$P_{ex}(V_k; \gamma_0, \gamma, N) \leq \frac{[V_0 + \Phi_N]}{\gamma} \tag{4.41}$$

substituting the value of Φ_N and bounding V_0, we obtain

$$P_{ex}(V_k; \gamma_0, \gamma, N) \leq \frac{[\gamma_0 + \gamma\lambda - \gamma_0]}{\gamma} = \lambda \tag{4.42}$$

and therefore finite-time stability follows.

■

The proposed design technique guarantees closed-loop finite-time stability under the theorem's assumptions. However, we actually designed to meet the specified bound by fixing ϕ_k. This is a constraint that makes the above conditions on the existence of the controller only sufficient.

4.3.5 Minimization of the Exit Bound

In the previous section we designed a controller in order to meet specific given bounds on the inclusion probability P_{in} of the stochastic system (4.36). Here we proceed to develop design techniques to maximize the inclusion probability of the system. Instead of directly designing for the objective P_{in} as was done in [47], where they designed to maximize the inclusion probability, we base our design on the minimization of some upper bound on the objective P_{ex}.

Consider the following optimization problem

$$\max_u P_{in}(x_k;\alpha,\beta,N) = \max_u P\{||x(k)|| < \beta : k \in [0,N] | ||x(0)|| \leq \alpha\}$$

given the system (4.36). This objective can be also achieved by considering the equivalent problem

$$\min_u P_{ex}(x_k;\alpha,\beta,N) = \min_{u_k} P\{\sup_{0\leq k\leq N} ||x_k|| > \beta; ||x_0|| \leq \alpha\} \tag{4.43}$$

We can indirectly solve this problem by minimizing an upper bound on the exit probability i.e.

$$\min_{u_k} L(u_k) \tag{4.44}$$

where $L(u_k)$ is a cost function such that

$$P_{ex}(x_k;\alpha,\beta,N) \leq L(u_k) \qquad \forall k = 0,\ldots,N,\ x_k \in S_\beta \tag{4.45}$$

In section 4.3.1 we provided some bounds on $P_{ex}(V_k;\gamma_0,\gamma,N)$ and consequently on $P_{ex}(x_k;\alpha,\beta,N)$. Here we use those bounds in order to design for finite-time stability for the system (4.3.1) with respect to $(\alpha,\beta,N,||.||,\lambda)$, with λ as small as possible.

Theorem 4.7 *Consider the system (4.3.1), and a function $V(x_k) = x_k^T x_k$. Then there exists a control law $u_{opt}(x_k)$ that minimizes $P_{ex}(V_k;\gamma_0,\gamma,N)$, i.e. stabilizes the system over a finite time with respect to $(\alpha,\beta,N,||.||)$, if for $g_{x_k}^T g_{x_k} \neq 0$, u_k minimizes the cost function $L_1(u_k)$, i.e.*

$$L_1(u_{k,opt}) \leq L_1(u_k),\ \forall u,\ \forall k = 1,\ldots,N \tag{4.46}$$

where

$$\begin{aligned} L_1(x_k,u_k) \;&= \\ &(\sigma_\theta f_{x_k}^T f_{x_k} - x_k^T x_k + g_{x_k}^T g_{x_k} u_k^2 + \mu_\theta(A_1) u_k) \\ &\forall k = 0,\ldots,N, \forall x_k \in S_\beta \end{aligned} \tag{4.47}$$

Finally, the optimal control law is given by

$$u_k = \begin{cases} 0 & g_{x_k}^T g_{x_k} = 0 \\ -\frac{\mu_\theta(g_{x_k}^T f_{x_k} + f_{x_k}^T g_{x_k})}{(2 g_{x_k}^T g_{x_k})} & g_{x_k}^T g_{x_k} \neq 0 \end{cases} \tag{4.48}$$

for all $k = 1,\ldots,N$.

Moreover choosing $\phi_k = L_1(u_k) = \frac{\gamma\lambda - \gamma_0}{N}$ *the minimum value* λ *is given by*

$$\lambda_{opt} = \frac{[\gamma_0 + Nb_f]}{\gamma} \tag{4.49}$$

where

$$(\sigma_\theta f_{x_k}^T f_{x_k} - x_k^T x_k) \leq b_f \tag{4.50}$$

Proof.

The control law that minimizes λ can be found by considering once again the upper bound on the exit probability presented in (4.3). The following sufficient conditions are given for the existence of such upper bound

$$P_{ex}(x_k; \gamma_0, \gamma, N) \leq \frac{[V_0 + \Phi_N]}{\gamma} \leq \frac{[V_0 + \sum_{k=0}^{N} L_1(u_k)]}{\gamma}$$

$$\mathbb{E}_{x_k}[V_{k+1} - V_k] \leq \phi_{k+1}, \ \forall x_k \in S_\gamma, \ \phi_k \geq 0 \tag{4.51}$$

where $\Phi_N = \sum_{k=0}^{N} \phi_k$. Since our objective is to maximize the inclusion probability or, equivalently, to minimize the exit probability, we can minimize the upper bound on the exit probability since γ, γ_0, N are independent of the input u_k, we can act on ϕ_k. We can then meet this requirement from inequality (4.51) by minimizing each of the terms $\mathbb{E}_{x_k}[V_{k+1} - V_k]$ for $x_k \in S_\gamma$ or equivalently for $V_k = x_k^T x_k$

$$\begin{aligned} L_1(x_k, u_k) &= \mathbb{E}[(\theta_k^2 f_{x_k}^T f_{x_k} - x_k^T x_k + g_{x_k}^T g_{x_k} u_k^2 \\ &\quad + \theta_k (g_{x_k}^T f_{x_k} + f_{x_k}^T g_{x_k}) u_k)] \end{aligned}$$

that is an upper bound on $\mathbb{E}_{x_k}[\Delta(V(x_k))]$. Since γ, γ_0 and $\mathbb{E}[\theta_k^2] = \sigma_\theta$, $\mathbb{E}[\theta_k] = \mu_\theta$ are fixed positive values we have

$$\begin{aligned} L_1(x_k, u_k) &= [(\sigma_\theta f_{x_k}^T f_{x_k} - x_k^T x_k + g_{x_k}^T g_{x_k} u_k^2 \\ &\quad +\mu_\theta(g_{x_k}^T f_{x_k} + f_{x_k}^T g_{x_k}) u_k)] \\ &\quad \forall k = 0, \ldots, N \end{aligned} \tag{4.52}$$

We then obtain u_k that minimizes $L_1(x_k, u_k)$ as follows

$$\frac{\partial}{\partial u_k} L_1(x_k, u_k) = 0 \tag{4.53}$$

solving for u we obtain the control law (4.48).

■

4.4 Finite-time stability Design: Example

In this section we present an example to show how our design techniques may be applied to a given nonlinear system.

Example 1 *Consider the system*

$$x_{k+1} = .5e^{(x_k)}\theta_k + sin(2\pi\frac{x_k}{5} - 7)u_k$$

where $\theta_k \in \{0, 1\}$, $\mu_\theta = 0.5$. *We want to choose the input* u_k *in such a way that the closed-loop system is finite-time stable with respect to* ($\alpha = 0.5, \beta = 1, N = 10, ||.||, \lambda = .3$). *We also want to minimize a bound on the exit probability* P_{ex}, *i.e. we want to minimize the value of* λ.

By applying theorem 4.6 with $\delta_1 = 1$, $\delta_2 = 1$ *and therefore* $\phi_k = -0.02$ *and choosing in the admissible range of controller* $u_k = -1.3$, *for* $sin(2\pi\frac{x_k}{5} - 7) \neq 0$, *and* $u_k = 0$, *for* $sin(2\pi\frac{x_k}{5} - 7) = 0$, *we obtain the closed-loop system*

$$x_{k+1} = 0.5e^{(x_k)}\theta_k + sin(2\pi\frac{x_k}{5} - 7)(-1.3sign(|sin(2\pi\frac{x_k}{5} - 7)|).$$

The closed-loop system is FTSS as in Figure (4.3).

Also applying the input u_{opt} *that minimizes* λ *we obtain the closed-loop dynamics,*

$$\begin{aligned} x_{k+1} &= 0.5e^{(x_k)}\theta_k + sin(2\pi\frac{x_k}{5} - 7))u_{opt}(k) \\ u_{opt}(k) &= sign(|(sin(2\pi\frac{x_k}{5} - 7))|)\frac{-.5e^{(x_k)}sin(2\pi\frac{x_k}{5} - 7)}{2(sin(2\pi\frac{x_k}{5} - 7))^2} \end{aligned}$$

In Figure (4.3) we compare the closed-loop system with the first controller u_k *designed for FTSS with respect to* $(\alpha = 0.5, \beta = 1, N = 10, ||.||, \lambda = 0.3)$, *with the open-loop controller and finally the closed-loop system with the second controller* u_{opt}.

4.5 Conclusions

In this chapter we presented some new results on finite-time stability for stochastic discrete-time nonlinear systems. Moreover, we explored how finite-time stability analysis techniques can be extended to control design.

