Limitations of Networked Controlled Systems

by

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Industrial Physics Engineering, Tecnológico de Monterrey, 1995
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DISSERTATION

Submitted in Partial Fulfillment of the
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Dedication

To my beloved wife Alma and my wonderful daughters Alma and Diana.

To my two little Angels in Heaven.

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I want to thank to my advisor, Prof. Chaouki T. Abdallah, who has been guiding me during these four years. Thanks for your time and for trusting in me. Thank you so much for your straight and reassuring encouragement when the results were not as good as expected. I want to acknowledge you, especially, for all the academic opportunities you have provided me that enriched my experience at UNM. Moreover, thanks for the multiple activities that you invited me to and that contribute enormously to financially support my studies and my family. It has been an honor to be part of your research group.

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Abstract

This dissertation studies the limitations of Networked Controlled Systems (NCS) caused by the presence of a finite capacity channel in the feedback loop.

First, we analyze the stabilization issue for an NCS that uses a Delta-Modulator scheme within the encoder/decoder structures. We also analyze the packet-loss issue, and determine a maximum allowable number of consecutive bits lost while keeping closed-loop stability. We then design a compensation scheme for regaining equimemory between the encoder and decoder after a bit is lost in a network without acknowledgment signals. We finally present a compensation scheme that ensures stability after a pre-determined number of bits is lost.

Second, we extend results from packet-based control theory and present sufficient conditions on the rate of a packet network to guarantee asymptotic stabilizability of unstable discrete-time linear invariant (DLTI) systems with less inputs than states.

We use a truncation-based encoder/decoder scheme and two types of NCS are considered in the absence of communication delays, then for one of the two NCS types, the case of a constant time delay is discussed. We also propose a new encoder/decoder scheme that is more complex than the truncation-based one, but that requires a lower stabilization rate.

Third, we obtain information theoretical conditions for tracking in DLTI control systems. The mutual information rate between the feedback signal and the reference input signal is used to quantify information about the reference signal that is available for feedback. This mutual information rate must be maximized in order to improve the tracking performance; however, the associated rate is shown to be upper bounded by a quantity that depends on the unstable eigenvalues of the plant and on the channel capacity. We also find a lower bound on the expected squared tracking error in terms of the entropy of a random reference signal.

Finally, we analyze the counterintuitive case where non-minimum zeros increase the mutual information rate between the feedback and reference signals. This analysis indicates that mutual information rate is an imperfect metric for specifying the performance for a DLTI tracking system. We also obtain a frequency interpretation of the tradeoffs that exist when tracking and disturbance rejection are required simultaneously and we obtain guidelines to satisfy the requirements on the controller in order to achieve good tracking and disturbance rejection.

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Glossary

C_s	Channel Capacity.
C_f	Feedback Channel Capacity.
$\mathrm{Cov}\{x\}$	Covariance of x
$\det(x)$	Determinant of x.
$\mathbf{det}(x)$	Absolute value of determinant of x.
$E\{x\}$	Expected value of x.
DLTI	Discrete-time linear invariant.
$h(x)$	Differential entropy of x.
$h_\infty(x)$	Differential entropy rate of x.
i.i.d.	Independent and Identical Distributed.
$I(x;y)$	Mutual information between x and y.
$I_\infty(x;y)$	Mutual information rate between x and y.
LTI	Linear Time Invariant.
LMI	Linear Matrix Inequality.

Glossary

NCS	Networked Controlled System.
UDP	User Datagram Protocol.
$\mathbf{sgn}(x)$	Sign function of x.
$\Delta - M$	Delta-Modulation.
$\lceil x \rceil$	Ceiling function of x.
$\lfloor x \rfloor$	Floor function of x.
$\|x\|$	Euclidean norm (norm-2) of x.

Chapter 1

Introduction

1.1 Motivation

Feedback control systems, where the control loops are closed through a real-time network, are called Networked Controlled Systems (NCS) [60], [56]. An architecture of a typical NCS is shown in Figure 1.1. The primary advantage of a NCS is that a reduced number of system components and connections are achieved, resulting in easier maintenance and diagnosis of the system. On the other hand, when controlling across networks, the assumptions of classical control theory may need to be revisited. For example, the delay from the sensor to the controller may be time-varying or random, and similarly for the path from the controller to the actuators. This issue has been analyzed in [13], [14], [29], and [34]. New problems thus arise because the sensed data and the control signals are no longer connected directly through a "dedicated wire", but rather through a data network that has finite bandwidth (finite data rate) and which may also be shared by many other systems. In recent years, much research and development has been focused in this area and, because of the attractive benefits of remote industrial control, several reliable protocols have been developed for robust real-time control. Meanwhile, the technologies based on general computer networks have also progressed. With the decrease in price and with the steady investment in infrastructure, the Internet is in fact becoming a suitable network for control applications. Without dedicated protocols, however, a new theory

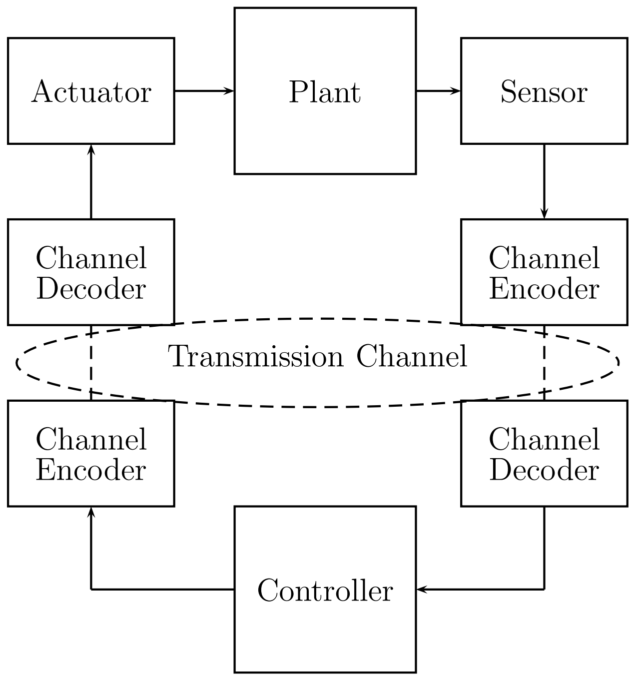

Figure 1.1: Architecture of a Typical Closed-Loop Networked Controlled System.

is needed for control design across the Internet. In particular, the communication channel between the plant and the controller may no longer be ignored, since the link can only carry a finite number of bits/s and the assumption of infinite capacity channels no longer holds. In addition to introducing both delay and quantization, the finite data rate channel brings forth the issue of how to best determine the usefulness of the sensed and control bits [32].

Back in 1999, Wong and Brockett [57] considered a digital channel with a finite capacity and found that unstable systems can never be asymptotically stabilized with a static quantizer across such a channel, and thus the concept of *containability* was introduced. Since then, Mitter [28] and some of his collaborators have contributed to the development of a new theory that matches classical control theory with traditional information theory [7], [39], [40], [52], [53]. Some of the first results within the information theoretic-control theoretic realm, were pointed out in [12]. The underlying concept is that the control loop can no longer be thought of as signal processors that interchange signals in a feedback configuration. A more accurate description is that several signals are transformed into packets of information. This information is transmitted through a rate constrained channel thus highlighting the quantization issue. These topics have also appeared in works [3], [8], [9], [15], [16], [27], [33], [59]. An extension of these ideas to nonlinear systems was presented in [31] and [37].

Since in a realistic setup, the feedback channel uses a packet network instead of a simple digital channel, the design of an NCS should also account for the packet-network limitations [30]. A complete understanding of the interaction between a control loop and a packet-based communication system requires the use of tools of information theory for real-time systems. The packet-based network control approach has been treated by Shi and Murray in [47] and [48].

Research efforts are also expanding in the development of a global theory linking control theory and communication theory. Examples of this effort may be found in [17] and [55].

All such studies aim at developing tools to deal with the issues introduced by the presence of an imperfect communication channel in the feedback loop. This dissertation is an effort in the same direction. It focuses on theoretical and practical issues such as the development of new encoding-decoding schemes and the analysis of the limitations of networked controlled tracking systems in terms of information theory.

Specifically, one of the main goals is to focus in the development of encoding schemes that are simple to implement while using a low data rate. Since simplicity is a main goal, then the rates may not reach the minimum possible. This goal finds an application when the rate is not the main issue, but the computational power is the problem.

The other important motivation is to find new interpretations of tracking systems in terms of information theory. In the past, control theory has been interpreted in terms of signal processing concepts or in terms of systems that interchange energy. With the new applications where the control systems interact with communication systems, a new interpretation in terms of information flow may be interesting for future applications. This dissertation is also focused in finding these new interpretations for the particular case of tracking systems. Although we do not pretend to come with ideas for immediate application, we are interested in finding fundamental limitations that are imposed by a limited communication medium and that any control system designer must be aware of when a constraint channel is present in the feedback loop of a tracking system.

1.2 Dissertation Outline

The research has several threads and this document is organized by focusing on a different topic in each chapter. We present in Chapter 2 an overview of recent results in control under limited communication and review the most important results that guided our research. Then, in Chapter 3, we present a Delta-Modulation encoding/decoding scheme that extends earlier results to the case where bits are dropped and plants may be highly unstable. After analyzing the limitations of the Delta-Modulation scheme we propose in Chapter 4 a different encoding/decoding scheme for higher-order plants that, although suboptimal in terms of the Data Rate Inequality, is simple to implement. Finally, in Chapter 5 we explore the limitations of networked controlled tracking systems. We analyze these systems in terms of information theoretical quantities. In particular, the mutual information rate is used to explain the limitations, and a frequency domain interpretation is obtained for the special case where asymptotic stationarity conditions hold.

1.3 Summary of Contributions

This dissertation introduces several contributions in Chapters 3, 4 and 5 as listed below:

- Chapter 3:
  - A Delta Modulation-like encoding scheme with a gain scheduling policy for discrete-time linear time invariant (DLTI) scalar system.
  - An algorithm to tolerate a specific number of lost bits in a NCS closed-loop system.

- Chapter 4:
  - A simple truncation-based encoding scheme for NCS stabilization considering a network between the sensors and the controller, and a network between the controller and the actuators.
  - Sufficient conditions in the network data rates considering a DTLI system without restrictions on the number of inputs of the system.
  - A dynamic quantizer that achieves a lower data stabilization rate than the truncation-based encoding scheme.

- Chapter 5:
  - Information theoretical conditions for DLTI tracking systems.
  - Analysis of the counterintuitive case where non-minimum phase zeros increase the mutual information rate between the feedback signal and the reference signal.
  - A frequency interpretation of the tradeoff that exists when we want to simultaneously achieve a good tracking and disturbance rejection.
  - Guidelines to satisfy the necessary requirements that the controller has to meet in order to achieve good tracking and reject disturbances simultaneously.

Chapter 2

Overview of Control Theory under Communication Constraints

2.1 Introduction

In this chapter we review fundamental results that form the background for the topics discussed in this dissertation. The chapter is divided into two sections and the results are presented in a chronological order. Section 2.2 presents the ideas that inspired most recent results. The section includes criteria for the necessity and sufficiency of stabilization. Section 2.3 presents the fundamental limitations of feedback control systems caused by causality and finite-capacity communication links, and includes the extensions of Bode-like integrals. These limitations provide some insight of what is possible and what is not in NCS. The results and theory reported in this chapter have been presented previously in our survey paper [18].

2.2 Limitations for Stabilizability

2.2.1 Feedback Scheme with a Noiseless Channel

The problem of state estimation and stabilization of an LTI system was originally introduced by Wong and Brockett [57], but it was Tatikonda and Mitter who generalized some of these ideas [52], [53]. In [53], limits were established for the channel data rate to achieve observability and stabilizability in a NCS. That work considered both a noiseless communication channel as well as a noisy one. The system with a noiseless digital communication channel is shown in Figure 2.1. The communication channel can transmit at each time, 2^R symbols, i.e., R bits of information per second without error. We assume that the encoder and decoder are equimemory at all times. By equimemory we mean that the decoder can invert the encoder map [51]. Consider

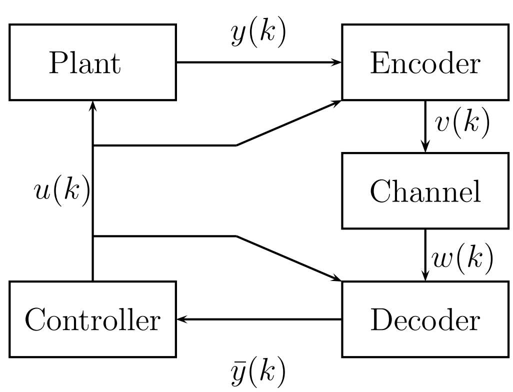

Figure 2.1: Closed-Loop System with Communication Channel.

then the DLTI system:

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k); \\
y(k) &= x(k).
\end{aligned}
\tag{2.1}
$$

where $x(k) \in \mathbb{R}^d$ is the state of the system and $u(k) \in \mathbb{R}^m$ is the control input, and $y(k) \in \mathbb{R}^l$ is the output of the system, all at time k. We assume that A and B have proper dimensions, and the state is available for measurement. In what follows, $\|x\|$ represents the Euclidean norm. We also introduce the following definition presented in [53].

Definition 2.2.1 *System 2.1 is asymptotically stabilizable if there exist an encoder, decoder and controller such that the following holds:*

1. *Stability:* $\forall \epsilon > 0$, $\exists\ \delta(\epsilon)$ *such that* $\|x(0)\| \leqslant \delta(\epsilon)$ *implies* $\|x(k)\| \leqslant \epsilon$, $\forall k \geqslant 0$.
2. *Uniform attractivity in* $x(0)$: $\forall \epsilon > 0$, $\forall \delta > 0\ \exists\ T(\epsilon, \delta)$ *such that* $\|x(0)\| \leqslant \delta$ *implies* $\|x(k)\| \leqslant \epsilon$, $\forall k \geqslant T$.