After discussing deterministic FTS, presenting existing work and a new approach to analysis, we considered a stochastic system and explored how finite-time stability can be studied. In particular, we described the concept of "inclusion probability" and "exit probability" which are crucial for the study of stochastic FTS. Also we showed how these quantities can be bounded by bounds that depend on the required

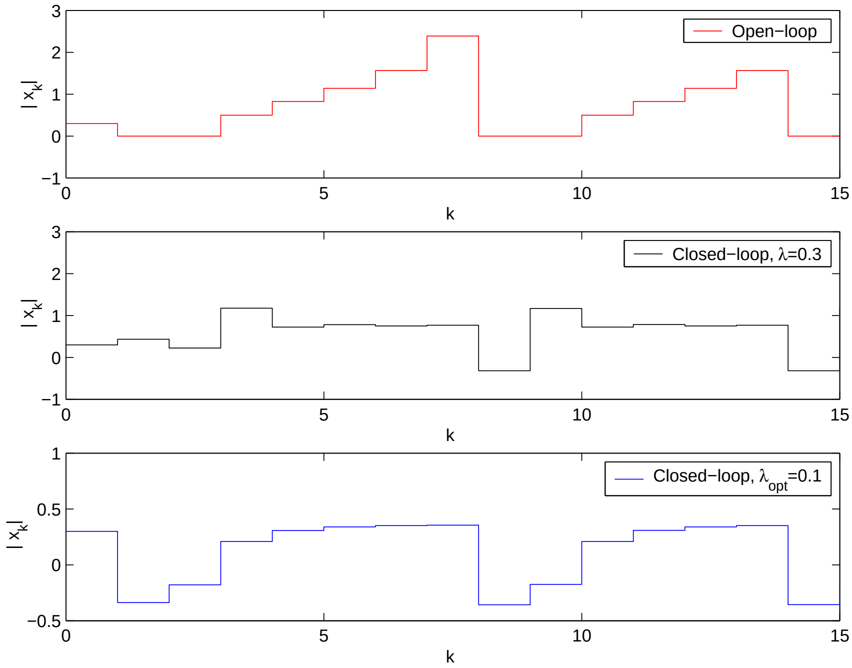

Figure 4.3: Open loop system versus closed loop systems with exit probability $P_{ex} = 0.3$ and minimal exit probability.

finite-time stability parameters. Moreover those bounds are used to analyze FTSS, and to design for closed-loop FTSS.

Also in the last section we described how an upper bound on the exit bound can be minimized, that is design for minimizing the probability of exceeding a bound over a finite time.

Chapter 5

Nonlinear Model Based NCS: Finite-Time Stochastic Stability

5.1 Introduction

In this chapter we study the finite-time stability a system controlled through a network, and therefore subject to the potential loss of information. To do so we use the tools described in chapter 4 to study FTS for a nonlinear, discrete-time dynamical system. In the current chapter, we use a stochastic model of the dropped packets, and therefore apply stochastic results in FTS. The packets loss is modelled as a binary independent random process with a known mean. While investigating how the loss of information affects the FTSS of a NCS, we observe first that the FTSS of the system depends on three main factors:

I. The controller performance i.e. whether or not the controller stabilizes the plant in the case of full information.

II. The accuracy of the model and its initial state with respect to the plant.

III. The amount of information received, or conversely the amount of packets dropped.

We address the first factor by assuming from now on the FTSS of the system when full information is available, and focus mainly on FTSS performance with respect to factors II and III. We first observe that they are complementary, i.e. that they can compensate for each other, and in fact if one is missing the other factor becomes more critical. It is well known that the state of a discrete-time system is uniquely determined by the dynamical equations and the initial conditions. Therefore, the accuracy of the model represents an important aspect, since the design is based on the model's state. In addition, the state of the model plays an important role since it contributes to the estimation of the plant's state when packets are dropped. From now on, we assume the initial state of the model equal to the initial state of the plant, i.e. that a packet is initially received. The amount of information needed becomes more crucial when the model is inaccurate. Let us consider the two extreme cases: assume first that we have a non-perfect model i.e., one that does not reproduce the plant's behavior correctly. After few time steps, the controller will not perform satisfactorily on the plant if packets are dropped. In this case, even with the correct model initial conditions, the system may eventually become unstable. It is necessary in this case to have frequent updates of the plant state. Moreover, by updating the state of the model, the divergence between the model and the plant's state will be limited. Eventually, in this extreme case, if there are no dropped packets, and therefore the plant's state is always available for feedback, the model becomes unnecessary and even when completely incorrect, will not affect the stability of the closed-loop system. On the other hand, if some of the packets are dropped, the accuracy of the model and its state become important, since the model will have to compensate for this lack of information. In the extreme case where the model represents the plant perfectly, then the plant's state becomes unnecessary and the system can be stabilized even when all packets are dropped.

Using the stochastic model of the NCS described in chapters 2 and 3 we first investigate conditions to guarantee FTSS of the system. We then proceed to design controllers in order to achieve FTSS. Moreover, we characterize the FTSS of the system in terms of the probability of dropping packets p, which relates to minimum attention control [53].

5.2 Some Preliminaries and Notation

Consider from chapter 2 a model-based NCS subject to the random loss of packets as described in chapter 3

$$z_{k+1} = H_1(z_k) + H_2(z_k)\varphi_k, \quad z_k \in \mathbb{R}^{2n},\ k = 0, 1, \ldots \tag{5.1}$$

in which the dropping sequence $\varphi_k = (1 - \theta_k)$ is a stationary independent random sequence, with mean $\mu_\varphi = (1 - p) = q$ and $\mu_{\varphi^2} = q$, where q is the probability of dropping a packet.

In chapter (4) we defined finite time stochastic stability for a generic discrete-time dynamical system with respect to $(\alpha, \beta, N, \lambda, ||.||)$. Here we reformulate the FTSS definition for NCS (6.1). In the following we let $||z_k|| = \sqrt{z_k^T z_k}$ be the Euclidian norm.

Definition 5.1 ***FTSS-NCS***

The NCS (5.1) is FTSS with respect to $(\alpha, \beta, N, \lambda, ||.||)$ *if*

$$P_{in}(z_k; \alpha, \beta, N) = P\{z_k^T z_k < \beta : k \in [0, N] \mid z_0^T z_0 \leq \alpha\} \geq (1 - \lambda) \tag{5.2}$$

▲

Let $V(z_k, k) = z_k^T M(k) z_k$ be a quadratic function where $M(k)$, is a given $2n \times 2n$ time-varying real-valued matrix, with

$$M(k) = \left[\begin{array}{c|c} m_1(k) & m_2(k) \\ \hline m_3(k) & m_4(k) \end{array}\right], m_i(k) \in \mathbb{R}^{n \times n}, \tag{5.3}$$
$$m_2(k)^T = m_3(k),\ M(k) > 0$$

then consider the following definition

Definition 5.2 ***Quadratically FTSS-NCS***

The NCS (6.1) is quadratically FTSS with respect to $(\alpha, \beta, N, \lambda, M)$ if for the quadratic function $V(z_k, k) = z_k^T M(k) z_k$ the following holds

$$P_{in}(V_k; \gamma_0, \gamma, N) =$$
$$P\{z_k^T M(k) z_k < \gamma : k \in [0, N] \mid z_0^T M(k) z_0 \leq \gamma_0\} \geq (1 - \lambda) \tag{5.4}$$

where $\delta_1 ||z_k||^2 \leq V(z_k, k) \leq \delta_2 ||z_k||^2$, $\delta_1(k) = \lambda_{min}\{M(k)\}$, $\delta_2(k) = \lambda_{max}\{M(k)\}$ are the minimum and maximum eigenvalue of $M(k)$ respectively. In addition we have $\delta_2(k)\alpha \geq \gamma_0$ and $\delta_1(k)\beta \geq \gamma$.

▲

We denote the sets of states with bounded V as follows

$$S_\gamma = \{z_k : V_z(z_k, k) \leq \gamma\} \tag{5.5}$$
$$S_\beta = \{x_k : V_x(x_k, k) \leq \beta\} \tag{5.6}$$

5.3 Finite-Time Stochastic Stability Analysis

We aim to study the behavior of the system over a finite time in the presence of packet dropping. In particular, assuming that with full information available, the system's state is constrained within a bound β over a finite time N, we want to find conditions for which the state remains within the given bound over the time interval when packets are being dropped. Moreover, we want these conditions to depend on the model's state and on the amount of packets dropped.

We are now ready to state the following theorem that considering a class C_{B-NCS} NCS, gives sufficient conditions on the bounds defined on the NCS for which FTSS holds.

Theorem 5.1 *Consider the NCS (6.1), and assume it belongs to class C_{B-NCS}, also consider the function $V_z(z_k, k) = z_k^T M(k) z_k$, in which $M(k)$, is a real-valued $2n \times 2n$ matrix, where $m_1(k) > 0$, $m_4(k) > 0$. Assume that $\forall z_k \in S_\gamma$ and $k \in [0, N]$*

$$B_{H_1}(\hat{x}_k) + 2B_{H_{1,2}}(\hat{x}_k)q + B_{H_2}(\hat{x}_k)q - \lambda_{min}\{M\}B_z(\hat{x}) \leq \phi_{k+1} \tag{5.7}$$

$$\frac{\alpha\delta_2 + \Phi_N}{\beta\delta_1} \leq \lambda \tag{5.8}$$

where $\Phi_N = \sum_{k=1}^{N} \phi_k$. Then the system is FTSS with respect to $(\alpha, \beta, N, M(k), \lambda)$.

Proof. The proof follows from theorem 4.3 in chapter 4, and using lemma 2.1 in chapter 2. The conditions in the theorem

$$\mathbb{E}_{z_k}[\Delta V_z(z_k, k)] = \mathbb{E}_{z_k}[(H_1(z_k) + H_2(z_k)\varphi_k)^T M(k+1)(H_1(z_k) + H_2(z_k)\varphi_k) - \lambda_{min}\{M\}B_z(\hat{x})] \leq \phi_k,\ \forall k = 0, \ldots, N,\ z_k \in S_\gamma \tag{5.9}$$

and

$$\frac{\alpha\delta_2 + \Phi_N}{\beta\delta_1} \leq \lambda \tag{5.10}$$

from which FTSS follows.