The first part of the definition implies that the state vector cannot grow unbounded for any bounded initial state, $x(0)$, while the attractivity property implies that the state decreases uniformly to zero. Considering this definition, the following result holds:

Proposition 2.2.1 *[53] Assuming (A, B) is a stabilizable pair; a necessary condition for system (2.1) to be asymptotically stabilizable is that the channel data rate R satisfies*

$$R \geqslant \sum_{\lambda(A)} \max \{ \, 0, \log_2 |\lambda(A)| \, \} \, ; \tag{2.2}$$

where $\lambda(A)$ are the eigenvalues of matrix A.

Actually, equation (2.2) considers the sum over the $\log_2$ of the unstable modes (since the $\log_2$ of the stable modes will be less than zero and will be discarded because of the max function). The lack of dependence on the stable modes of the system is easily explained since such modes decay to zero on their own. This result indicates that stabilizability can only be achieved if we have a minimum data rate which is related to the dynamics of the plant. In other words, if the system has fast unstable dynamics, the channel data rate must be faster to overcome the effects of the unstable dynamics. It was also shown that equation (2.2) provides exactly the same conditions for asymptotic observability. These ideas were extended by the same authors for noisy communication channels [52].

2.2.2 Feedback Scheme with a Noisy Channel

When a noisy channel is present in the feedback loop given in Figure 2.1, there must be a restatement of the concept of asymptotic stabilizability, since the deterministic definition no longer holds. There have been three main concepts of stability in the stochastic case, the first one involving a mean-squared convergence criterion, the second using the almost-sure convergence criterion while the third considers the m-th moment convergence criterion (to recall the differences in the convergence criteria, see Appendix A). Depending on the particular application, one may use one convergence criterion versus another. The mean-squared criterion is a good candidate for situations where large deviations occur and must be penalized, whereas the almost-sure convergence is more appropriate when almost all realizations are typical [50]. The resulting conditions are not the same because one convergence criterion does not necessarily imply any of the other two [49]. In what follows, we will only provide the result for the almost-sure convergence criteria.

Almost-sure Convergence Criteria

In [52], almost-sure convergence is defined as follows:

Definition 2.2.2 *System (2.1) is asymptotically stabilizable if there exist an encoder, a decoder, and a controller such that* $\|x(k)\| \to 0$ *almost surely.*

For Definition 2.2.2, the bound for R given by equation (2.2) can no longer be assumed valid. A new framework is needed to guarantee almost sure asymptotic stabilizability. The approach used in [52] was the use of Shannon's channel capacity, C_s, interpreted in [5] as a measure of channel quality and defined as follows:

Definition 2.2.3 *[5] The "information" channel capacity of a discrete memoryless channel is*

$$\mathrm{C}_s = \max_{p(x)} \; I(x;y);$$

where the random process x is the channel input, the random process y is the channel output and the maximum is taken over all possible input distributions $p(x)$.

A result from [52] is given by the following proposition:

Proposition 2.2.2 *For system (2.1) with (A,B) a stabilizable pair, a necessary condition for almost sure asymptotic stabilizability is that*

$$\mathrm{C}_s \geqslant \sum_{\lambda(A)} \max\{\, 0, \log_2 |\lambda(A)|\,\}$$

.

Although we do not include in this chapter the proofs of these results, the reader may notice that these conditions make no assumptions on specific architectures for the encoder, decoder, or controller. The results hold independently of these elements. A fictitious difference between the noisy and the noiseless cases is that for the noiseless channel, the results are given in terms of R, while in the noisy case, the conditions are given in terms of C_s. Relations in terms of R may however be obtained, depending on the type of noisy channel.

2.2.3 Feedback Scheme including a Packet-Based Network

The previous results taken from [26], [52], [53] and [54], considered noisy and noiseless discrete communication channels. However, work by Shi and Murray [47] took the initial steps towards the development of a packet-based control theory. They consider the problem of stabilizing an unstable, but controllable and observable, linear time-invariant system when the feedback path includes a packet-based network. The LTI system is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t); \\ y(t) &= x(t). \end{aligned} \tag{2.3}$$

The following assumptions for system (2.3) are made: the $n \times n$ matrix A has at least one eigenvalue in the right half plane (unstable), and the pair (A, B) is controllable while the pair $(A, I_{n\times n})$ is obviously observable. It is also assumed that the packet network has a finite data rate R bits/s and that no packets are lost, no reordering of packets may occur, and a packet length of l bits is considered. The transmission delay in the network from encoder to decoder is δ, and the total delay induced by the network, in addition to the transmission delay, is constant and equal to D. The signals are discretized $(y(t) \rightarrow y(k))$ and the sampling rate is chosen as $\delta + D$. Limits are obtained for the minimum data rate needed to stabilize the closed-loop system for three different bit allocation schemes.

Equal Bit Allocation

In the first case, bits were allocated equally to each discrete output $y(k)$, i.e., the allocation of bits in a packet was such that l/n bits were used for the ith component of $y(k)$. For this scheme, the following sufficient condition was obtained [47] in order to guarantee exponential stability of the closed-loop system:

$$R > \frac{l \log(\|e^A\|)}{\frac{l}{n} - 1 - D \log(\|e^A\|)};$$

where $\|e^A\|$ is the induced L_2 norm or the largest singular value of e^A.

Proportional Bit Allocation

Assuming that $A = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. A proportional bit allocation is intuitively a smarter scheme than equal bit allocation, because instead of using the same number of bits for each component, $\frac{\lambda_i}{\lambda} l$ bits are used for the ith component of $y(k)$, where $\lambda = \sum_{i=1}^{n} \lambda_i$. Therefore, the bit allocation is proportional to the size of the eigenvalues so that more bits in a packet are allocated to the more unstable modes. For this scheme, reference [47] showed that the limit for the data rate to achieve exponential stability is given by:

$$R > \frac{l\lambda \log_2 e}{l - \lambda D \log_2 e - \frac{\lambda}{\lambda_n}}.$$

However, this result is counter-intuitive since a proportional bit allocation scheme is expected to give a dependence of R on the largest eigenvalue (the most unstable), rather than the smallest eigenvalue. This contradictory condition led the authors of [47] towards a third approach, where they considered optimal bit allocation.

Optimal Bit Allocation

The idea here is to give variable bit allocation portions for each individual subsystem (ith component), and then perform an optimization algorithm on those variable portions. The optimization problem is stated as follows:

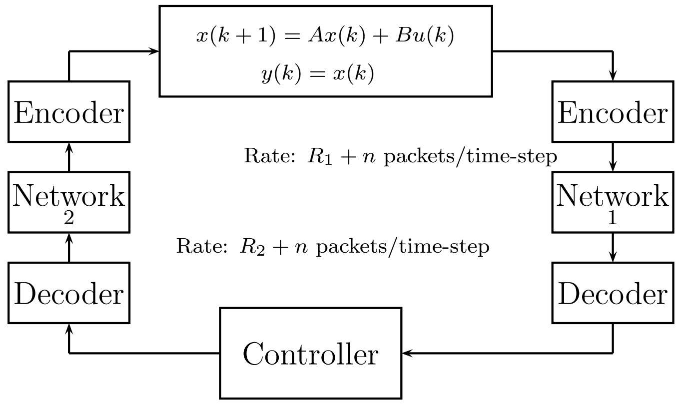

Figure 2.2: Packet-based NCS.

Let $\beta = [\beta_1, \cdots, \beta_n]$, then find:

$$\min_{\beta}\{R_{min}\}$$

subject to

$$\sum_{i=1}^{n} \beta_i = 1, \beta_i > 0,\ 1 \leq i \leq n;$$
$$\beta_i l > 1 + \lambda_i D \log_2 e,\ 1 \leq i \leq n;$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n > 0$ are given. The disadvantage of this approach is that there is no analytical solution to the bits allocation in terms of the size of the eigenvalues since the answer is based on the solution of a Linear Matrix Inequality (LMI). The scheme tries to give more weight to the most unstable eigenvalue and less weight to the least unstable one. For this allocation scheme, reference [47] shows that if $l \ll n$, the following approximation holds:

$$R_{inf\text{optimal}} \approx \sum_{i=1}^{n} \lambda_i \log_2 e.$$

The concepts discussed so far provide initial steps towards a deterministic theory of packet-base control. However, there is also a robust result for a stochastic case where the system model is given by:

$$\begin{aligned} x(k+1) &= (A+\Delta_k)x(k) + \gamma_k B u(k); \\ y(k) &= \lambda_k \mathbb{C} x(k); \end{aligned}$$

where λ_k and γ_k (used to model packet drops) are Bernoulli independent and identically distributed (i.i.d.) random variables with $E\{\lambda_k\} = \lambda$ and $E\{\gamma_k\} = \gamma$ for all k, respectively. In addition, the uncertainty matrix Δ_k satisfies $\Delta_k^T \Delta_k \leqslant \mathbb{K}^2 I$ for all k, where $\mathbb{K}$ is a constant. The network on the right side of Figure 2.2 has a data rate $R_1 + n$ while the network on the left side of figure has a data rate $R_2 + n$. Note that R_i bits are used to allocate the magnitude of the state and n bits are used for the sign of the n state vector components. The problem was formulated in [45] to guarantee almost-sure stability and the result is given by the following theorem:

Theorem 2.2.1 *Assume B and $\mathbb{C}$ are invertible and the system dimension is n. Then a sufficient condition for the closed-loop almost sure stability (if there are no packet drops, i.e., $\lambda = 1$ and $\gamma = 1$, this notion changes to exponential stability) is that the network and system parameters satisfy the inequality*

$$(\|A\|\mathbb{K})^{1-\lambda\gamma} \left(\|A\| 2^{-\frac{R_1}{n}} + \|B\| \|B^{-1}A\| 2^{-\frac{R_2}{n}} + \mathbb{K} \right)^{\lambda\gamma} < 1. \tag{2.4}$$

The importance of this result is that it is equivalent to the one by Tatikonda, equation (2.2), when considering a single packet-based network (by letting $R_2 = \infty$, $\lambda = 1$, $\gamma = 1$ and $\mathbb{K} = 0$ in equation (2.4)).

2.3 Fundamental Limitations on a Causal and Finite Capacity Feedback Scheme

Recently, Martins and Dahleh [25] proposed exciting ideas on fundamental limitations on the performance of a finite capacity feedback channel. The results in [25] are a relevant achievement for NCS theory similar to the the Bode Integral results for the classic control theory [43]. Before proceeding, we propose the following notation:

- We use the notation in [49] where bold letters represent stochastic processes.
- Let $\mathbf{a}(k)$ be a time sample of the stochastic process $\mathbf{a}$.
- The expectation of $\mathbf{a}$ is given by $E\{\mathbf{a}\}$ and $\mathbf{a}^k = \{\mathbf{a}(1), \mathbf{a}(2), \ldots, \mathbf{a}(k)\}$.
- The covariance matrix of a stochastic process $\mathbf{a}$ is:
  $R_a(i,j) = E\{(\mathbf{a}(i) - E\{\mathbf{a}(i)\})(\mathbf{a}(j) - E\{\mathbf{a}(j)\})^T\}$, where i, j are integers.
- Let $I(\mathbf{z}; \mathbf{w})$ denote the mutual information rate between the random variables $\mathbf{z}$ and $\mathbf{w}$.
- Let $I_\infty(\mathbf{z}; \mathbf{w})$ denote the mutual information rate between the stochastic processes $\mathbf{z}$ and $\mathbf{w}$.
- Let $|.|$ denote the absolute value.
- The eigenvalues of an $n \times n$ matrix A are denoted by $\lambda_i(A)$, with $1 \leq i \leq n$.

With this notation we introduce the feedback structure presented in [25]. The closed-loop system is shown in Figure 2.3, where $\mathbf{d}$ is the disturbance sequence, $\mathbf{e}$ is the plant input, $\mathbf{y}$ is the plant output, $\mathbf{z}$ is the channel output and the transfer function between $\mathbf{d}$ and $\mathbf{e}$ is the sensitivity function represented by $S_{e,d}(\omega)$. Under this feedback scheme, two limitations were found for the closed loop performance. These

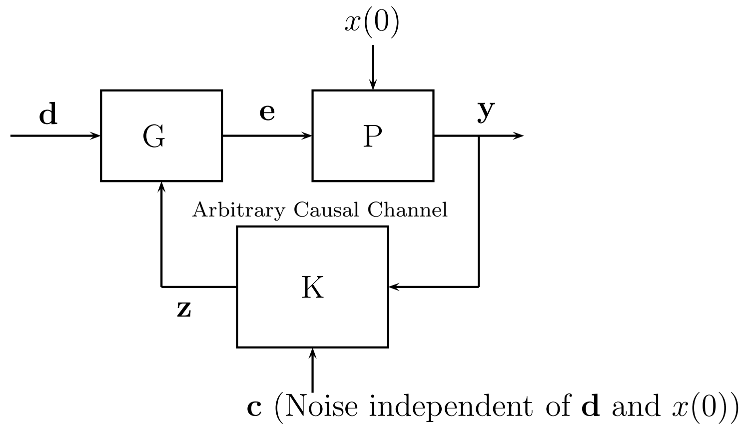

Figure 2.3: Feedback Scheme including Finite Capacity Channel.

limitations are due to causality and the finite capacity in the feedback channel K. We define next the concept of feedback capacity of a channel.

Definition 2.3.1 *[24] In the setup given by Figure 2.3, the feedback capacity, C_f, is defined as the least upper bound C_f that satisfies*

$$\sup_{k \in \mathbb{N}_+} \frac{I((\mathbf{d}^k, \mathbf{x}(0)); \mathbf{z}^k)}{k} \leq C_f.$$

A particular case of the closed-loop system in Figure 2.3 is the block diagram in Figure 2.4, where $\mathbf{z}$ is relabeled as $\mathbf{u}$ and the block G is replaced by an adder.

2.3.1 Limitations Caused by Causality

Consider a single input DLTI plant with the following state-space realization:

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{e}(k); \\ \mathbf{y}(k) &= \mathbb{C}\mathbf{x}(k). \end{aligned} \tag{2.5}$$

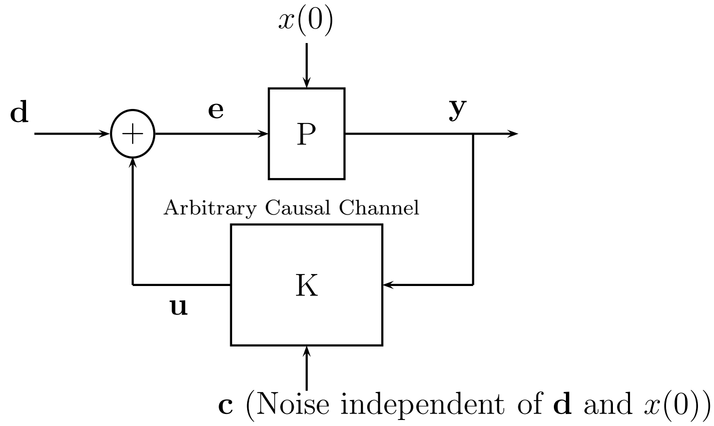

Figure 2.4: Equivalent Feedback Scheme including Finite Capacity Channel.

The maps G and K in Figure 2.3 are related to $\mathbf{e}(k) = G(k, \mathbf{d}^k, \mathbf{z}^k)$ and to the channel output $\mathbf{z}(k) = K(k, \mathbf{y}^k, \mathbf{c}^k)$. A particular case of the system in Figure 2.3, is the closed-loop system in Figure 2.4, where block G is an adder and $\mathbf{z}$ is relabeled as $\mathbf{u}$. With these definitions, an information flux inequality is derived from the assumption of causality. Mathematically, the relationship is expressed in the following theorem:

Theorem 2.3.1 *[25] Let $\mathbf{x}(k)$ be the solution of the state-space equation (2.5) of system in Figure 2.3. If the system is stable, i.e., $E\{\mathbf{x}(k)^T\mathbf{x}(k)\} < \infty$, then*

$$I_\infty(\mathbf{z}; \mathbf{d}) \leq I_\infty(\mathbf{y}; \mathbf{z}) - \sum_{i=1} \max\left\{0, \log_2\left(|\lambda_i(A)|\right)\right\}.$$

This theorem implies that the mutual information rate (i.e., the information flow) from the input $\mathbf{d}$ to the output $\mathbf{z}$ is upper-bounded and that the unstable eigenvalues of the open-loop system decrease this upper bound.

2.3.2 Limitations Caused by Finite Feedback Capacity

We notice that the bound expressed by Theorem 2.3.1 does not depend on the feedback channel capacity. Using Definition 2.3.1 and Theorem 2.3.1 the following result is obtained:

Theorem 2.3.2 *[25] If the closed-loop system in Figure 2.3 is stable, i.e.,* $\sup_k E\{\mathbf{x}(k)^T\mathbf{x}(k)\} < \infty$*, then*

$$I_\infty(\mathbf{z};\mathbf{d}) \leq C_f - \sum_{i=1} \max\{0, \log_2(|\lambda_i(A)|)\}.$$

Using the result in Theorem 2.3.2, we note that the feedback capacity can be used as a universal upper-bound on $I_\infty(\mathbf{z};\mathbf{d})$. As C_f approaches $\sum_i \max\{0, \log_2(|\lambda_i(A)|)\}$, the mutual information rate $I_\infty(\mathbf{z};\mathbf{d})$ decreases to zero. Before proceeding, we recall some definitions from random processes theory:

Definition 2.3.2 *[24], [44] A given zero mean real stochastic process* $\mathbf{a}$ *is asymptotically stationary if the following limit exists for every* $\gamma \in \mathbb{N}$*:*

$$\overline{R}_a(\gamma) = \lim_{k\to\infty} E\{\mathbf{a}(k+\gamma)\mathbf{a}(k)\}. \tag{2.6}$$

Using equation (2.6) we have a second definition.

Definition 2.3.3 *[24] The asymptotic power spectral density,* $\widehat{\Phi}_a(\omega)$*, is defined as:*

$$\widehat{\Phi}_a(\omega) = \sum_{k=-\infty}^{\infty} \overline{R}_a(k)e^{-j\omega k}.$$

Finally, the last definition that we will use later is:

Definition 2.3.4 *[24] Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be asymptotically stationary processes. Then* $\mathbf{a}$ *and* $\mathbf{b}$ *are jointly asymptotically stationary if the following limit exists for every* $\gamma \in \mathbb{N}$*:*

$$\overline{R}_{a,b}(\gamma) = \lim_{k\to\infty} E\{\mathbf{a}(k+\gamma)\mathbf{b}(k)\}.$$

In some particular cases, Theorem 2.3.2 may be extended in some important directions. Assuming that $\mathbf{e}$ is asymptotic stationary, then the following results may be obtained:

Theorem 2.3.3 *[25] Consider the scheme of Figure 2.3, where* $\mathbf{e}$ *and* $\mathbf{d}$ *are jointly asymptotically stationary, with* $\mathbf{d}$ *also Gaussian auto-regressive then, the following holds:*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \min\{0, \log_2(S_{e,d}(\omega))\}d\omega \geq -I_\infty(\mathbf{d};\mathbf{u}); \tag{2.7}$$

where $S_{e,d}(\omega)$ *is a sensitivity-like function,* $S_{e,d}(\omega) = \sqrt{\frac{\widehat{\Phi}_e(\omega)}{\widehat{\Phi}_d(\omega)}}$ *and* $\widehat{\Phi}_e(\omega)$ *and* $\widehat{\Phi}_d(\omega)$ *are the asymptotic power spectral density (Definition 2.3.3) of* $\mathbf{e}$ *and* $\mathbf{d}$*, respectively.*

Theorem 2.3.4 *[25] Consider the scheme in Figure 2.3, where* $\mathbf{e}$ *and* $\mathbf{d}$ *are assumed asymptotically stationary, with* $\mathbf{d}$ *Gaussian auto-regressive. If the state of the plant satisfies* $\sup_k E\{\mathbf{x}^T(k)\mathbf{x}(k)\} < \infty$ *then the following holds:*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \min\{0, \log_2(S_{e,d}(\omega))\}d\omega \geq \sum_i \max\{0, \log_2(|\lambda_i(A)|)\} - C_f. \tag{2.8}$$

This sensitivity-like function $S_{e,d}(\omega)$ is a measure of the disturbance amplification or rejection property of the closed-loop system. Specifically,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \min\{0, \log_2(S_{e,d}(\omega))\}d\omega$$

measures the rejection of disturbances and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \max\{0, \log_2(S_{e,d}(\omega))\}d\omega$$

measures the amplification of disturbances. In other words, to achieve disturbance rejection we would like to have

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi} \min\{0, \log_2(S_{e,d}(\omega))\}d\omega \tag{2.9}$$

be as large as possible. Moreover, (2.8) shows that the disturbance rejection ability is limited by:

$$C_f - \sum_i \max\left\{0, \log_2\left(|\lambda_i(A)|\right)\right\}.$$

Theorem 2.3.4 provides a universal bound on disturbance attenuation in the presence of communication constraints. If the channel does not have a minimum feedback capacity it is impossible to reject disturbances. The limitation is a direct effect of the finite feedback capacity consideration. This result that cannot be predicted by existing results and it is completely independent of the Bode integral results [25]. We note that Theorem 2.3.4 is valid for any channel as it depends only on the feedback capacity and on the unstable eigenvalues of A.

2.4 Summary

In this chapter we have reviewed recent results pertaining to the analysis of NCS from an information theoretic setting. Limitations on stabilization in the deterministic sense and in several stochastic frameworks have been presented. The results dealt with the case of a discrete channel as well as a packet-based network. Under some assumptions, the results are equivalent for both approaches. The packet-based theory remain, however, very preliminary. The most recent results on fundamental limitations on the performance of a feedback scheme have also been presented as well as the new Bode Integral Formula interpretation for these limitations. The incorporation of the Bode Integral results shows the equivalence of feedback control schemes and feedback communications schemes, bringing new interpretations for areas such as biology [6]. Future work is expected in order to extend these ideas to distributed systems with multiple channel communication schemes.

In the next chapter we present our first results where a Delta-Modulation scheme system is used to encode/decode the output signal of the system. It will be shown that the Data Rate Theorem presented in this chapter has a direct consequence in the systems that can be controlled with the proposed encoding scheme and in the number of lost packets that can be tolerated before reaching unstability.

Chapter 3

Delta Modulation Scheme

3.1 Introduction

In this chapter we analyze stabilization issues for a NCS that uses a Delta-Modulator Scheme within its encoder/decoder structures as shown in Figure 3.1. These results were presented in [21]. In [4] a differential coding with a Delta-Modulation ($\Delta - M$) scheme was used since such a scheme provides the simplest form of differential coding. Basically, one bit is transmitted every time-step through the communication channel. This translates into a low cost design, since the $\Delta - M$ algorithm is a simple two-level dynamic quantizer. This is important for applications where the transmission is very expensive and instead of sending a network packet (with several bits for data and protocol) the communication channel only allows to transmit very small amounts of information. Recalling the Data Rate Theorem (Proposition 2.2.1), the minimum required rate for stabilization is given by

$$R > \sum_{i=1}^{n} \log_2(|\lambda_i(A)|); \tag{3.1}$$

where $\lambda_i(A)$ are the eigenvalues of the open-loop discrete linear system:

$$x(k+1) = Ax(k) + Bu(k).$$

We then note that in the case of scalar systems, a $\Delta - M$ algorithm is limited to stabilizing linear systems of the form

$$x(k+1) = ax(k) + bu(k);$$

with $|a| \leq 2$. Since more unstable systems may need to be stabilized, we analyze modifications for the $\Delta - M$ algorithm proposed in [4] in order to stabilize systems with $a > 2$. This is motivated by the previously mentioned cost issues associated with the simple $\Delta - M$ scheme. We also consider the packet-dropping problem and the issues of recovering equimemory and stabilizability. This analysis is innovative since most previous work on the subject dealt with the limited-rate and the packet losses separately. Recent works considered packet losses but assumed unlimited channel rate, see [10] and [46], while research dealing with limited-rate channels have not included packet losses (see for example [52] where the minimum channel capacity is derived when an erasure channel is present). Our results show that the maximum number of bits that can be sequentially lost depends on the region where a certain estimation error lies. Using this fact, we redesign the $\Delta - M$ scheme used in [4] so that the system can handle at least a minimum number of bit losses.

3.2 Problem Setup

We consider the same system described in [4] and shown in Figure 3.1. To better understand our proposed scheme, we first analyze the details and limitations of the original scheme. In what follows, we assume:

- The transmitted bit at time k is $b_k \in \{-1, 1\}$.

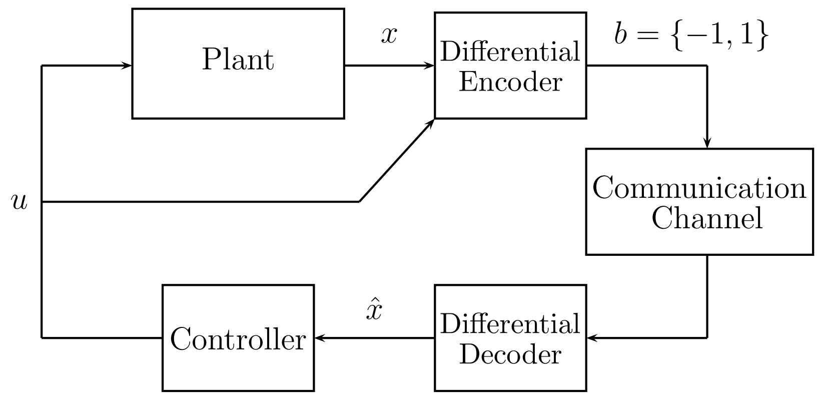

Figure 3.1: Closed-Loop System with Differential Encoding Scheme.

- The encoder has access to the control signal (this condition is only needed for the case when bit losses may occur and not acknowledgment messages can be sent from the decoder to the encoder).
- No packets are lost (this assumption will be relaxed later).
- All the elements in the loop, including the communication channel, are noiseless.

The assumption that the encoder has access to the control signal is necessary to guarantee equimemory and it is a strong assumption. If the system does not have bit losses this condition can be relaxed for the purpose of stabilization. Similarly, if the system has bit losses but acknowledgment messages between encoder and decoder are allowed, the condition can also be relaxed. The plant is modeled as a scalar discrete linear time invariant system

$$x(k+1) = ax(k) + bu(k); \tag{3.2}$$

with the linear feedback $u(k) = -K_c\hat{x}(k)$, where $\hat{x}(k)$ is the encoded state. For simplicity sake, we consider systems with $a \geq 1$. In system (3.2) we disregard

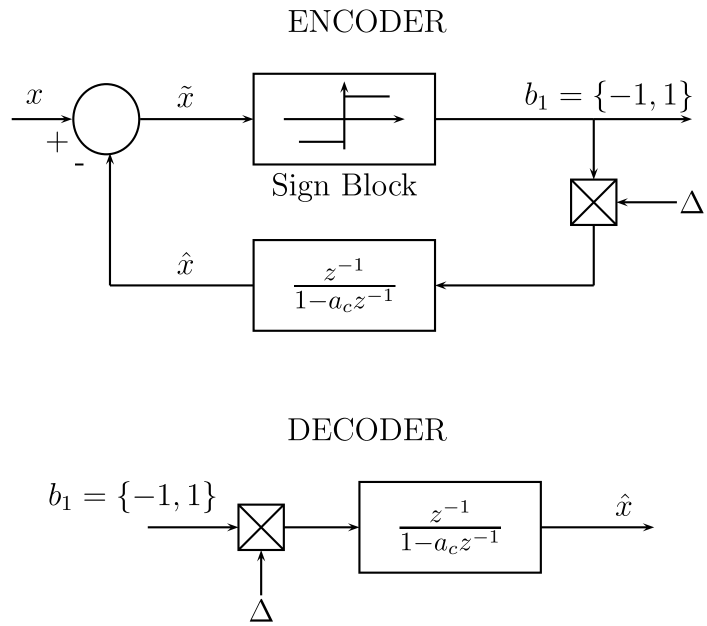

Figure 3.2: Original Encoder-Decoder Scheme.

eigenvalues $0 \leq a < 1$ since they imply an already stable plant. For $a \leq 0$ a similar approach may be used. In the original $\Delta - M$ scheme of [4], the encoder and decoder shown in Figure 3.2 are described by

$$\hat{x}(k+1) = a_c\hat{x}(k) + \Delta\mathbf{sgn}(\tilde{x}(k)); \tag{3.3}$$

where $a_c = a - bK_c$, Δ is a real number constant (Delta Gain), $\tilde{x}(k)$ is the error $(x(k) - \hat{x}(k))$ and $\mathbf{sgn}()$ denotes the **sign** function defined as:

$$\mathbf{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0; \\ -1 & \text{if } x < 0. \end{cases}$$

Combining equations (3.2) and (3.3) we obtain the following dynamics for the system and the error:

$$x(k+1) = a_c x(k) + bK_c\tilde{x}(k); \tag{3.4}$$

$$\tilde{x}(k+1) = a\tilde{x}(k) - \Delta\mathbf{sgn}(\tilde{x}(k)). \tag{3.5}$$

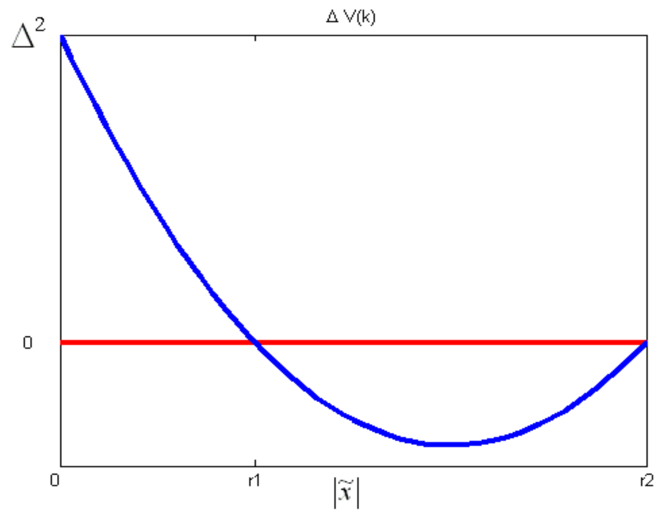

Figure 3.3: $\Delta V(k)$.