■

Roughly speaking, the theorem restates the conditions for FTSS described in theorem 4.5, in a NCS context. Moreover, in order to make the analysis dependent only on the model's state that is assumed to be always available, it uses the fact that the NCS belongs to class C_{B-NCS}. Finally those bounds are used to specify FTSS conditions.

5.4 Finite-Time Stochastic Stability Design

In the previous chapter we presented sufficient conditions for FTSS of the NCS in the presence of packet dropping. We now investigate the possibility of designing a controller to guarantee the FTSS of the system. We therefore consider a network model in which the input function $u_k = K(\hat{x}_k)$ is not fixed i.e.

$$z_{k+1} = (F_1(z_k) + F_2(z_k)\varphi_k) + (G_1(z_k) + G_2(z_k)\varphi_k)u_k, \qquad k \geq 0 \tag{5.11}$$

Where the functions F_1, F_2, G_1, G_2 were defined in chapter 2 and $u_k : \mathbb{R}^n \rightarrow \mathbb{R}$ is a scalar input. Although we will only focus on the case of scalar inputs, the results may be easily extended to multidimensional inputs.

Theorem 5.2 *The class C_{B-NCS} NCS (5.11), is quadratically finite-time stochastically stabilizable with respect to $(\alpha, \beta, M, N, \lambda)$ and $\phi_k = \phi = \frac{\gamma\lambda - \gamma_0}{N}$ if for the function*

$V(z_k,k) = z_k^T M(k) z_k$, where $M(k)$ satisfies the conditions, (5.3), there exists an input law $u_k = K(\hat{x}_k)$ such that

1. *The system is FTSS with respect to $(\alpha, \beta, M, N, \lambda)$ for the time in which the input cannot affect it, i.e. if*

$$\begin{aligned}
&\mathbb{E}_{z_k}[(G_1(z_k) + G_2(z_k)q)^T M(k+1)(G_1(z_k) + G_2(z_k)q)] = 0, && (5.12)\\
&\mathbb{E}_{z_k}[(F_1(z_k) + F_2(z_k)q)^T M(k+1)(G_1(z_k) + G_2(z_k)q)] = 0\\
&\Rightarrow \mathbb{E}_{z_k}[(F_1(z_k) + F_2(z_k)q)^T M(k+1)(F_1(z_k) + F_2(z_k)q) - z_k^T M(k) z_k] \leq \phi_k
\end{aligned}$$

2. *We have for all $\hat{x}_k \in S_\beta$*

$$\begin{aligned}
&\mathbb{E}_{z_k}[\Delta V_{z_k}(z_k,k)] = (B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)q)u_k^2 && (5.13)\\
&2((B_{F1G2}(\hat{x}_k) + B_{F2G1}(\hat{x}_k) + B_{F1G2}(\hat{x}_k))q + B_{F1G1}(\hat{x}_k))u_k\\
&+(B_{F2}(\hat{x}_k))q + 2(B_{F1F2}(\hat{x}_k)q) + B_{F1}(\hat{x}_k) - \lambda_{min}\{M\}B_z(\hat{x}) \leq \frac{\gamma\lambda - \gamma_0}{N}
\end{aligned}$$

The set of controllers is given by:

$$\begin{aligned}
&u_1(\hat{x}_k) \leq u(\hat{x}_k) \leq u_2(\hat{x}_k), \quad for\ (B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)) \neq 0 && (5.14)\\
&u = 0, \qquad for\ (B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)) = 0 && (5.15)
\end{aligned}$$

$$\begin{aligned}
&B_{FG} = ((B_{F1G2}(\hat{x}_k) + B_{F2G1}(\hat{x}_k) + B_{F1G2}(\hat{x}_k))q + B_{F1G1}(\hat{x}_k))\\
&B_F = B_{F2}(\hat{x}_k))q + 2(B_{F1F2}(\hat{x}_k)q) + B_{F1}(\hat{x}_k) && (5.16)
\end{aligned}$$

$$\begin{aligned}
u_{1,2} = &\frac{-|B_{FG}|}{(B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)q)}\\
&\frac{\pm\sqrt{(B_{FG})^2 - (B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)q)(B_F - \lambda_{min}\{M\}B_z(\hat{x}) - \frac{\gamma\lambda-\gamma_0}{N}))}}{(B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)q)}
\end{aligned}$$

with

$$
\begin{aligned}
&(B_{FG})^2 - (B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)q)(B_F - \hat{x}_k^T m_4(k)\hat{x}_k - \frac{\gamma\lambda - \gamma_0}{N})) \geq 0 \\
&(B_{G1}(\hat{x}_k) + B_{G2}(\hat{x}_k)q) \neq 0
\end{aligned} \tag{5.17}
$$

Proof.

The proof follows from theorem 4.4 in chapter 4. In particular the control law with the conditions above imply

$$
\begin{aligned}
&\mathbb{E}_{z_k}[\Delta V(z_k, k)] \leq \phi_k \\
&\frac{\alpha\delta_2 + \Phi_N}{\beta\delta_1} \leq \lambda,
\end{aligned} \tag{5.18}
$$

$$\forall k = 0, \ldots, N, \; z_k \in S_\gamma$$

and therefore FTSS follows.

■

The theorem uses the FTSS conditions for the NCS in theorem 5.3, to generate a control law that will satisfy those conditions, and therefore will stochastically stabilize the NCS in a finite time with respect to the specified conditions.

5.5 FT Stability and Rate-Limit of Packet Dropping

In [53] the idea of relating the stability of a linear system controlled by a network to the rate of information (measured in packet per seconds) necessary to achieve stability is proposed. In particular the concept of “minimum attention control” was introduced, where the objective is to minimize the amount of bandwidth used, i.e. the amount of information transmitted through the channel in order to stabilize the system. The concept of minimum attention control is based on the following idea: if the system is stable (either in the classical or FT sense), then no controller nor feedback information are needed. Conversely if the system is unstable, then the need for feedback information increases. A relation between the rate of information and the eigenvalues associated with the unstable modes of a linear system is then obtained. We can then conclude that qualitatively, the amount of information needed is inversely related to the degree of stability of the system.

In the theory developed so far, we showed how the stability of the NCS is dependent on three main factors: the accuracy of the model and initial conditions, the amount of information available,(the number of packets dropped), and finally to the efficiency of the controller in stabilizing the plant when complete information is available.

This section focuses on the robust stability of the system with respect of the loss of packets, i.e. how the controller performs when the plant state is not received. We want to directly relate the stability of the system to the probability of packet drops. Let us consider once more the model of the networked control system in the form

$$z_{k+1} = H_1(z_k) + H_2(z_k)\varphi_k,\ k \geq 0 \tag{5.19}$$

in which $H_1(z_k)$ and $H_2(z_k)$ are dependent on the model, the system, and the error between the model and the system as described previously. We investigate the rate of lost packets that the system can support while remaining FTSS. To do so, we consider the sufficient conditions for FTSS of the system (5.19), by applying theorem 5.3, and investigate the level of probability of packet lost $P\{\varphi_k = 1\} = q$ for which stability is guaranteed. This results in the following.

Theorem 5.3 *A class C_{B-NCS} NCS (5.19) is FTSS through the network with respect to $(\alpha, \beta, M, N, \lambda)$ and $\phi_k = \frac{\gamma\lambda - \gamma_0}{N}$ if the rate of packets lost, i.e. probability $\mu_\varphi = q$ of dropping a packet, is such that*

$$0 \leq q \leq (2B_{H_{1,2}}(\hat{x}_k) + B_{H2}(\hat{x}_k))^{-1}(\lambda_{min}\{M\}B_z(\hat{x}) + \phi_k - B_{H1}(\hat{x}_k))$$
$$\forall \hat{x}_k \in S_\beta,\ \hat{x}_k \in S_\beta,\ k = [0, N] \tag{5.20}$$

■

Note that the amount of packets the system can afford to lose while remaining FTSS, is inversely dependent on the bounds of the NCS, i.e. the errors introduced by the model and the initial conditions. Therefore as the errors become larger, in order to maintain FTSS, the probability q of dropping a packet needs to be smaller. Moreover the bound on q directly depends on ϕ, therefore small values of γ and λ will lead to a small bound on the dropping probability q. Note that because of the sufficiency of the conditions, the bound on q may be conservative. While in the linear case the required rate depends on the unstable eigenvalues of the system, in the nonlinear finite-time stability setting, the relation is given by the Lyapunov-like function and depends on the accuracy of the model and the finite-time stability parameters through ϕ_k.

The above result shows how it is possible to "design" for FTSS of the NCS not only through the feedback controller, but also by modifying the networks dynamics when they are accessible. Moreover, in case more information is available about the network, the packet dropping may eventually be modelled deterministically, and therefore, controlling the network's traffic will help in controlling the NCS. This will become important in the next chapter where we will be dealing with deterministic dropout in NCS and where we link the network's dynamics to the control dynamics.

5.6 Examples

This section provides a set of examples of NCS, for which we study FTSS for different amounts of dropped packets. Though we only analyzed the case of scalar inputs, the results presented can be easily extended to the vector input case. In this section we present vector inputs examples. In particular, we consider

$$\begin{aligned} x_1(k+1) &= x_1(k) + u_1(k) \\ x_2(k+1) &= x_2(k) + u_2(k) \\ x_3(k+1) &= x_3(k) + (x_1(k)u_2(k) - x_2(k)u_1(k)) \end{aligned} \tag{5.21}$$

which is the discrete-time version of the non-holonomic integrator proposed by Brockett in [54].