If $V(k) = \tilde{x}^T \tilde{x}$ is chosen as a Lyapunov function candidate, reference [4] shows that the change $V(k+1) - V(k) = \Delta V(k)$ is given by

$$\Delta V(k) = \begin{cases} \geq 0 & \text{if } |\tilde{x}(k)| \leq r_1; \\ < 0 & \text{if } r_1 < |\tilde{x}(k)| < r_2 \text{ ;} \\ \geq 0 & \text{if } r_2 \leq |\tilde{x}(k)| \text{ ;} \end{cases} \tag{3.6}$$

where $r_1 = \frac{\Delta}{(a+1)}$ and $r_2 = \frac{\Delta}{(a-1)}$. The region where $\Delta V(k) < 0$, i.e, $r_1 < |\tilde{x}(k)| < r_2$ is denoted by R_1. We already know from inequality (3.1) that the rate of the feedback channel will limit the absolute value of a, but it is important to study what happens if we try to stabilize systems with $a > 2$. This is important because in real scenarios the plant may be very unstable. This analysis will provide ideas for redesigning the Delta Modulator scheme.

3.3 Stability Analysis for Systems with $a > 2$

In [4] it was shown that the expansion of $\Delta V(k)$ is given

$$\Delta V(k) = \left(a^2 - 1\right) \tilde{x}^2(k) - 2a\Delta \left|\tilde{x}(k)\right| + \Delta^2. \tag{3.7}$$

If we study the plot of this function in Figure 3.3 in the range $0 \leq |\tilde{x}(k)| \leq r_2$, we notice that the maximum value of $\Delta V(k)$ occurs at $|\tilde{x}(k)| = 0$. If at some time instant k_f we have $|\tilde{x}(k_f)| = 0$, then $\tilde{x}(k_f + 1) = -\Delta$ and this may push $\tilde{x}(k_f + 1)$ outside the region R_1. Once there, the error state will not return to the region of attraction since the Lyapunov function is increasing outside R_1 ($|\tilde{x}(k_f + 1)| > r_2$). Let us examine when such an event takes place. We know that $r_2 = \frac{\Delta}{(a-1)}$, so that if $\Delta > \frac{\Delta}{(a-1)}$, the error $\tilde{x}(k)$ will be ejected from the region $r_1 < |\tilde{x}(k)| < r_2$ and can never return to it. We note that $a > 2$ is exactly the condition that forces the inequality $\Delta > \frac{\Delta}{(a-1)}$. Therefore, for $a > 2$ it is not possible to stabilize the system using a rate of only one bit per time-step as predicted by the minimum rate given by equation (3.1). Although this result was expected, this analysis motivates us to think of ways to solve the problem for systems with $a > 2$ using more bits.

3.4 Gain Scheduling Scheme

To overcome the limitations imposed by the scheme of Section 3.3, we obviously need to increase the bit-rate in the closed-loop system. This can be done either by sending the same number of bits in less time (i.e. by increasing the sampling frequency) or by increasing the number of bits in the same time period. Let us analyze each of these two options separately.

3.4.1 Increasing Sampling Frequency

Let us assume that the discrete-time linear system was obtained by discretizing a continuous time system. The scalar continuous time, linear, time-invariant system with eigenvalue α has the form $\dot{x}(t) = \alpha x(t) + gu(t)$. Then, the discretized system is given by

$$x(k+1) = e^{\alpha T} x(k) + g \int_0^T e^{\alpha \eta} d\eta \; u(k) = ax(k) + bu(k).$$

If we allow the sampling time, T, to be decreased (increasing the sampling frequency), we can move the discrete pole to the desired position $1 \leq a \leq 2$. This will be accomplished if $e^{\alpha T} < 2$, or equivalently $T < \log_e(2)/\alpha$.

3.4.2 2-Bit Delta-Modulation-like Scheme

On the other hand, if we cannot increase the sampling time, we may try sending more information (bits per unit of time) across the channel. In [11], it was shown that using a Differential Pulse Code Modulation (DPCM) communication scheme (that is a generalization of the Delta Modulation Scheme) instead of a 1-bit quantizer may solve the problem. Thus, a multilevel quantizer is used, where the number of levels is determined by the relation $r_{min} = \log_2(a)$. In this chapter, a different idea is proposed to conserve the general simple structure of a Delta Modulator. Again, the purpose is to achieve simplicity of the encoding algorithms rather than to achieve the minimum possible rate. We propose to add an extra bit containing information pertinent to the "size" of the prediction error. This information allows us to schedule the value of the modulation gain Δ. The scheme is illustrated in Figures 3.4 and 3.5. The idea behind our proposed scheme is to use a comparator that determines whether we are inside or outside the region $|\tilde{x}(k)| < r_2$. If we are inside this region, we use the Δ_1 gain, and switch to the other gain, Δ_2, once outside. The gain Δ_2

allows us to increase the region of attraction as will be explained later. The **sgn**() function provides one bit, and the comparator an extra bit that allows us to handle more instability in the system, i.e. to stabilize scalar systems whose eigenvalues are greater than 2 in magnitude.

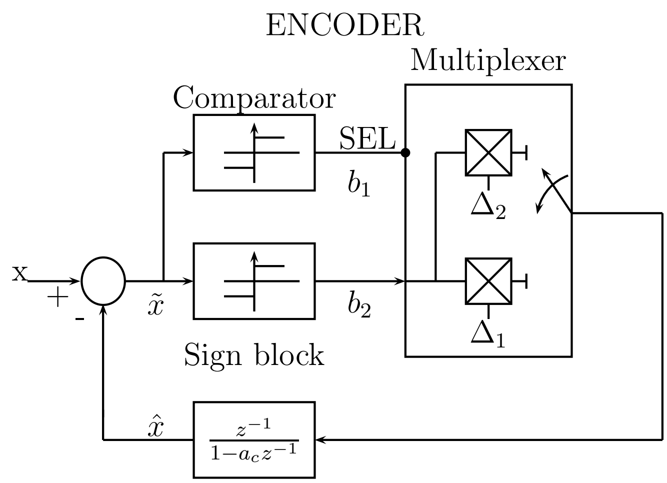

Figure 3.4: 2-Bit Encoder with Multiplexer.

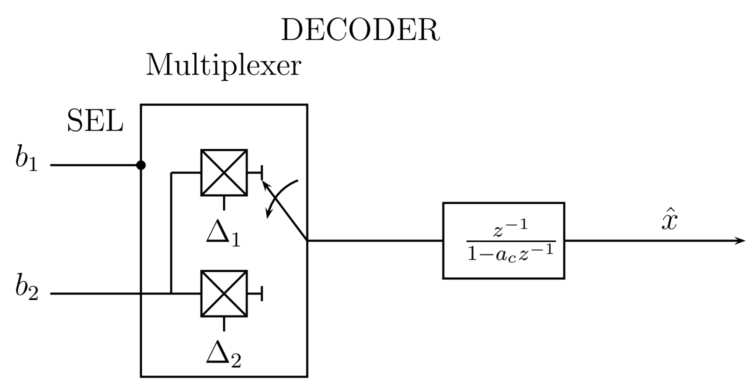

Figure 3.5: 2-Bit Decoder.

3.5 Design of the Gain Scheduler Scheme with 2 Bits

Let us explain in greater detail the proposed 2-Bit-Delta-Modulator. First, we build a Delta Modulator similar to [4]. This creates the region R_1 where $\Delta V(k) < 0$. As shown before, the problem for systems with $a > 2$ arises when $|\tilde{x}(k)| = 0$ since $\tilde{x}(k+1)$ will move outside the region R_1. That is exactly where the gain scheduler is activated. The new value $\Delta_2 > \Delta_1$ creates a second region in which $r_3 < |\tilde{x}(k)| < r_4$ and where we can enforce $r_2 = r_3$ by a suitable selection of Δ_2. We denote this second region by R_2 as shown in Figure 3.6.

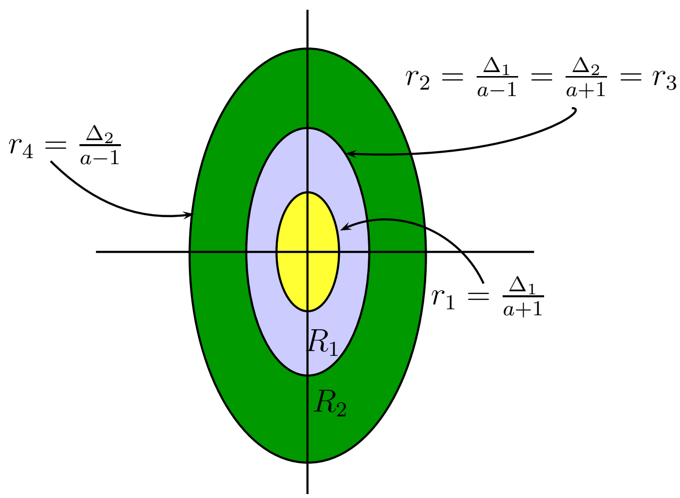

Figure 3.6: Different possible regions for $\tilde{x}(k)$.

Remark 3.5.1 *Although we are dealing with a scalar system, in Figure 3.6 we use two-dimensional balls, to better illustrate the proposed concepts.*

Reviewing our Lyapunov analysis, we know that if the initial condition is such that

$$|\tilde{x}(0)| < r_2$$

then $\Delta V(k) < 0$. There will however be a moment when $|\tilde{x}(k)|$ will be less than r_1 and eventually, when it is near 0, it will be ejected to a value greater than r_2 if $a > 2$. The comparator in Figure 3.4 then provides the signal to switch the modulator gain, which makes $\Delta V(k) < 0$ in the region $r_3 \leq |\tilde{x}(0)| \leq r_4$. To force $r_2 = r_3$ we select Δ_2 as follows:

$$\Delta_2 = \frac{a+1}{a-1}\Delta_1. \tag{3.8}$$

With this relation we know that the region where $\Delta V(k)$ is negative is within

$$r_1 < |\tilde{x}| < r_4$$

where $r_2 = \frac{\Delta_1}{(a-1)}$, $r_3 = \frac{\Delta_2}{(a+1)}$, $r_3 = r_2$ and $r_4 = \frac{\Delta_2}{(a-1)} = \frac{\Delta_1(a+1)}{(a-1)^2}$. If we analyze the case where $\Delta V(k)$ reaches its maximum, i.e., when $|\tilde{x}(k)| = 0$, we see that $|\tilde{x}(k+1)| = \Delta_1$. If we compare this jump with the boundary $|\tilde{x}(k)| = r_4 = \frac{\Delta_1(a+1)}{(a-1)^2}$ it is clear that for $\Delta_1 > \frac{\Delta_1(a+1)}{(a-1)^2}$, then $a > 3$. In other words, by modifying the modulation scheme and using a second bit, we can now stabilize systems with an unstable eigenvalue $a \leq 3$. Note however that the 2-bit $\Delta - M$ modulation scheme is conservative, since with 2 bits we should be able to handle eigenvalues $a \leq 4$ as established in equation (3.1). Following the approach in [4], we see that from equation (3.4), $X(z) = \frac{bK_c}{z-a_c}\tilde{X}(z)$. Then, if we define $H(z) = \frac{bK_c z}{z-a_c}$, we get $h(k) = bK_c a_c^k, k \geq 0$ and we write $X(z) = H(z)z^{-1}\tilde{X}(z)$. In the time domain we get $x(k) = \sum_{m=0}^{\infty} \tilde{x}(m-1)h(k-m) = \sum_{m=0}^{\infty} \tilde{x}(m)h(k-m-1)$ and since $|\tilde{x}(m)| \leq r_4$, $\forall m > 0$, then we conclude that $|x(k)| \leq r_4 \sum_{m=0}^{\infty} |h(k-m-1)|$ which is bounded since $h(k)$ is BIBO stable and $\sum_{m=0}^{\infty} |h(k-m-1)|$ equals $\frac{bK_c}{(1-a_c)}$. These ideas are summarized in the following lemma.

Lemma 3.5.1 *Consider the scalar discrete-linear system given by equations (3.2)-(3.4), with constant* Δ_1 *and constant* Δ_2 *given by equation (3.8), and let* $a \leq 3$. *If the initial condition of the coding error is such that* $\tilde{x}(0) < r_2$ *then, the following holds* $\forall k \geq 0$:

- $|\tilde{x}(k)| < r_4$;

- $|x(k)| \leq \frac{r_4 b K_c}{1-a_c}$;

where $r_2 = \frac{\Delta_1}{(a-1)}$ *and* $r_4 = \frac{(a+1)\Delta_1}{(a-1)^2}$.

■

Remark 3.5.2 *If we extend the approach to an M-bit-Modulator Scheme, it will become clear that this scheme is far from optimal. It can be easily shown that if we keep adding concentric rings, the gain* Δ_M *to satisfies* $\Delta_M = \frac{(a+1)^{M-1}\Delta_1}{(a-1)^M}$ *and the maximum eigenvalue that can be stabilized is the one that solves the inequality* $(a-1)^M - (a+1)^{M-1} > 0$. *In Table (3.1) we show the number of bits needed and compare the maximum eigenvalue, a, that can be stabilized by the maximum theoretical number of bits given by equation (3.1) and the one of our proposed scheme.*

M bits	$a_{Theoretical}$	$a_{\Delta-MScheme}$
1	2	2
2	4	3
3	8	3.87
4	16	4.67
5	32	5.43
6	64	6.15
7	128	6.84

Table 3.1: Comparison of Maximum Eigenvalue ($a_{\Delta-MScheme}$) vs Theoretical Maximum ($a_{Theoretical}$).

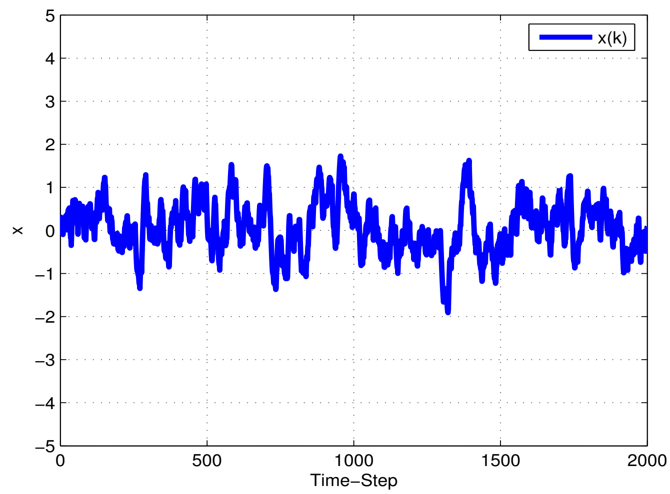

Figure 3.7: NCS response with 2-Bit-$\Delta - M$ Scheme and $a = 2.15 \leq 3$.

3.5.1 Example 1

To test the modified scheme we present the following system that cannot be stabilized with the original Delta-Modulation scheme of [4]:

$$x(k+1) = 2.1x(k) + u(k);$$

with $u(k) = -1.15\hat{x}(k)$. Assume that $\Delta = 0.2$, $x_0 = 0.12$; therefore, $r_1 = 0.0645$, $r_2 = 0.1818$, $r_4 = 0.5124$ and $r_1 < |\tilde{x}_0| < r_2$. In the simulation shown in Figure 3.7 we see that the system has been stabilized.

3.5.2 Example 2

We see now that as predicted for $a > 3$ the system can no longer be stabilized by the 2-Bit-Delta-Modulation. The system that was simulated is $x(k+1) = 3.05x(k) + u(k)$ and $u(k) = -2.1\hat{x}(k)$. The result is shown in Figure 3.8.

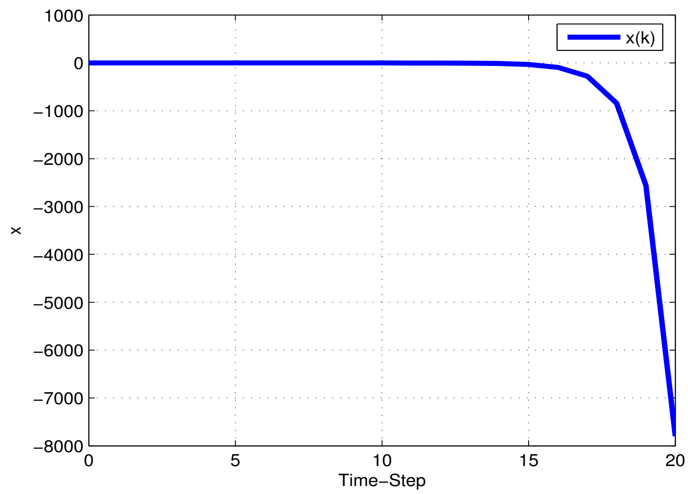

Figure 3.8: NCS response with 2-Bit-$\Delta - M$ Scheme and $a = 3.05 > 3$.

3.6 Issue of Packet Losses

The previous sections dealt with a communication channel that does not suffer any packet losses. The loss of packets is however a common problem in packet-based networks, and is dealt with using a variety of approaches (re-transmission, for example). Note that in this particular case, by packet losses, we mean bit losses since the information sent through the channel is a bit and not a set of bits with a header (commonly known as packet). One question that naturally arises is how robust is the Delta-Modulation scheme in the face of lost bits. We study the case where the bits transmitted from the encoder through the channel do not reach the decoder. For our analysis, we consider a User Datagram Protocol (UDP)-like channel, i.e., we do not allow any acknowledgement packet flowing from the decoder back to the encoder. The reason for this choice is that UDP-like channels have been used in several experiments to avoid long, and potentially destabilizing delays, see [22] and [41]. Before continuing with the discussion and recalling that we are using a binary alphabet in the transmitted bit at time k ($b_k \in \{-1, 1\}$), we assign a value of 0 to the decoder inputs in the case where the transmitted bit, b_k, is lost and no bit is present at the decoder site when the sampling occurs. The first effect of dropping a bit, even before considering its impact on stability and performance, is the loss of equimemory between the encoder and decoder. Recall (see equation (3.3)) that both encoder and decoder use a predictor that updates with the bit transmitted. Therefore, a dropped bit causes the encoder and decoder to lose equimemory, since there is no acknowledgment signal to inform the encoder that a bit has not reached its intended destination. We note that the plant and encoder are physically collocated and as such, the encoder has access to the control input $u(k)$ for all k. This fact has been shown in [53] to be enough for conserving the equimemory property of the encoder and decoder, i.e, it can guarantee their equimemory, even in the absence of an acknowledgement signal. To use this advantage we modify the structure of the encoder and decoder. We then introduce a new notation: let $\hat{x}_e$ be the encoder estimate and $\hat{x}_d$ be the decoder estimate of x. If no packets are lost, their dynamics will be given by

$$\hat{x}_e(k+1) = a_c\hat{x}_e(k) + \Delta\mathbf{sgn}(\tilde{x}_e(k)); \tag{3.9}$$

for the encoder, and

$$\hat{x}_d(k+1) = a_c\hat{x}_d(k) + \Delta\mathbf{sgn}(\tilde{x}_e(k)); \tag{3.10}$$

for the decoder, where the $\mathbf{sgn}()$ function in both equations is actually the transmitted bit b. Let us suppose that at some instant, k, the transmitted bit is lost. The encoder prediction will continue to evolve according to equation (3.9). However, since the bit with the information of $\mathbf{sgn}(\tilde{x}_e(k))$ never reaches the decoder, the decoder estimator will no longer follow equation (3.10) but will instead evolve according to equation

$$\hat{x}_d(k+1) = a_c\hat{x}_d(k). \tag{3.11}$$

Therefore, the control law that is applied in the next sampling instant is given by $u(k+1) = -K_c\hat{x}_d(k+1)$. With this event, the equimemory between encoder and decoder is lost and has to be recovered. According to the approach given in [52], we propose to modify the encoder as follows. At instant $k+1$, the encoder compares the control signal u that was received from the controller, with the expected control signal $u_e = -K_c\hat{x}_e$. If no bits were lost, u will be equal to u_e, since the estimates $\hat{x}_e$ and $\hat{x}_d$ will be equal. However, if a bit is lost, then $u(k)$ will be different from $u_e(k)$ and that will trigger a reset action for the encoder estimator. The reset action consists of the following steps: before the encoder generates its next estimate, $\hat{x}_e(k+1)$, it replaces the current value of $\hat{x}_e(k)$ (which was previously calculated using the information that did not arrive to the decoder) by the value given by $a_c\hat{x}_e(k-1)$. This expression has the same value that the decoder calculated previously because of the lost bit. The approach works because just before the first bit is lost the estimates $\hat{x}_e$ and $\hat{x}_d$ are equal. Then, after doing the replacement, the encoder calculates the next prediction $\hat{x}_e(k+1)$.

In summary, both encoder and decoder will be once more in equimemory and proceed thereafter considering the fact that $\hat{x}_e = \hat{x}_d = \hat{x}$. We note that we are assuming noiseless elements between the controller and the actuator so we can use the equality between u and u_e without major concerns. We may, however, robustify the scheme in the presence of some noise between the controller and actuator by considering the following compensation. When a bit is lost, we note from equations (3.10) and (3.11), that the difference between the expected signal u_e and the received u is given by $|K_c\Delta|$. Therefore, we place a threshold in the comparison of u_e and u: if $|u - u_e| < \left|\frac{K_c\Delta}{2}\right|$ we consider them equal, i.e., the bit arrived to the decoder. If $|u - u_e| > \left|\frac{K_c\Delta}{2}\right|$ then we assume that the bit was lost. Therefore, any additive noise with magnitude strictly less than $\left|\frac{K_c\Delta}{2}\right|$ does not cause problems. While this solves the equimemory problem when bits are lost, we have yet to analyze what happens to the stability of the closed-loop system. When one bit is lost, we need to alter the prediction form of the encoder as explained above in order to regain equimemory. This however implies that the error $\tilde{x}(k)$ is not longer given by the recursive equation $\tilde{x}(k+1) = a\tilde{x}(k) - \Delta\mathbf{sgn}(\tilde{x}(k))$, but instead by:

$$\begin{aligned}\tilde{x}(k+1) &= x(k+1) - \hat{x}(k+1); \\ &= ax(k) - bK_c\hat{x}(k) - a\hat{x}(k) - bK_c\hat{x}(k); \\ &= a\tilde{x}(k). \end{aligned} \tag{3.12}$$

This last expression may be easily generalized to l consecutive lost bits as

$$\tilde{x}(k+1) = a^l\tilde{x}(k). \tag{3.13}$$

We know from Section 3.2 that the stability region is bounded on the outside by $r_2 = \frac{\Delta}{a-1}$. This limit allows us to determine the number of consecutive bits that may be lost before losing stability. In fact, by setting $|\tilde{x}(k)| \leq r_2$, from equation (3.13) and the expression for r_2 we obtain

$$l \leq \left\lfloor \frac{\log_2\left(\frac{\Delta}{(a-1)|\tilde{x}(k)|}\right)}{\log_2(a)} \right\rfloor; \tag{3.14}$$

where $\lfloor . \rfloor$ is the **floor** function. We note that the allowable number of lost future bits depends on the current error $\tilde{x}(k)$. This implies that there is a region within R_1 that does not allow for any bits to be lost. This region is given by $\tilde{x}(k) > \Delta/(a(a-1))$. In summary, the Delta-Modulator system in the original design can no longer guarantee stabilization for the whole of region R_1 when bits are lost. We present next some simulations that show the behavior of both the error, $\tilde{x}(k)$, and the state, $x(k)$, when bits are lost.

3.6.1 Example 3

Consider the system given by $x(k+1) = 1.5x(k) + u(k)$ with $u(k) = -0.8\hat{x}(k)$. Suppose $x(0) = 0.24$ and $\Delta = 0.2$, then $r_1 = 0.08$, $r_2 = 0.4$. Now let us suppose

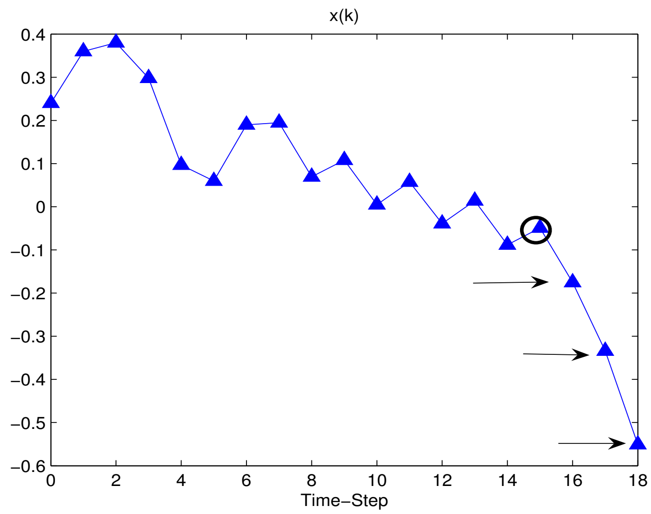

Figure 3.9: $x(k)$ with 3 Packet Losses from $k = 15$ to $k = 17$.

that in the time interval $0 \leq k \leq 15$ no bits are lost. The error at $k = 15$ is $\tilde{x}(k) = -0.1761$, i.e, it is within the region R_1 and suppose next that 3 consecutive bits are lost. Equation (3.14) gives us a maximum of 2 consecutive bit losses before we leave the stability region. In fact, we see in Figure 3.10 that if 3 bits are lost, $\tilde{x}(k)$ goes outside $-r_2$ and, therefore, outside region R_1. We show the state evolution in Figure 3.9.

3.6.2 Example 4

We want to clarify that equation (3.14) is actually valid for all the region $|\tilde{x}(k)| < r_2$ and not just for the stability region. In this example we consider the same system as before but we assume that no bits are lost in the time interval $0 \leq k \leq 23$. Therefore, at $k = 23$ we have $\tilde{x}(k) = 0.05059 < r_1$, i.e, we are in the interior ball where the change in the Lyapunov function is positive. However, equation (3.14) predicts that more than 5 consecutive bit losses will cause the error to reach the instability region.