5.6.1 Family of Controllers for FTS of Discrete-Time Brockett Integrator

The continuous-time non-holonomic integrator represents a challenging system studied by many authors. It was proven by Brockett that the non-holonomic integrator is not smoothly stabilizable with time-invariant controller. In discrete-time, the system is less challenging but remains interesting, especially in the new context of FTSS through the network. We aim to achieve FTSS with respect to $(\alpha = .09, \beta = 1.5^2, N = 10, M = I_{3\times3}, \lambda = .3)$, by using the function $V(x_k, k) = x_k^T x_k$. In particular we propose two families of controller u_a and u_b for the case $\frac{\lambda\gamma - \gamma_0}{N} > 0$.

Theorem 5.4 *Consider the discrete-time non-holonomic integrator (5.21), then the following classes of controllers finite-time stabilize the system with respect to the parameters* (α, β, N, M), *and therefore finite-time stochastically stabilize the system with respect to the parameters* $(\alpha, \beta, N, M, \lambda = 1)$, *with* $\frac{\lambda\gamma - \gamma_0}{N} > 0$.

Class a

$$u_a(k) = -\begin{bmatrix} a_1(k) & 0 \\ 0 & a_2(k) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{5.22}$$

where $a_1(k)$, $a_2(k)$ are positive functions defined on $\mathbb{N}^+$, and $a_1(k) < a_2(k)$ for $1 < a_1(k), 1 < a_2(k)$.

Class b

$$u_b(k) = -\begin{bmatrix} 0 & b_1(k)x_1^2(k) \\ b_2(k)x_2^2(k) & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{5.23}$$

In which , $b_1(k)$, $b_2(k)$ are positive functions defined on $\mathbb{N}^+$, and $b_1(k) < b_2(k)$ and $1 < a_1(k), 1 < a_2(k)$ such that

$$\begin{aligned} &(x_1(k) - b_1(k)x_1^2(k)x_2(k))^2 \leq x_1^2(k) \forall k = 0, \ldots, N,\ x_1, x_2 \in S_\gamma \\ &(x_2(k) - b_2(k)x_2^2(k)x_1(k))^2 \leq x_2^2(k) \end{aligned} \tag{5.24}$$

Proof.

Consider the FTS condition

$$\begin{aligned} &\Delta V(x_k, k) = \\ &(x_1(k) + u_1(k))^2 + (x_2(k) + u_2(k))^2 + (x_3(k) + x_1(k)u_2(k) \\ &-x_2(k)u_1(k))^2 - x_1^2(k) - x_2^2(k) - x_3^2(k) \leq \frac{\lambda\gamma - \gamma_0}{N} \end{aligned} \tag{5.25}$$

	class a	class b
constant	$a_1(k) = a_2(k) = c$	$b_1(k) = b_2(k) = c$
linear	$a_1(k) = a_2(k) = k$	$b_1(k) = b_2(k) = k$
exponential	$a_1(k) = a_2(k) = e^{-k}$	$b_1(k) = b_2(k) = e^{-k}$

Table 5.1: Classes of controller for the discrete-time Brocket integrator.

using the controller of class (a) we have

$$\begin{aligned}
&\Delta V(x_k, k) = \\
&(x_1(k) - a_1(k)x_1(k))^2 + (x_2(k) - a_2(k)x_2(k))^2 + \\
&(x_3(k) + x_1(k)(-a_2(k)x_2(k)) - x_2(k)(-a_1(k)x_1(k)))^2 \\
&-x_1^2(k) - x_2^2(k) - x_3^2(k) \leq 0 \leq \frac{\lambda\gamma - \gamma_0}{N}, \\
&\forall(\gamma, \gamma_0, N, I_{3\times 3}, \lambda), \frac{\lambda\gamma - \gamma_0}{N} > 0, \, and \, a_1 > 1, \, a_2 > 0
\end{aligned} \tag{5.26}$$

and with class (b)

$$\begin{aligned}
&\Delta V(x_k, k) = \\
&(x_1(k) - b_1(k)x_1^2(k)x_2(k))^2 + (x_2(k) - b_2(k)x_2^2(k)x_1(k))^2 + \\
&(x_3(k) + x_1(k)(-b_2(k)x_2^2x_1(k)) - x_2(k)(-b_1(k)x_1^2(k)x_2(k)))^2 \\
&-x_1^2(k) - x_2^2(k) - x_3^2(k) \leq \frac{\lambda\gamma - \gamma_0}{N}
\end{aligned} \tag{5.27}$$

■

In Table 5.1 we propose different controllers with the defined structures, and study their performance. Theorem 5.4 provides control laws that stochastically finite-time stabilize the discrete-time Brockett integrator, in case of available state feedback, that is in case the state is sent across a network, with no packet dropping. In order to analyze how the controller performs as some packets are dropped, we consider, in the next section, the NCS composed by the Brockett integrator (5.21) and its approximate model.

Control of Brockett Integrator trough a Network by Class (a) Constant Controller

Consider again the discrete version of the Brockett integrator (non-holonomic integrator) (5.21) and its approximate model

$$\begin{aligned} \hat{x}_1(k+1) &= 10\hat{x}_1(k) + 3u_1(k) \\ \hat{x}_2(k+1) &= 50\hat{x}_2(k) + 7u_2(k) \\ \hat{x}_3(k+1) &= 50\hat{x}_3(k) - 8(\hat{x}_1(k)u_2(k) + 7\hat{x}_2(k)u_1(k)) \end{aligned} \tag{5.28}$$

We want to use the class (a) linear controller to FT control the Brockett integrator trough the network.

$$u(k) = -\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} \tag{5.29}$$

where $a_1 = 1.7$, $a_2 = 2.3$. At first we check using the multi-input version of theorem 5.3 that the proposed controller FT stabilizes the system trough the network. We obtain with $M = I_{3\times 3}$

$$\mathbb{E}_{z_k} \Delta V(z_k, k) \leq \frac{\lambda\gamma - \gamma_0}{N} = 0.0585 \tag{5.30}$$

where z_k is given as in (2.3) with θ_k being an independent random sequence. In Figures 5.1, 5.2 and 5.3 we show simulations of the system controlled across a network, using a linear class (a) controller in which $a_1 = 1.7$, $a_2 = 2.3$, and with packets loss of $0\%, 20\%, 50\%$ respectively. Note how in the case of a class (a) controller with full

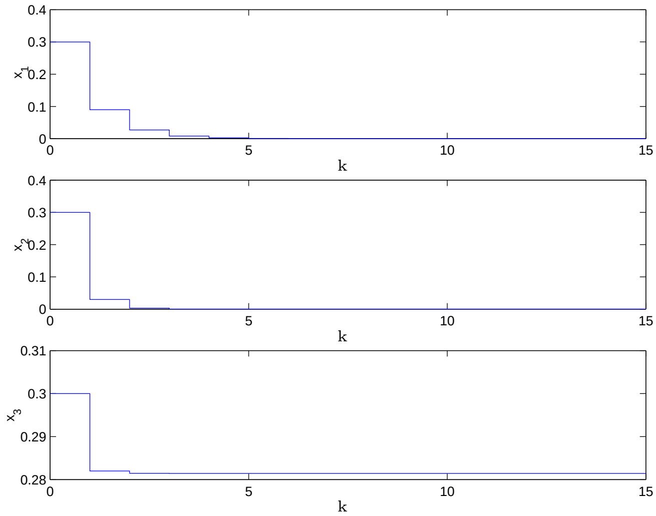

Figure 5.1: Brockett integrator controlled through the network with linear class (a) controller with $a_1 = 1.7$, $a_2 = 2.3$ and packet lost rate of 0%.

information available, finite-time stability is guaranteed for every set of parameters $(\alpha, \beta, N, M, \lambda)$ since the system is contracting. This property is however lost when the network starts dropping packets. Next we repeat the experiment using the same family of linear controllers (a) with different parameters $a_1 = .7$, $a_2 = .9$. Now due to the fact that $a_1 < 1$, $a_2 < 1$ the FTSS of the system is not absolute but depends on the parameters. In particular we are interested in FTSS with respect to $(\alpha = 0.09, \beta = 1.5^2, N = 10, M = I_{3\times 3}, \lambda = 0.3)$. In Figures 5.4, 5.5 and 5.6 we observe how the second mode goes unstable when the network starts losing packets.

5.7 Conclusions

Finite-time stochastic stability of model based NCS has been studied. In particular sufficient conditions for FTSS of the NCS were given. We showed how the FTSS of

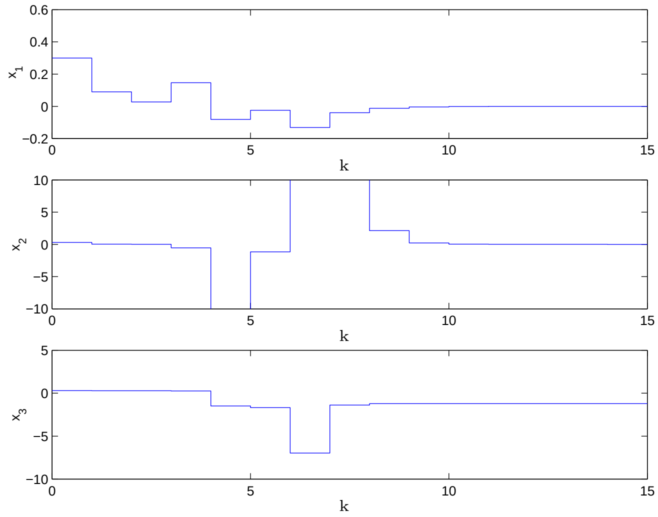

Figure 5.2: Brockett integrator controlled through the network with linear class (a) controller with $a_1 = 1.7$, $a_2 = 2.3$ and packet lost rate of 20%.