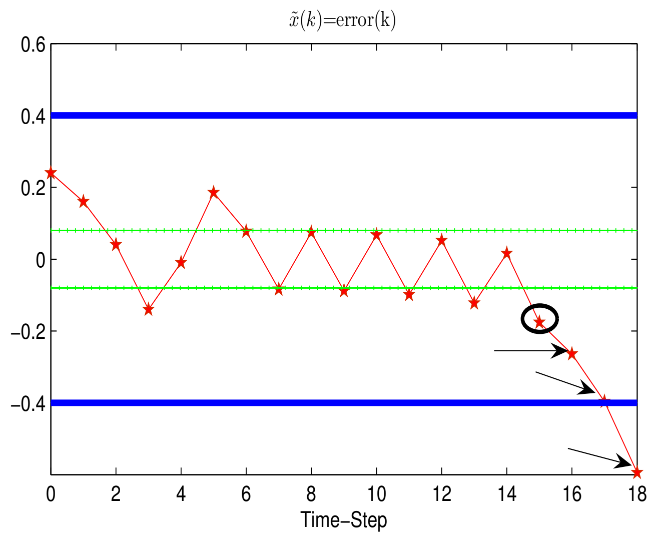

Figure 3.10: $\tilde{x}(k)$ with 3 Packet Losses from $k = 15$ to $k = 17$.

Figures 3.11 and 3.12 illustrate this.

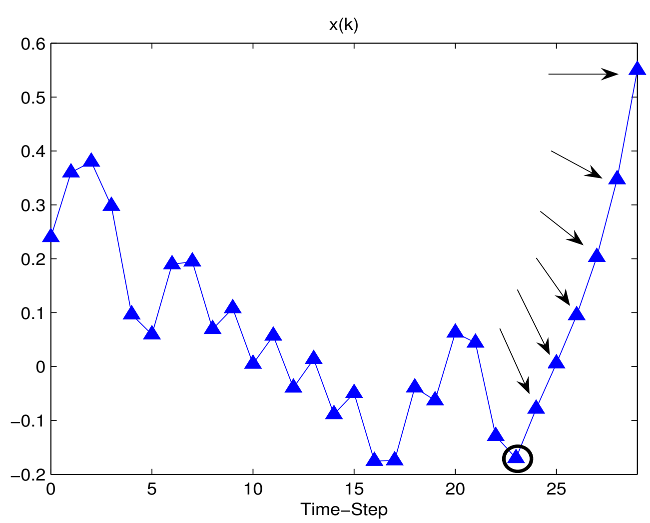

Figure 3.11: $x(k)$ with 6 Packet Losses from $k = 24$ to $k = 29$.

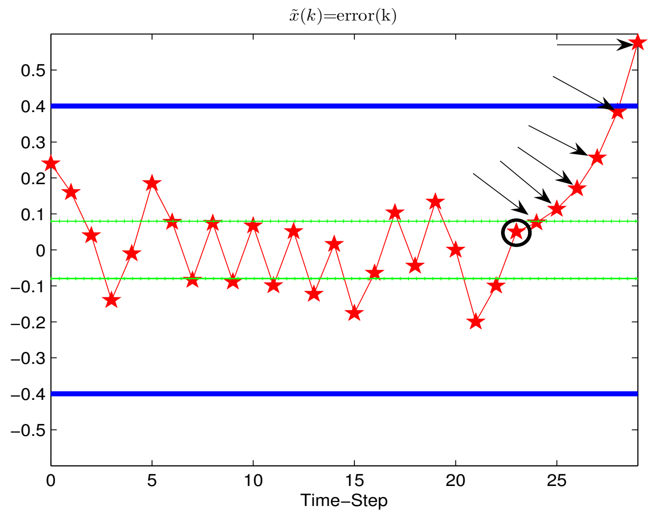

Figure 3.12: $\tilde{x}(k)$ with 6 Packet Losses from $k = 24$ to $k = 29$.

3.7 Compensation for Packet-Losses

We know from the results in [52] that in the case of noisy channels (for example, an erasure channel), the rate (or more accurately the capacity) of the channel is no longer limited by $R = \log_2(a)$ but instead by $R = \frac{\log_2(a)}{\gamma}$, where γ is the probability that a packet was received. In other words, we have to increase the data rate in order to guarantee stabilization of the system. For the purpose of our design, we assume that $1 \leq a \leq 2$ and study whether the 2-Bit-Delta-Modulator scheme can help with the lost packets issue. The reason behind this assumption is that we want to use the extra rate provided by the second bit to compensate for lost bits instead of accommodating more unstable systems. We want to clarify that our "packet" is now composed by the concatenation of the two bits, b_1 and b_2, that are sent through the channel. Let us consider the same 2-Bit-Delta-Modulator Scheme of Figures 3.4 and 3.5. If $\tilde{x}(k) \in R_1$, the multiplexer selection input, b_1, is multiplied by Δ_1. If we are outside the region R_1, b_1 is multiplied by Δ_2. Recall from Section 3.5 that

$\Delta_2 > \Delta_1$. Obviously, this scheme provides guarantees only if we assume that no packets are dropped when $\tilde{x}(k)$ is outside R_1 (actually, the scheme tolerates some packet drops outside R_1 that will be dependent of $\tilde{x}$ as we will see later). Let us then establish the following condition for our design: the maximum consecutive number of packets that can be lost when we are in **any** subregion within R_1 is β, where $\beta \in \mathbb{N}$. With this condition, we can guarantee that in any subregion R_1, β packets lost may be tolerated, but for some of these subregions even more packets may be handled. As we see in the examples, for any error in the region R_1, the worst case scenario in terms of packet losses is when $\tilde{x}(k)$ is "near" r_2. Let us therefore quantify how "close" must $\tilde{x}(k)$ be to r_2 in order to go outside the region after β lost packets. From equation (3.13), this is given by

$$\begin{aligned} a^{\beta}|\tilde{x}(k)| &< \Delta_1/(a-1); \\ |\tilde{x}(k)| &< \Delta_1/(a^{\beta}(a-1)). \end{aligned} \tag{3.15}$$

Then, the region

$$R_{\beta} = \left\{ \tilde{x} : \frac{\Delta_1}{a^{\beta}(a-1)} < |\tilde{x}(k)| < \frac{\Delta_1}{(a-1)} \right\} \tag{3.16}$$

is the one where β lost packets force the system into instability. Moreover, the extreme cases occur when $\tilde{x}(k) = \Delta_1/(a-1)$ and $\tilde{x}(k) = \Delta_1/(a^{\beta}(a-1))$ and we name these quantities as $\tilde{x}_{sup}$ and $\tilde{x}_{inf}$, respectively. If either of these two extreme cases occur, then after β packet losses we obtain:

$$\tilde{x}(k+\beta) = \begin{cases} \frac{a^{\beta}\Delta_1}{a-1} & \text{for } \tilde{x}_{sup}; \\ \frac{\Delta_1}{a-1} & \text{for } \tilde{x}_{inf}. \end{cases}$$

But this implies that the 2-Bit Delta Modulator will change the value of b_1 and use the product with Δ_2. Since we assume that β is the maximum number of packets that may be consecutively lost, then we know that $b_1(k+\beta+1)$ and $b_2(k+\beta+1)$ will arrive to the decoder and this will also switch to Δ_2. Recalling from Section

3.5, the value of Δ_2 that can match the regions of the two modulators is given by equation (3.8). If the state is located in the worst part of the region, $\tilde{x}(k) = r_2$, and we want to allow β consecutive packet losses without encountering instability, we must guarantee that $a^\beta r_2 \leq r_4$. That is equivalent to

$$a^\beta \frac{\Delta_1}{a-1} \leq \frac{(a+1)\Delta_1}{(a-1)^2}.$$

From this last inequality, the maximum number of packets loss is given by

$$\beta = \left\lfloor \frac{\log_2(\frac{a+1}{a-1})}{\log_2(a)} \right\rfloor. \tag{3.17}$$

Equation (3.17) provides the maximum number of consecutive packets that may be lost when we are in the worst case (subregion of R_β) without losing stability. Obviously, this is an inverse relation in the eigenvalue of the system: the number β of packets that we can afford to lose, is small for a approaching 2 (faster dynamics) and is large for a approaching 1 (slower dynamics). This may be seen in Figure 3.13. It is important to note that because $a \leq 2$, the only way that the state moves out of region R_1 is due to packet losses. Moreover, if we are within R_1, the system may lose up to β packets without going unstable. We know that after β packet losses, the system may end up in region R_2. In R_2 however, we can no longer guarantee that more lost packets are tolerated. The number of lost packets that may be accomodated will obviously vary, according to $\tilde{x}(k)$ and may be determined by an equation similar to (3.14) as follows

$$l_2 \leq \left\lfloor \frac{\log_2\left(\frac{\Delta_2}{(a-1)|\tilde{x}(k)|}\right)}{\log_2(a)} \right\rfloor = \left\lfloor \frac{\log_2\left(\frac{(a+1)\Delta_1}{(a-1)^2|\tilde{x}(k)|}\right)}{\log_2(a)} \right\rfloor. \tag{3.18}$$

This will force the system to return to region R_1 in order to guarantee that β packets may be lost again without losing stability. We illustrate these ideas with the following example.

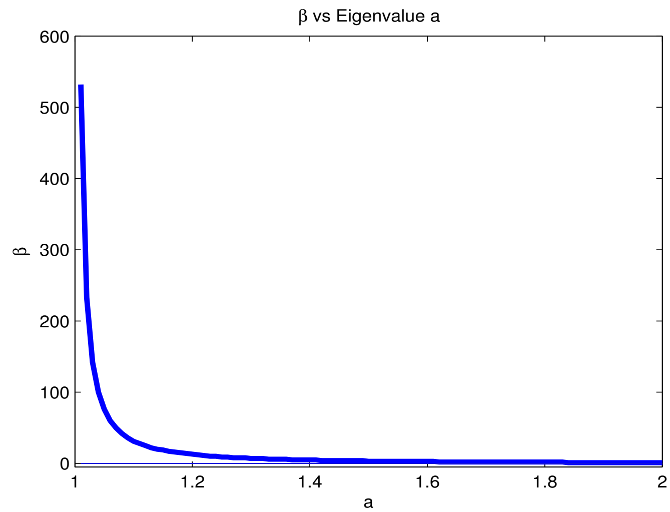

Figure 3.13: β versus Eigenvalue a.

3.7.1 Example 5

Consider the system given by $x(k+1) = 1.2x(k) + u(k)$ with $u(k) = -0.8\hat{x}(k)$. Suppose that $x(0) = 0.547$, $\Delta_1 = 0.2$ and $\Delta_2 = 2.2$ then $r_1 = 0.0909$, $r_2 = 1$, $r_3 = r_2 = 1$, $r_4 = 11$. Then, $R_1 : r_1 \leq \tilde{x}(0) = 0.547 \leq r_2$ and $R_\beta : 0.093 \leq |\tilde{x}| \leq 1$. According to equation (3.17), the β number of packets that may be lost is 13 in any subregion of R_1. We assume that starting at $k = 0$ the system loses its first packet and consecutively loses one packet per sampling time until $k = 12$ (13 lost packets in total). The system then loses no more packets until $k = 20$, where $\tilde{x}(k) \approx 0.02 < 0.093 < 1$ (less than r_2 and out of subregion R_β) where it starts losing a packet per sampling time until $k = 42$, i.e, 22 packets in total, then it continues its operation without suffering any more lost packets. The behavior of the state, $x(k)$ and the estimation error, $\tilde{x}(k)$, are illustrated in Figures 3.14 and 3.15. The circles indicate the instants when the packet losses start, and the rectangles indicate when the packet losses have ended. In Figure 3.15 the limits for region R_1, $\pm r_1$ and $\pm r_2$, are shown as well as the regions added by the second packet, $\pm r_3 = \pm r_2$ and $\pm r_4$. We see that $\beta = 13$ is not the maximum for subregions that are different from R_β.

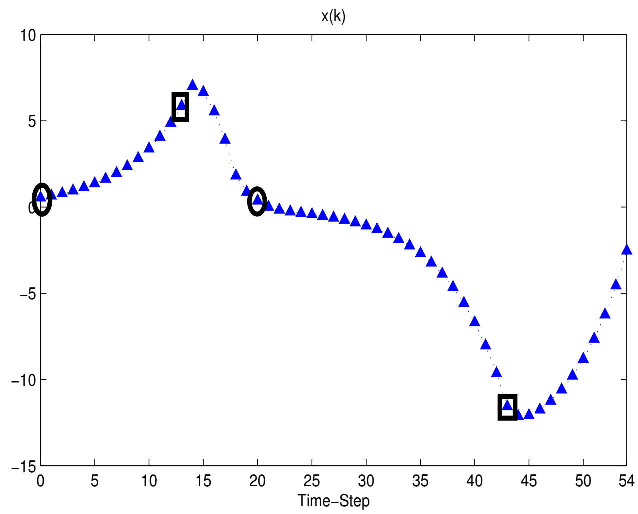

Figure 3.14: $x(k)$ with 13 Packets Losses from $k = 0$ to $k = 12$ and 22 Packets Losses from $k = 20$ to $k = 42$.

3.8 Summary

This chapter has provided extensions to previous results that make it possible to stabilize scalar systems with eigenvalues greater than 2 in magnitude. We then presented a new design of a 2-Bit Delta-Modulator-Like encoder/decoder scheme that keeps the simplicity and desirable characteristics of the 1-Bit scheme. We have also included the effects of lost packets in the channel and showed how to regain equimemory of the encoder and decoder. We determined the number of consecutive packets that may be lost before going into unstability and finally, we presented a 2-Bit Delta-Modulator-Like encoder/decoder scheme that allows us to handle a specific and predetermined number of lost packets.

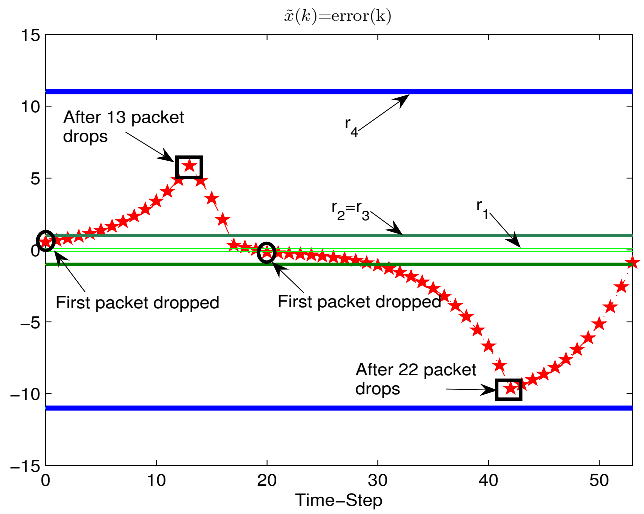

Figure 3.15: Evolution of $\tilde{x}(k)$ with 13 Packet Losses from $k = 0$ to $k = 12$ and 22 Packets Losses from $k = 20$ to $k = 42$.

In the next chapter we present more general encoding schemes that take into account the presence of a network in the controller-actuator path and consider time delays. These schemes will be suitable when the plant is an n-dimensional linear time invariant system instead of a scalar (or diagonizable) one. Moreover, these new schemes achieve asymptotic stabilization and not only boundedness like in the Delta-Modulation scheme proposed in this chapter.

Chapter 4

Rate-Limited Stabilization for Network Control Systems

4.1 Introduction

In Chapter 3 we present a low-complex encoding scheme that is limited by the degree of instability of the plant. In other words, we noted the tradeoff between encoding complexity (measured in terms of the number of bits) and the difficulty of stabilizing an unstable system. In this chapter, we show this tradeoff also exists between the complexity of the algorithm to encode a the transmitted signals versus the number of bits that are transmitted in the channel. The main purpose is to find simple encoding schemes for multidimensional systems that achieve a low data rate. While these schemes may not guarantee the minimum data rate for stabilization, this sacrifice will be compensated by the simplicity of the algorithms.

In [53], an efficient encoder/decoder scheme is proposed to guarantee stabilization of a class of DLTI system using the minimum rate imposed by the Data Rate Theorem. Reference [16] described an encoder/decoder scheme that also achieved the minimum data rate while also considering packet losses. Similarly, reference [29] presents an encoder/decoder scheme that deals with uncertainty in the plant model. It is clear in all of these schemes that the cost of reducing the data rate is the increase in the complexity of the stabilization algorithm and the computational power required for the encoding/decoding operations. There may however be situations where simpler algorithms are preferred, at the expense of requiring a higher data rate. The purpose of this chapter is to provide such simple encoder/decoder schemes that may require higher data rates in order to guarantee asymptotic stability.

The first scheme presented in this chapter is based on ideas proposed in [45], [47] and [48]. The authors of those papers considered a general DLTI system and found a sufficient rate for exponential stabilization of an unstable plant of order n, under the rather limiting assumption that the system has n inputs (where n is the number of states) and an invertible input matrix B. The work addressed finite rate issues, packet dropping, as well as uncertainties in the plant model. Moreover, the authors assumed the existence of a truncation-based encoder/decoder without providing its specific structure.

We extend the results of [45] to the case of DLTI systems with m inputs such that $m \leq n$, where n is the order of the system. We also relax the condition of the invertibility of the B matrix, and extend the stabilizability results to systems with a constant time-delay induced by the sensor-to-controller network. Moreover, we present an easily implementable encoder/decoder structure. As was considered in [45], we discuss two types of NCS: one that includes a network between the sensors and the controller, and another that models two networks in the loop, one between the sensors and controller, and another between the controller and the actuator. Sections 4.2-4.8 of this work are an extension to the preliminary results we presented in [19] and [20].

Finally, we also propose a zoom-in-type dynamic quantizer scheme with lower data rate but a more complex encoding scheme than the truncation-based one. The new dynamic quantizer requires a lower data rate to achieve stabilization, and while it does not achieve the minimum data rate given by the Data Rate Theorem, it uses an encoding algorithm that is simpler than others reported in [16], [29] and [53]. Examples and simulations are provided in Section 4.8 to illustrate the results.

4.2 Problem Setup

We consider the two configurations for the packet-based NCS presented in [45]. They are general systems that can modeled several real scenarios such as the Internet. The first system is referred to as *Network Control System Type I* has a rate of R_{p_1} *packets/time-step*. This packet-based network accomodates a packet size of D_{Max} bits used for data (although the protocol information requires extra bits in the packet, it is not needed for this analysis). Let us consider the discrete LTI system shown in Figure 4.1 and described by

$$x(k+1) = Ax(k) + Bu(k); \tag{4.1}$$

where A is $n \times n$, B is $n \times m$ and $u(k)$ is $m \times 1$. The second type of packet-based network, referred to as *Network Control System Type II*, consists of the same discrete LTI system given by equation (4.1), but with the addition of a second network between the controller and the actuator with rate R_{p_2} as shown in Figure 4.2. From here on, the following notations are adopted. The norm symbol ($\|.\|$) denotes the Euclidean norm and $\lceil . \rceil$ is the **ceiling** function. In addition, we use the variable μ to denote the controllability index which for multivariable linear systems [1] is defined as the least integer k such that

$$\text{rank} \begin{bmatrix} B| & AB| & \dots| & A^{k-1}B \end{bmatrix} = n. \tag{4.2}$$

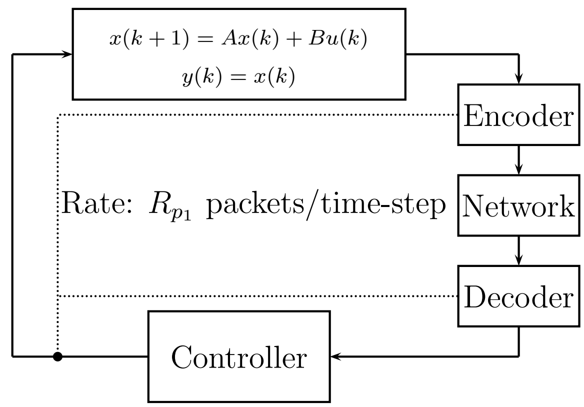

Figure 4.1: Closed-Loop NCS: Type I.

We assume that the controller does not saturate, and that the packet-network does not drop packets nor is it subjected to disturbances (noise). For both NCS types, we assume that the states can be measured. We also assume the decoder knows exactly the encoding scheme used by the encoder at all times (equimemory property), as described in Section 4.3. The last assumption is that the encoder and decoder know a value $L_0 > 0$ such that $\|x(0)\| < L_0$ and that both have access to the control signal or can compute it as represented by a dotted line in Figures 4.1 and 4.2.

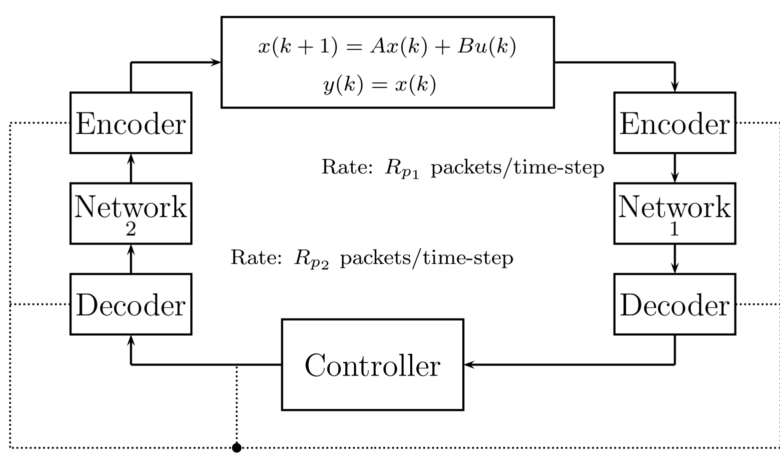

Figure 4.2: Closed-Loop NCS: Type II.

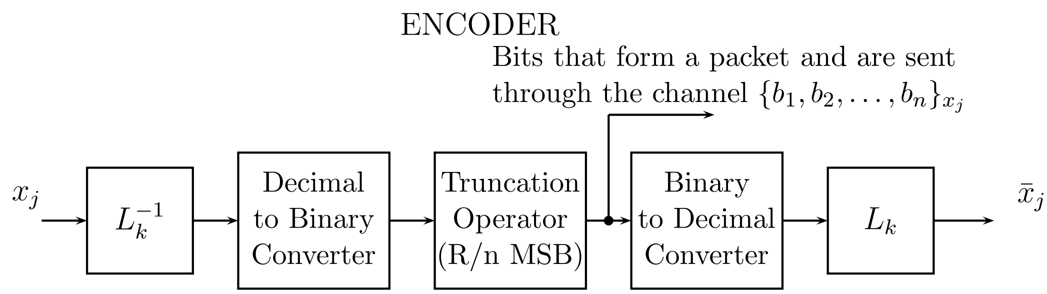

Figure 4.3: Encoder Scheme.

4.3 Encoder-Decoder Design

Several approaches for the design of an encoder/decoder scheme were presented in previous works and in the previous chapter. Most of them are based on some type of predictor that emulates the evolution of the plant state and the difference between this prediction and the actual state of the plant, i.e., the error. The quantized error is sent through the channel, then decoded at the receiver and used to obtain an approximation of the state, which is used to generate the control signal. In our case however, we send a truncated version of every state component rather than the error using a modified version of the encoder/decoder scheme proposed in [42]. One of the advantages of this approach is that it can be applied not only to the case of a constraint transmission channel (or network) but to the quantization issue in the data acquisition system attached to the sensors. Figures 4.3 and 4.4 illustrate our scheme which is described next in detail. At the first instant, $k = 0$, the sensor measures the state exactly. Since we assume that both the encoder and decoder know L_0, each component x_j of the measured state is divided by L_0 which gives a number x_j/L_0 that is less than or equal to 1 in magnitude. We assume for now that x_j/L_0 is strictly less than one and positive (in Section 4.4 we describe how to proceed if x_j/L_0 is exactly 1 or negative). The encoder converts this x_j/L_0 to its binary representation

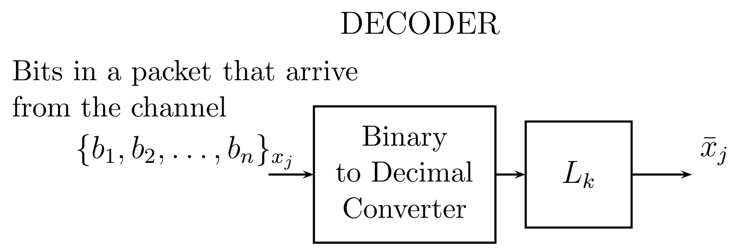

Figure 4.4: Decoder Scheme.

and keeps only the r_j most significant bits (MSB). This truncated version is labeled as $\left(\frac{x_j(0)}{L_0}\right)_{T_{r_j}}$, where the symbol $()_{T_{r_j}}$ denotes the truncation operation that retains the r_j most significant bits. The quantity r_j will be calculated in Section 4.4. The decimal representation of these r_j bits is multiplied by L_0 resulting in an estimate $\bar{x}_j(0) = \left(\frac{x_j(0)}{L_0}\right)_{T_{r_j}} L_0$ which is stored in the encoder. By grouping into a vector the j truncated state components, we obtain the state estimate $\bar{x}(0)$. The bits in each truncated state component form a packet (or packets depending on D_{Max}) that is sent through the channel. On the receiver side, the decoder receives a packet (or packets) and separates the bits that correspond to each state component. Assuming perfect transmission, the decoder then converts the binary representation of the bits received into a decimal representation and multiplies by L_0 which gives the value $\bar{x}_j(0)$. This should result in the same value stored in the encoder and, therefore, the equimemory property between encoder and decoder is preserved. Since the control signal at time $k = 1$ only depends on $\bar{x}(0)$, we can show that at time $k = 1$, $x_j(1)$, is bounded as follows. Using the triangle inequality and matrix norm properties we have:

$$\begin{aligned}
\|x(1)\| &\leq \|Ax(0) + Bu(\bar{x}(0))\|; \\
&\leq \|A\|\|x(0)\| + \|Bu(\bar{x}(0))\|; \\
&\leq \|A\|L_0 + \|Bu(\bar{x}(0))\|; \\
&= L_1.
\end{aligned}$$

Since the control algorithm is predefined, the encoder and decoder can both calculate this value L_1 right after they have calculated the value $\bar{x}(0)$. The stored L_1 will then be used at instant $k = 1$ to keep the ratio $|x(1)/L_1| \leq 1$. By carefully examining the above steps, we obtain the following scalar difference equation to bound the norm of each state component:

$$L_k = \|A\|L_{k-1} + \|Bu(\bar{x}(k-1))\|, \ \forall k = \{1, \ldots, \mu\}. \tag{4.3}$$

Since equation (4.3) only depends on the terms L_{k-1} and $\bar{x}(k-1)$, all signals needed to compute this equation are available at the encoder and the decoder. In Section 4.4 we will see that L_k only evolves for μ time-steps, before it is reset to a new starting value for the next μ time-steps and this is the reason to limit k to a maximum of μ in equation (4.3).