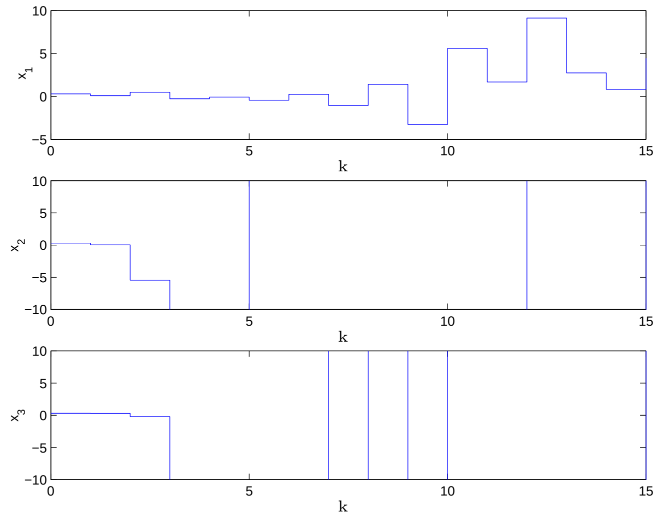

Figure 5.3: Brockett integrator controlled through the network with linear class (a) controller with $a_1 = 1.7$, $a_2 = 2.3$ and packet lost rate of 50%.

the system depends on three main factors: the stability of the closed-loop system in the case of available full information, the received information (packets transmitted),

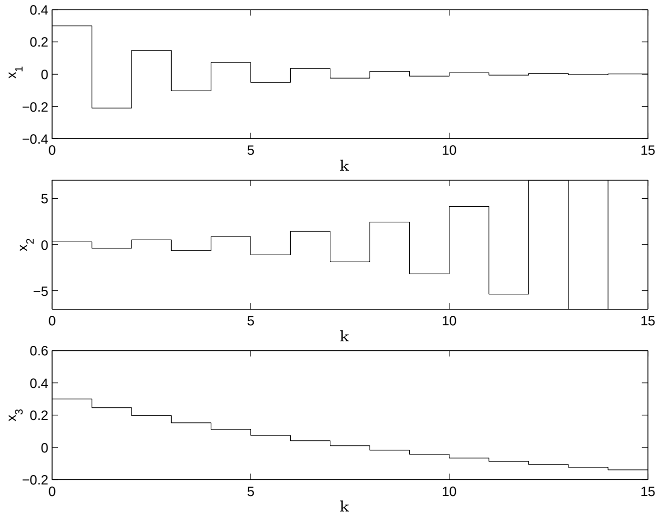

Figure 5.4: Brockett integrator controlled through the network with linear class (a) controller with $a_1 = .7$, $a_2 = .9$ and packet lost rate of 0%.

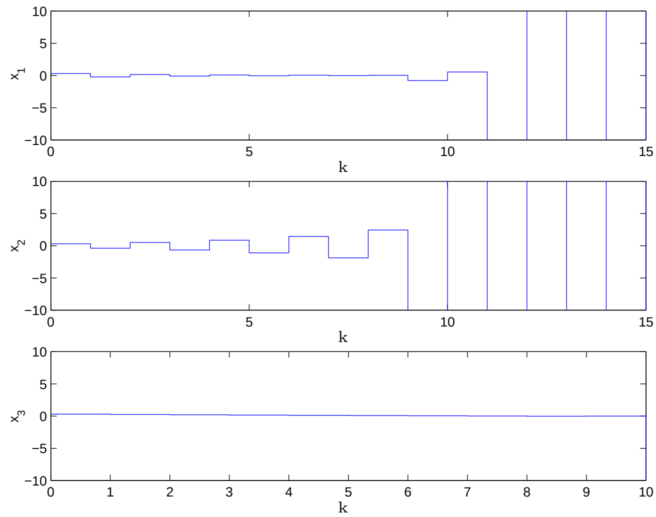

Figure 5.5: Brockett integrator controlled through the network with linear class (a) controller with $a_1 = .7$, $a_2 = .9$ and packet lost rate of 20%.

and the accuracy of the model and the initial conditions.

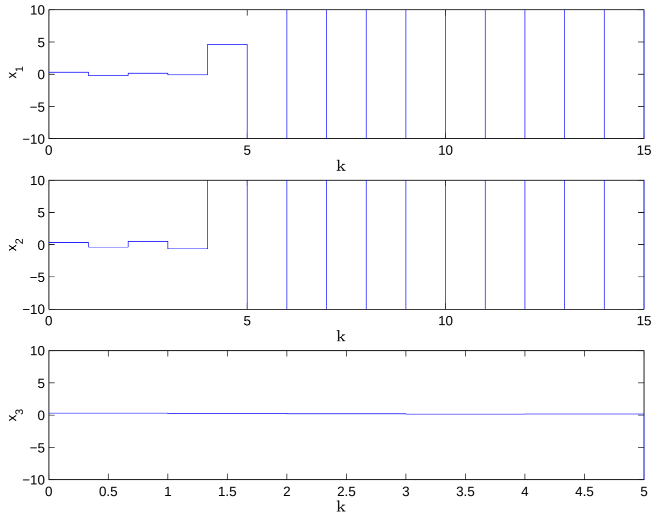

Figure 5.6: Brockett integrator controlled through the network with linear class (a) controller with $a_1 = 0.7$, $a_2 = 0.9$ and packet lost rate of 50%.

We also investigated the possibility of designing for FTSS, and a set of admissible controllers were proposed for a specific system. In particular we presented a class of controllers that only depends on the model state. Since the conditions used for design are only sufficient, the set of controllers might be conservative. Finally we characterized the FTSS of the system in terms of the amount of dropped packets.

Chapter 6

Nonlinear Model Based NCS: Deterministic Finite-Time Stability

6.1 Introduction

In chapter 5, we studied the effect of random packet dropping, on the FT stability of a nonlinear discrete-time system. The resulting NCS was a stochastic system, which allowed us to study its finite-time stochastic stability (FTSS).

In the present chapter, we consider packet dropping as a deterministic event, and make use of the possibility of knowing when a packet drop may occur. To accomplish this, the network dynamics must be incorporated in the control loop as was done in chapter 3, providing a more accurate model of packet dropping. The model asserts that the dropping of packets containing sensor measurements, is due to network congestion.

As mentioned earlier, congestion is caused by sources that transmit at an aggregate rate exceeding the channel capacity. It may therefore be possible to control packet dropping by appropriately managing the sources' rates in specific paths. Since, the stability of NCS depends on the controller performance, model accuracy, and the amount of packets dropped, we can design for FTS, not only through direct state feedback, but also by implementing a network controller to reduce packets dropping by controlling path capacities and sources' rates.

Therefore, the model studied in this chapter is extended to include network dynamics, such as link's capacities and sources rates. In this new deterministic setting we redefine FTDS for NCS and moreover, supply a sufficient condition for NCS to be FTDS.

In Section 6.2 we present a brief discussion of two different fields, namely, Congestion control, as well as NCS. Section 6.3 presents some notations about the system under investigation, and redefines FTS in this new context of deterministic packets dropping in NCS. Analytic results that provide sufficient conditions for EFTDS of NCS are developed in Section 6.4, while Section 6.5 explores some characteristics of the dropping sequence in relation to EFTS. Finally, Section 6.6 demonstrates examples of EFTS for a discrete-time Brockett integrator controlled across a network experiencing packet loss.

6.2 Networked Control Systems and Networks

As discussed in chapters 2 and 3, several studies have been conducted in modelling and controlling Networked-Control Systems (NCS),[1],[2],[12], mostly to study the stability of a system whose control loop has been closed across a network.

The introduction of a network in a control loop brings about problems such as packet drops, delays, and so on. These issues have been analyzed individually although some studies have combined the effects of sampling and delay [1]. However, to the best of our knowledge, the network model itself has not yet been directly incorporated into the NCS model, but only through the effects that arise as a result of the network's conditions.

This is the missing link between Networked-Control System and Network-Control. Models of networks have been developed in Network-Control to study delays and packet drops caused by congestion. Therefore, there is a gap between the network dynamics, covered in Network-Control, and the effects that these dynamics have on a control system, which Networked-Control Systems focuses on. Until now, we have emphasized the effects of packet dropping on stability, particularly finite-time stability, and developed a stochastic model for packet dropping. In this chapter, we merge the two aspects of research, in order to combine into a single model the NCS and the network model. In chapter 3 we developed such a combined model, particularly taking into consideration packet dropping within the model-based NCS. Here, we will study the finite-time stability of the system based on this combined model. Due to the deterministic nature of the model, we will not employ FTSS, but rather finite-time deterministic stability.

6.3 Preliminary Notation

We consider the deterministic MB-NCS, described in Chapters 2 and 3

$$z_{k+1} = H_1(z_k) + H_2(z_k)\varphi_k,\ z_k \in \mathbb{R}^{2n},\ k = 0, 1, \ldots \tag{6.1}$$

More details about the model were given in Chapter 2. Recall from chapter 3 the dropping sequence $\varphi_k = (1 - \theta_k) \in \{0, 1\}$ is defined as follows: assume the state of the system is sent across a network path T of n_l links, each of which has a maximum allowed packet rate (or capacity $c_i(k)$), and a global capacity $G_i(k)$ determined by the sources' rates at time k, then the dropping sequence is given by

$$\varphi_k = 1 - \prod_{j=1}^{n_l} \frac{sign(c_j(k) - G_j(k)) + 1}{2} \tag{6.2}$$

where the sequence of variables $\{\varphi_k\}$ identifies whether a packet has been dropped due to congestion ($\varphi_k = 1$), or not ($\varphi_k = 0$).

The NCS described in equation (6.1) is a deterministic system, and we are interested in investigating its stability over a finite time in the event of packet dropping. In the stochastic case, bounds may be exceeded with low probability. A deterministic definition of EFTS is given next, which also allows bounds to be exceeded, but over limited intervals.

Definition 6.1 ***Extended Finite-Time Stable NCS (EFTS-NCS)***

The NCS (6.1) is EFTDS with respect to $(\alpha_x, \beta_x; \alpha_z, \beta_z; N, N_o)$*, if the following conditions hold*

(I.) the system is FTS with respect to (α_x, β_x, N)*, if no packet dropping occurs*

$$\{z_k^T z_k < \beta_z : k \in [0, N] | z_0^T z_0 \leq \alpha_z\} \tag{6.3}$$

(II.) for $\varphi_k = 1$*, and some* $k \in [0, N]$ *either*

$$\{z_k^T z_k < \beta_z : k \in [0, N] | z_0^T z_0 \leq \alpha_z\} \tag{6.4}$$

or

$$\{\forall j \in [0, N] : z_j^T z_j > \beta_z, \Rightarrow \min_{j+1 \leq i \leq j+N_o+1} x_i^T x_i \leq \beta_x\},\ N_o < N \tag{6.5}$$

where N_o *is the number of consecutive steps the system state is allowed to exceed the FT bound due to packet dropping.*

▲

In particular, FTS for NCS is redefined so that if packet dropping occurs, the system state may exceed the bound β_x for a fixed finite number of consecutive steps N_o. Note that the above definition requires the knowledge of future states to ensure FTS at each step. We will also redefine quadratic FTDS in case it is desired to bound

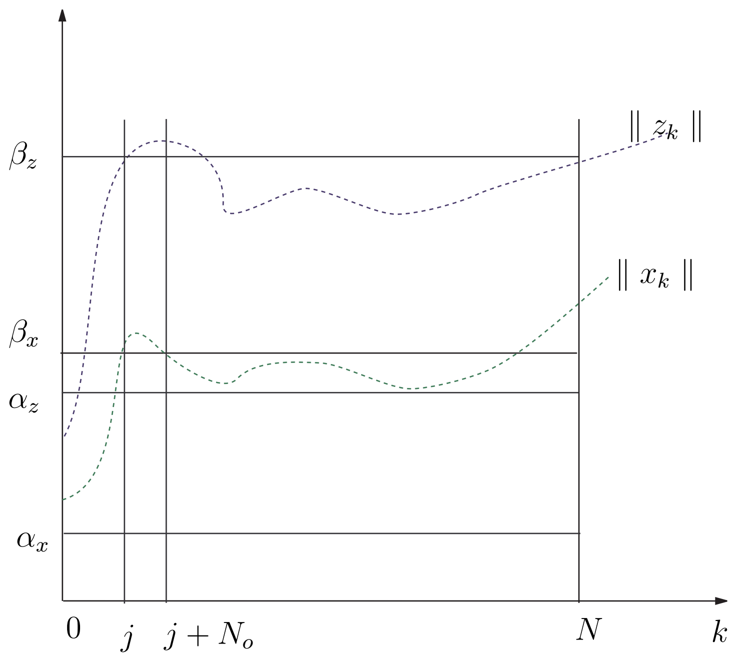

Figure 6.1: Extended definition of FTDS for NCS: the global state norm $||z_k||$ may exceed the bound β_z and $||x_k||$ the bound β_x as long as the plant state norm $||x_k||$ contracts in a finite number of steps N_o, back to β_x.

a given quadratic function of the state.

Definition 6.2 ***Quadratically EFTS-NCS***

The NCS (6.1) is quadratically EFTS with respect to $(\gamma_x, \gamma_{x0}; \gamma_z, \gamma_{z0}; N, N_o, M)$, *if for the choice of quadratic Lyapunov functions* $V_z(z_k, k) = z_k^T M(k) z_k$, $V_x(x_k, k) =$

$x_k^T m_1(k) x_k$ and $V_{\hat{x}}(\hat{x}_k, k) = \hat{x}_k^T m_4(k)\hat{x}_k$, in which $M(k) = M^T(k)$ is a $2n \times 2n$ time-varying matrix, with $m_1(k) > 0,\ m_4(k) > 0$, we have

(I.) for $\varphi_k = 0$

$$\{V_z(z_k, k) < \gamma_z : k \in [0, N] | V_z(z_0, 0) \leq \gamma_{z0}\} \tag{6.6}$$

(II.) for $\varphi_k = 1$ either

$$\{V_z(z_k, k) < \gamma_z : k \in [0, N] | V_z(z_0, 0) \leq \gamma_{z0}\} \tag{6.7}$$

or

$$\{\forall j \in [0, N] : V_z(z_j, j) > \gamma_z, \Rightarrow \min_{j+1 \leq i \leq j+N_o+1} V_x(x_i, i) \leq \gamma_x\}$$

▲

Theorem 6.1 *Every NCS that is quadratically EFTS with respect to the parameters $(\gamma_x, \gamma_{x0}; \gamma_z, \gamma_{z0}; N, N_o, M)$, is also EFTS with respect to $(\alpha_x, \beta_x; \alpha_z, \beta_z; N, N_o)$.*

Proof.

The proof easily follows by considering the fact that $\delta_1 ||z_k||^2 \leq V_z(z_k, k) \leq \delta_2 ||z_k||^2$, $\delta_1(k) = \lambda_{min}\{M(k)\}$, $\delta_2(k) = \lambda_{max}\{M(k)\}$ are the minimum and maximum eigenvalues of $M(k)$, respectively.

■

6.4 Extended Finite-Time Deterministic Stability Analysis

In this section, we consider sufficient conditions that will guarantee FTDS for the NCS. In the new setting, if the NCS state exceeds the bound specified at time j, then, in order to predict the future values of the state, it is required to have an estimate of the plant state for the successive $N_o + 1$ steps. This is presented in the following theorem by using the model to predict future states.

Recalling from Chapter 5, the sets of bounded states are denoted as follows

$$S_{\gamma_z} = \{z_k : V_z(z_k, k) \leq \gamma_z\} \tag{6.8}$$

$$S_{\gamma_x} = \{x_k : V_x(x_k, k) \leq \gamma_x\} \tag{6.9}$$

$$S_{\gamma_{\hat{x}}} = \{\hat{x}_k : V_{\hat{x}}(\hat{x}_k, k) \leq \gamma_{\hat{x}}\} \tag{6.10}$$

Theorem 6.2 *Consider the class C_{B-NCS} NCS (6.1), and the state prediction using the model*

$$\hat{x}_{k+(j+1)} = \hat{f}(\hat{x}_{k+j}) + \hat{g}(\hat{x}_{k+j})u_{k+j}, \quad k+1 \leq j \leq k+1+N_o \tag{6.11}$$

and assume for all $x_k \in S_{\gamma_x}$ and $k = 1, \ldots, N$

$$\Delta V_z \leq \Delta V_{B_z} = B_{H_2}(\hat{x}_k)\varphi_k^2 + 2(B_{H_{1,2}}(\hat{x}_k))\varphi_k + B_{H_1}(\hat{x}_k) - \hat{x}_k^T M(k)\hat{x}_k \tag{6.12}$$

$$\Delta V_x \leq \Delta V_{B_x} = B_{H_1}(\hat{x}_k) - \lambda_{min}\{M\}B_z(\hat{x}) \tag{6.13}$$

then if

$$[\rho_k V_z(z_k,k) - \Delta V_{B_z}(z_k,k))] \geq 0 \tag{6.14}$$

$$\frac{\gamma_z}{\gamma_{z0}} \geq \sup_{0\leq k\leq N} \prod_{j=0}^{k-1}(1+\rho_j) \tag{6.15}$$

$$[\rho_k V_z(z_k,k) - \Delta V_{B_z}(z_k,k))] \leq 0 \tag{6.16}$$

$$\min_{k+1\leq i\leq k+N_o+1} [\rho_i^{'} V_x(\hat{x}_i,i) - \Delta V_{B_x}(\hat{x}_i,i)] \tag{6.17}$$

$$\frac{\beta_{\hat{x}}}{\alpha_{\hat{x}}} \geq \sup_{0\leq k\leq N} \prod_{j=0}^{k-1}(1+\rho_j^{'}) \tag{6.18}$$

and finally

$$B_e(\hat{x}_k) + \beta_{\hat{x}} \leq \beta_x, \quad \forall \hat{x}_k \in S_{\gamma_{\hat{x}}}, \tag{6.19}$$

then the NCS (6.1) is FTDS with respect to $(\alpha_x, \beta_x; \alpha_z, \beta_z; N, N_o)$

Proof.

From condition (6.15) we have that if

$$[\rho_k V_z(z_k,k) - \Delta V_{B_z}(z_k,k)] \geq 0 \tag{6.20}$$

then using the fact that the NCS belongs to class C_{B-NCS} together with condition 6.15 and theorem 4.1, chapter 4 we can show the FTDS for the NCS. Let us study the case in which $[\rho_k V_z(z_k,k) - \Delta V_{B_z}(z_k,k)] \leq 0$, then inequality (6.18) reduces to

$$\min_{j+1\leq i\leq j+N_o+1} [\rho_k V_x(\hat{x}_{i)},i) - \Delta V_{B_x}(\hat{x}_i,i))] \geq 0,$$

from which it follows that there exists a $j+1 \leq i \leq N_o+1$ for which

$$[\rho_i V(\hat{x}_i, i) - \Delta V(\hat{x}_i, i)] \geq 0 \tag{6.21}$$

that combined with condition (6.18) with theorem 4.1, chapter 4, implies FTS for the model state $\hat{x}$ with respect to $(\alpha_{\hat{x}}, \beta_{\hat{x}}, 1)$. Also consider the following

$$||x_k|| = ||x_k - \hat{x}_k + \hat{x}_k|| \leq ||e_k|| + ||\hat{x}_k|| \leq B_e(x_k) + ||x_k|| \tag{6.22}$$

then considering the condition (6.19), and the FTS of $\hat{x}_k$, from which it follows $||x_k|| \leq \beta_x$ for at least one $k \in [j+1 \leq i \leq N_o+1]$ and moreover FTDS for the NCS.

■

6.5 Rate Limit of Packet Dropping

Let us consider once more the model of the networked control system in the form (6.1). In chapter 5 we investigate the rate of lost packets that system can support while maintaining FTSS. To do so, we consider the sufficient conditions for FTS of the system (6.1), given in theorem 6.2, and investigate the level of dropping rate $\varphi_k = 1$ for which stability is guaranteed. This results in the following.

Definition 6.3 ***Attractive System*** *A discrete-time system of the form*

$$z_{k+1} = f(z_k),\ z_k \in \mathbb{R}^n,\ z(0) = z_0 \tag{6.23}$$

is an attractive system with respect to $(\alpha_{z1}, \beta_z, \alpha_{z2}, N, N_r)$, $\alpha_{z1} \leq \beta_z \leq \alpha_{z2}$ *if it is FTS with respect to* $(\alpha_{z1}, \beta_z, N,)$ *and contracting with respect to* $(\alpha_{z2}, \beta_z, N_r)$, *i.e.*

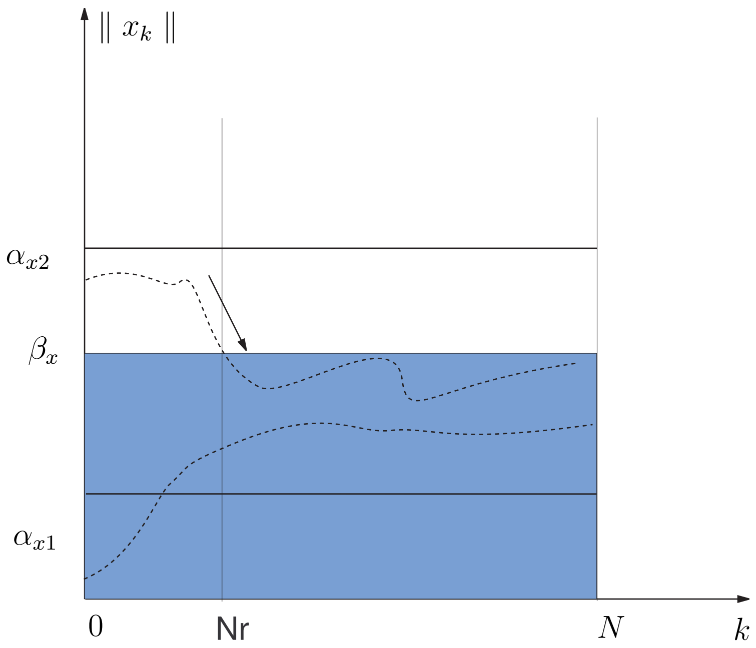

Figure 6.2: Attractive system.

$$||z_0|| \leq \alpha_{z1} \Rightarrow ||z_k|| \leq \beta_z,\ k = [0, N]$$

$$\alpha_{z2} \geq ||z_0|| \geq \beta_z \Rightarrow ||z_k|| \leq \beta_z,\ k = [N_r, N]$$

▲

Theorem 6.3 *Consider a class C_{B-NCS} NCS (5.19), also assume the controller $K(\hat{x})$ is such that in case of successful packet reception the system is attracted with respect to $(\alpha_{z1}, \beta_z, \alpha_{z2}, N, N_r)$*

$$||z_0|| \leq \alpha_z, \Rightarrow ||z_k|| \leq \beta_z,\ k = [0, N]$$

or

$$\alpha_{z2} \geq ||z_0|| \geq \beta_z \Rightarrow ||z_k|| \leq \beta_z,\ k = [N_r, N]$$

with $\alpha_{z1} \leq \beta_z$ *i.e. the region* $[-\beta_z, \beta_z]$ *is a global region of attraction for the state. Let* N_r *be the number of steps needed by the controller to pull the state back into the ball of radius* β_z *from a distance* α_{z2}*. Consider* N_r *number of time steps needed to pull the state from* $[\beta_z, \alpha_{z2}]$ *to* $[0, \beta_z]$*. Also we have that* N_0 *is the number of FT unstable steps the system can support, and* δB_z *is the bound on maximum value that the state norm* $||z_k||$ *exceed the bound* β_z *for each dropped packet* ($\varphi_k = 1$)*. Then we have that the number of packets that need to be received (* $\varphi_k = 1$) *in order to recover* δB_z *the loss of* N_e *packets is*

$$N_c = \frac{B_z(\hat{x})N_e}{\alpha_{z2}N_r.} \tag{6.24}$$

and the maximum number of admissible consecutive drops that needs to be followed by N_c *received packets in order to achieve EFTS through the network with respect to* $(\alpha_x, \beta_x; \alpha_z, \beta_z; N, N_o)$ *is*

$$N_d = \frac{N_0}{1 + N_r}. \tag{6.25}$$

Proof. We have that the number of packets that need to be received ($\varphi_k = 1$) in order to recover δB_z the loss of one packet is

$$N_g = \frac{\delta B_z}{\alpha_{z2}} N_r \tag{6.26}$$

then for the loss of N_e packets the number of packets that need to be received ($\varphi_k = 1$) in order to recover δB_z is

$$N_c = \frac{B_z(\hat{x})N_e}{\alpha_{z2}N_r.} \tag{6.27}$$

From the above consideration we can then evaluate the maximum number of admissible consecutive drops $N_d = \frac{N_0}{1+N_r}$ the system can support while being EFTS through the network with respect to $(\alpha_x, \beta_x; \alpha_z, \beta_z; N, N_o)$ assuming that are followed by N_c received packets.

■

The proof of the theorem follows from the assumption that the system is EFTDS if no packets are dropped. Also note that since the system belongs to class C_{B-NCS} a bound $B_z(\hat{x})$ on the state increments can be defined. Note that the condition is once more, only sufficient.

6.6 Examples

Exponential Class-b Controller for Brockett Integrator trough a Network

Recalling from section 5.6 in chapter 5 the discrete-time Brockett integrator, we investigate in a deterministic setting how packets losses, affect the closed-loop EFTS of the system. Consider again the discrete version of the Brockett integrator (5.21) and the model

$$\hat{x}_1(k+1) = -23\hat{x}_1(k) - 17u_1(k) \tag{6.28}$$

$$\hat{x}_2(k+1) = -19\hat{x}_2(k) + 3.33u_2(k) \tag{6.29}$$

$$\hat{x}_3(k+1) = -5\hat{x}_3(k) - 8(\hat{x}_1(k)u_2(k) - 7\hat{x}_2(k)u_1(k)) \tag{6.30}$$

We study EFTS with respect to $(\alpha_z = 1, \beta_z = 3, \alpha_x = 0.6, \beta_x = 1.5, N = 10, N_o = 2)$. Let us use the exponential class (a) controller

$$u(k) = -\begin{bmatrix} e^{-ak} & 0 \\ 0 & e^{-bk} \end{bmatrix} y(k) \tag{6.31}$$

with parameters $a = 1.3$, $b = 0.7$. Then the conditions of theorem 6.2 are satisfied if full information is available, i.e. $\varphi_k = 1$, $\forall k = 0, \dots 10$. In order to simulate the system, we consider the path used to the NCS composed of three links l_1, l_2, l_3, each with limit capacity $c_i(k)$. The links are used by five sources $s_1, \dots, s_5$ as follows

$$\begin{aligned} & l_1 \rightarrow s_1, s_4 \\ & l_2 \rightarrow s_1, s_3 \\ & l_3 \rightarrow s_2, s_3, s_5 \end{aligned} \tag{6.32}$$

meanwhile the sources send at the following respective rates

$$\begin{aligned} r_1(k) &= 1(sin(k) + 1)\,[packett/s] \\ r_2(k) &= 3(cos(k) + 1) \\ r_3(k) &= 1.7exp^{-k} \\ r_4(k) &= 8(cos(k) + 1) \\ r_5(k) &= 9exp^{-k} \end{aligned} \tag{6.33}$$

from which we can calculate the global rates at each link as follows

$$G_1(k) = r_1(k) + r_4(k); \tag{6.34}$$

$$G_2(k) = r_1(k) + r_3(k); \tag{6.35}$$

$$G_3(k) = r_5(k) + r_2(k) + r_3(k); \tag{6.36}$$

We study the closed-loop behavior of the NCS as the limit rate of the link, and therefore the amount of packets dropped vary. Starting from initial conditions $x_i(0) = \hat{x}_i(0) = 0.3$, $i = 1, 2, 3$, we first consider a fixe limit capacity $c_i = c = 17$ *packets/second* that will lead to a dropping sequence $\{\varphi_k\}$ of all zeros, that is all the packets are received (and therefore a receiving sequence $\{\theta_k\}$ of all ones). Figure

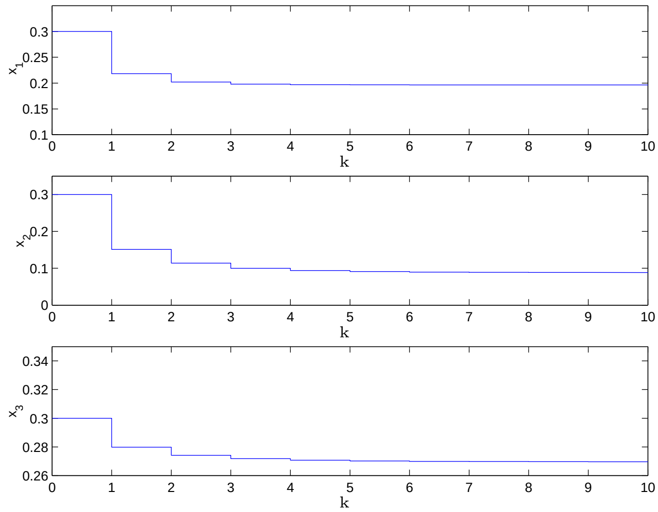

Figure 6.3: Brockett integrator controlled through the network with exponential class (b) controller with $a_1 = 1.3$, $a_2 = 0.7$, capacity $c = 17$.

(6.3) shows the evolution of the system state over time. If we lower the limit capacity to $c = 13$ $packets/second$, the receiving sequence becomes

$$\theta = [0\,1\,1\,1\,1\,0\,0\,1\,1\,1\,1\,0\,0\,1\,1] \tag{6.37}$$

for which FTDS conditions are still satisfied, as shown in Figure (6.4). For $c = 1$ we obtain a dropping sequence of all ones and the state dynamics are depicted in Figure 6.5. Finally in Figure 6.6, we show the norms for the three values of capacities for $x_i(0) = \hat{x}_i(0) = 0.3$.

6.7 Conclusions

We studied $MB - NCS$ with a deterministic model for the packet dropout. This model was realized by including the network in the NCS. This allowed us to obtain a deterministic model for the packet dropping and therefore for the complete NCS. The

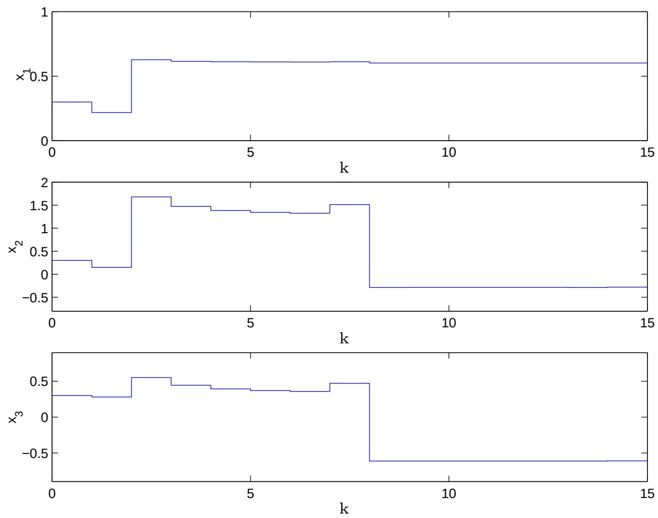

Figure 6.4: Brockett integrator controlled through the network with exponential class (a) controller with $a_1 = 1.3$, $a_2 = 0.7$, static capacity $c = 13$

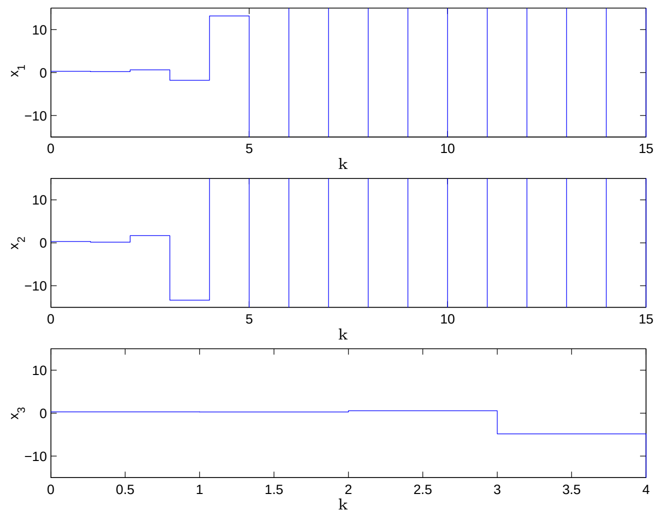

Figure 6.5: Brockett integrator controlled through the network with exponential class (a) controller with $a_1 = 1.3$, $a_2 = 0.7$, static capacity $c = 1$

EFTS for such systems was explored, and in particular redefined for the deterministic NCS. This has allowed the system to possibly exceed the specified bounds for a

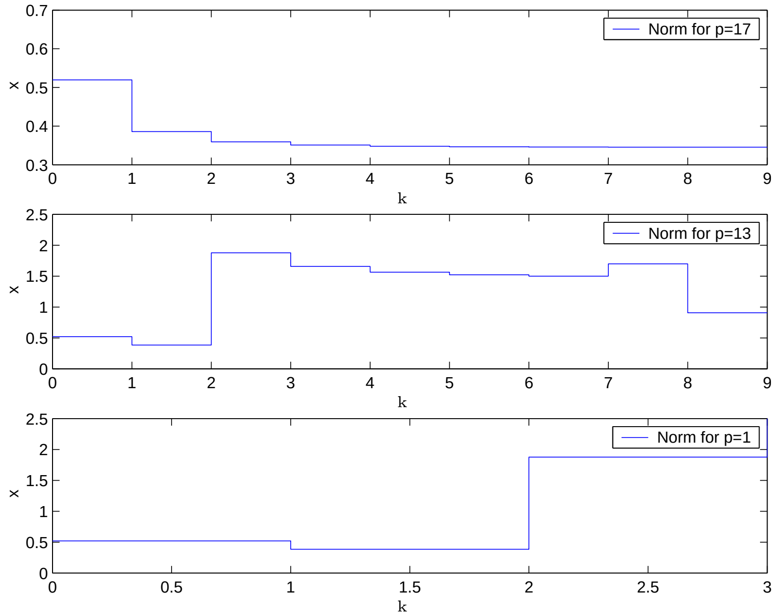

Figure 6.6: Norm of the state x for values of capacities $c = 17$, $c = 13$ and $c = 1$

finite number of steps, which was otherwise unacceptable under the classical FTS definition.

Chapter 7

Conclusions and Future Work

The aim of this thesis was to explore stochastic and deterministic finite-time stability of nonlinear discrete-time systems, within a networked communication systems framework. Moreover, we intended to provide a link between NCS and communication networks, by including the network model in the NCS closed-loop system.

Introducing a network in the control loop gives rise to several side issues that we tried to address in this thesis. In particular we focused on the issue of packets dropout, and assumed available a plant model on the controller's side, that is a model-based NCS approach. In this model the state measurements are sent through a network, while the control signal is directly applied to the plant. With such a model we showed how the stability, (specifically the finite-time stability) of the system is affected by the loss of state information. Moreover, we extended the existing model-based NCS approach to encompass nonlinear systems, and we characterized a class of model-based NCS with bounded errors between plant and model. Next, we completed the NCS model by providing stochastic and deterministic models for the packet dropout. For the stochastic case we used some existing results, while we studied the deterministic case by introducing the network dynamics in the NCS model.

A model for NCS was proposed for both stochastic and deterministic packet dropout. The extension to a deterministic setting has been done through the inclusion of the network dynamics in the NCS model. In particular, by detailing the network dynamics and including them in the NCS model we provided a deterministic model for the packet dropping. For the stochastic counterpart we considered a homogenous Markov chain and independent Markov chain, i.e. independent sequence, to address the issue of packet drops.

We then studied FTS in a stochastic and deterministic setting while extending some of the existing results in analysis and design. Furthermore, we extended the classical FTS concept to a more general one in which exceeding the bounds is allowed for finite number of consecutive steps.

Then, using the NCS model and the FTS results we proceeded to study FTS and FTSS of NCS. Finally, using the deterministic model for NCS, we studied FTDS for such a system.

7.1 Future Work

We are considering three possible extensions of our research. The first addresses the inclusion of delay in the control loop. A second extension involves the parallel solution of the two problems of stabilizing the NCS and the network, which may be realized, for example, by including priority schemes between sources and dynamically changing the priority of the NCS depending on the level of stability of the plant. We also plan on incorporating existing optimal solutions into the stability studies of the network in the control of NCS. This leads to two possibilities for stabilizing the two interconnected systems (network and NCS). The controller of the network may serve to stabilize the network and indirectly the plant, as will the controller of the networked system. Finally, a third extension includes the analysis and design for EFTS that was newly introduced in this thesis.

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