4.4 Results: Truncation-Based Encoding Scheme

4.4.1 Network Control System: Type I

In the case of NCS Type I, the state vector $x(k)$ is given by

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}.$$

We assume below that $x_j(k) > 0, \ \forall j$ since the sign of each state component may later be accounted for by adding n extra bits to the rate (one extra bit per each state component sign). We then obtain the following binary representation of $\frac{x(0)}{L_0}$ at the encoder side:

$$\frac{x(0)}{L_0} = \begin{bmatrix} \frac{x_1(0)}{L_0} \\ \frac{x_2(0)}{L_0} \\ \vdots \\ \frac{x_n(0)}{L_0} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{\infty} \alpha_{1i} 2^{-i} \\ \sum_{i=1}^{\infty} \alpha_{2i} 2^{-i} \\ \vdots \\ \sum_{i=1}^{\infty} \alpha_{ni} 2^{-i} \end{bmatrix}; \tag{4.4}$$

where $\alpha_{ij} \in \{0, 1\}$. This binary representation is truncated keeping only the r_j most significant bits for state component x_j. The truncated representation is given by:

$$\left(\frac{x(0)}{L_0}\right)_{T_{r_j}} = \begin{bmatrix} \left(\frac{x_1(0)}{L_0}\right)_{T_{r_j}} \\ \left(\frac{x_2(0)}{L_0}\right)_{T_{r_j}} \\ \vdots \\ \left(\frac{x_n(0)}{L_0}\right)_{T_{r_j}} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{r_1} \alpha_{1i} 2^{-i} \\ \sum_{i=1}^{r_2} \alpha_{2i} 2^{-i} \\ \vdots \\ \sum_{i=1}^{r_n} \alpha_{ni} 2^{-i} \end{bmatrix}; \tag{4.5}$$

where $\alpha_{ij} \in \{0, 1\}$. The r_j bits per state component j are sent through the channel and, at the receiver site, the decoder transforms the bits back into decimal numbers, and multiplies them by L_0 in order to obtain $\bar{x}(0)$. With this encoding/decoding process, we guarantee that the error between the actual state component and its encoded version, $\epsilon_j(0) = x_j(0) - \bar{x}_j(0)$, is limited by $\|\epsilon_j(0)\| < 2^{-r_j} L_0$, $\forall\ j \in \{0, 1, \ldots, n\}$. Using the triangle inequality, the norm of the total error is bounded by

$$\|\epsilon(0)\| \leqslant \sqrt{\sum_{j=1}^{n} 2^{-2r_j}} L_0. \tag{4.6}$$

Let us then consider the evolution of the system starting at time $k = 0$:

$$\begin{aligned} x(1) &= Ax(0) + Bu(0); \\ x(2) &= Ax(1) + Bu(1); \\ &= A^2x(0) + ABu(0) + Bu(1); \\ &\vdots \\ x(l) &= A^l x(0) + \sum_{i=1}^{l} A^{l-i}Bu(i-1); \quad \forall l \geq 3. \end{aligned}$$

Recalling that μ represents the controllability index, after μ steps we have

$$x(\mu) = A^\mu x(0) + A^{\mu-1}Bu(0) + A^{\mu-2}Bu(1) + \ldots + Bu(\mu-1).$$

This equation may be re-arranged as $x(\mu) = A^\mu x(0) + \zeta_\mu \mathbb{U}$, where

$$\begin{aligned} \zeta_\mu &= \begin{bmatrix} B| & AB| & \ldots| & A^{\mu-1}B \end{bmatrix}; \\ &= \begin{bmatrix} \delta_1| & \delta_2| & \ldots \delta_j| & \ldots| & \delta_\mu \end{bmatrix}; \end{aligned}$$

and

$$\mathbb{U} = \begin{bmatrix} u(\mu-1) \\ \vdots \\ u(0) \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_j \\ \vdots \\ u_\mu; \end{bmatrix};$$

noting that δ_j is the jth column in ζ_μ and u_j is the jth element in the vector $\mathbb{U}$. Let us select the first n independent columns of ζ_μ and build a new matrix, called ζ_n. Let us also select the elements of $\mathbb{U}$ corresponding to the columns chosen from ζ_μ and form a new vector, called $\mathbb{U}_n$. Recalling that $x(0) = \bar{x}(0) + \epsilon(0)$ we have $x(\mu) = A^\mu \bar{x}(0) + A^\mu \epsilon(0) + \zeta_\mu \mathbb{U}$. If we choose the control law

$$\mathbb{U}_n = -\zeta_n^{-1} A^\mu \bar{x}(0); \tag{4.7}$$

we may reconstruct $\mathbb{U}$ by replacing u_j with the corresponding values of $\mathbb{U}_n$ in the proper order and letting $u_j = 0$ for the remaining elements. After μ steps, and by applying the control sequence $\mathbb{U}$ we obtain

$$x(\mu) = A^{\mu}\epsilon(0). \tag{4.8}$$

Then, from equations (4.6), (4.8), and the properties of matrix norms, we obtain

$$\begin{aligned} \|x(\mu)\| &= \|A^{\mu}\epsilon(0)\|; \\ &\leqslant \|A^{\mu}\|\|\epsilon(0)\|; \\ &\leqslant \|A^{\mu}\|\sqrt{\sum_{j=1}^{n} 2^{-2r_j}}L_0. \end{aligned}$$

In order to force the state to decrease in the norm (after μ steps), we shrink the upper bound of the state $x(\mu)$ by forcing it to be less than a fraction of the upper bound of the state $x(0)$, i.e., $\|A^{\mu}\|\sqrt{\sum_{j=1}^{n} 2^{-2r_j}}L_0 < \frac{L_0}{\delta}$, for some $\delta > 1$. At this point, we have to decide on the value of each r_j. This may be formulated as an optimization problem whose objective is to minimize the total rate given by $\sum_{j=1}^{n} r_j$. In other words, let us consider the optimization problem:

$$\min_{r_j} \sum_{j=1}^{n} r_j \tag{4.9}$$

subject to

$$\sqrt{\sum_{j=1}^{n} 2^{-2r_j}} < \frac{1}{\delta\|A^{\mu}\|} = C_*. \tag{4.10}$$

This problem may be solved by applying the Karush-Kuhn-Tucker (KKT) conditions [35] to the Lagrangian function $L(r_1, r_2, \ldots, r_n, l)$ with Lagrange multiplier l as given by

$$L = r_1 + r_2 + \ldots + r_n - l(C_* - \sqrt{2^{-2r_1} + 2^{-2r_2} + \ldots + 2^{-2r_n}}).$$

The KKT conditions are then:

$$\begin{aligned}
\frac{\partial L}{\partial r_1} &= 1 - l\frac{2^{-2r_1}\ln(2)}{\sqrt{2^{-2r_1}+2^{-2r_2}+\ldots+2^{-2r_n}}} = 0; \\
\frac{\partial L}{\partial r_2} &= 1 - l\frac{2^{-2r_2}\ln(2)}{\sqrt{2^{-2r_1}+2^{-2r_2}+\ldots+2^{-2r_n}}} = 0; \\
&\vdots \\
\frac{\partial L}{\partial r_n} &= 1 - l\frac{2^{-2r_n}\ln(2)}{\sqrt{2^{-2r_1}+2^{-2r_2}+\ldots+2^{-2r_n}}} = 0.
\end{aligned}$$

Solving this system of n equations, we obtain:

$$r_1 = r_2 = \ldots = r_j = \ldots = r_n.$$

Therefore, an equal allocation of bits per each state component actually guarantees the minimum total rate. Using the constraint (4.10) we obtain the optimal rate allocation $r_n > \left\lceil \log_2(\|A^\mu\|) + \frac{1}{2}\log_2(n) + \log_2(\delta) \right\rceil$. We notice that δ is a parameter that determines the fraction by which the upper bound of $\|x(0)\|$ is shrinking. Therefore, it is sufficient to consider the *infimum* of this quantity to obtain $r_n > \left\lceil \log_2(\|A^\mu\|) + \frac{1}{2}\log_2(n) \right\rceil$. Note that the $\lceil . \rceil$ function was introduced since r_n must be an integer denoting the number of bits for each state component. We can therefore define the total R bits in a packet (or packets) as $R = nr_n + n$ where the second n term may be used to code the sign of each state component.

For the next μ steps, we repeat the same steps above but using $x(\mu)$ as the initial condition. To stop the growth of L_k, and noting that $\|x(\mu)\| < n\|A^\mu\|2^{-r_n}L_0$, we assign $L_\mu = n\|A^\mu\|2^{-r_n}L_0$ as the new L_0 for the next μ time steps in equation (4.3). We repeat this procedure every μ steps. Using the same algorithm to generate the control sequence and the same rate R, the state $x(2\mu)$ will be a shrunken version of $x(\mu)$. Proceeding in the same fashion, $x(t\mu)$ will tend to zero as $t \in \mathbb{N}$ grows and, therefore, the state x will tend to zero and asymptotic stabilizability will be achieved. Note that R is the sufficient number of effective bits that we need to transmit for the whole state to guarantee stabilization, but since a packet has a maximum length D_{Max}, if $R \leq D_{Max}$, we need a packet rate of $R_{p_1} = 1$ packet/sample-time. If on the other hand, $R > D_{Max}$ then, a minimum of $\left\lceil \frac{R}{D_{Max}} \right\rceil$ packets/time-step are needed. Note that the last expression actually covers both cases, since $\frac{R}{D_{Max}} < 1$ gives a 1 packet/sample-time when the ceil function is applied.

■

This analysis may be summarized in the following theorem.

Theorem 4.4.1 *Assuming an equal allocation of bits per state component, a network rate R_{p_1} packets/time-step, and assuming that (A, B) is a controllable pair with controllability index μ, a sufficient condition for system (4.1) to be asymptotically stabilizable is*

$$R_{p_1} = \left\lceil \frac{R}{D_{Max}} \right\rceil,$$

where $R > n \left\lceil \log_2 (\|A^\mu\|) + \frac{1}{2}\log_2(n) \right\rceil + n$ and every state allocates $\frac{R}{n}$ bits/time-step.

An immediate consequence of Theorem 4.4.1 in the specific case of a single input system is given in the following corollary.

Corollary 4.4.1 *Assuming an equal allocation of bits per state component, a network rate R_{p_1} packets/time-step, (A, B) is a controllable pair, and B is $n \times 1$, a sufficient condition for system (4.1) to be asymptotically stabilizable is*

$$R_{p_1} = \left\lceil \frac{R}{D_{Max}} \right\rceil;$$

where $R > n \left\lceil \log_2 (\|A^n\|) + \frac{1}{2}\log_2(n) \right\rceil + n$ and every state allocates $\frac{R}{n}$ bits/sample.

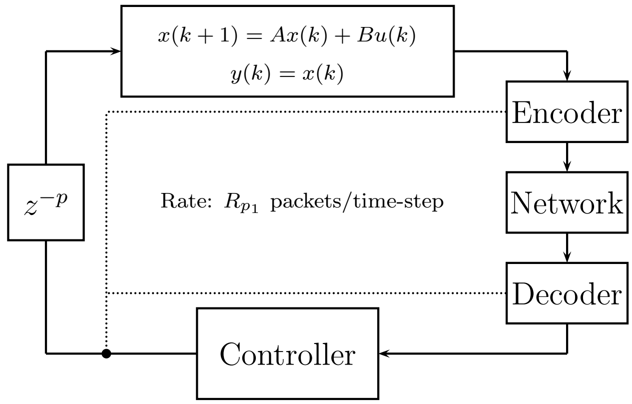

Figure 4.5: Closed-loop NCS Type I with time-delay.

Proof: The proof is the same as that of Theorem 4.4.1. If B is $n \times 1$ and $u(k)$ is 1×1, then $\mu = n$. Substituting μ in R in the proof of Theorem 4.4.1, we obtain the rate given by the corollary.

■

4.4.2 Network Control System Type I with Time Delay

One of our motivations for extending the results of [45], is to account for the effects of time delays that may be present in the network. As mentioned earlier, even for the scalar case, the invertibility requirement of B would not allow the traditional augmentation of the state by its delayed versions. Let us consider the modified NCS type I shown in Figure 4.5 and the DLTI system given by the following equation:

$$x(k+1) = Ax(k) + Bu(k-p); \tag{4.11}$$

where A is $n \times n$, B is $n \times 1$ and $u(k)$ is 1×1. We assume here that the control signal to actuator delay is a constant equal to $p \in \mathbb{N}$ time-steps. Under such conditions, we obtain the following theorem:

Theorem 4.4.2 *Assuming an equal allocation of bits per state component, a network rate of* $R_{p_1} = \left\lceil \frac{R}{D_{Max}} \right\rceil$ *packets/time-step, and*

$$\mathbb{A} = \begin{bmatrix} A & B & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & & & & 1 \\ 0 & 0 & \vdots & \dots & 0 \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

such that $(\mathbb{A}, \mathbb{B})$ *is a controllable pair. A sufficient condition for system (4.11) to be asymptotically stabilizable is*

$$R_{p_1} = \left\lceil \frac{R}{D_{Max}} \right\rceil;$$

where $R > (n+p) \left\lceil \log_2 \left(\|\mathbb{A}^{n+p}\| \right) + \frac{1}{2} \log_2(n+p) \right\rceil + (n+p)$*, and each state component of the augmented system allocates* $\frac{R}{n+p}$ *bits/time-step.*

Proof: Similarly to [58] and [63], we start out by augmenting the state vector, considering as new states the last p previous inputs. We then obtain

$$\begin{aligned} \mathbb{X}(k+1) &= \begin{bmatrix} \mathbf{x}(k+1) \\ x_{n+1}(k+1) \\ x_{n+2}(k+1) \\ \vdots \\ x_{n+p}(k+1) \end{bmatrix}; \\ &= \begin{bmatrix} A & B & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ & & & & 1 \\ 0 & 0 & \vdots & \dots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ x_{n+1}(k) \\ x_{n+2}(k) \\ \vdots \\ x_{n+p}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k). \end{aligned}$$

This may be written as

$$\mathbb{X}(k+1) = \mathbb{A}\mathbb{X}(k) + \mathbb{B}u(k).$$

We now have a system similar to the one treated in Corollary 4.4.1 with a state dimension $n+p$ instead of n. Therefore, in order to shrink the upper bound of the state $\mathbb{X}(k+n+p)$ we need a rate R given by

$$\frac{R}{n+p} \geqslant \left\lceil \log_2(\|\mathbb{A}^{n+p}\|) + \frac{1}{2}\log_2(n+p) \right\rceil + 1.$$

Similarly to previous proofs, we find a minimum rate of $R_{p_1} = \left\lceil \frac{R}{D_{Max}} \right\rceil$ packets/time-step.

■

4.4.3 Network Control System: Type II

We now consider an NCS Type II and present the following result.

Theorem 4.4.3 *Assume an equal allocation of bits per state component, network rates of* $R_{p_1} = \left\lceil \frac{R_1+n}{D_{Max}} \right\rceil$ *packets/time-step and* $R_{p_2} = \left\lceil \frac{R_2+1}{D_{Max}} \right\rceil$ *packets/time-step for network 1 and 2, respectively. Assuming also that* (A, B) *is a controllable pair, where* B *is* $n \times 1$*, the controllability matrix is given by* $\zeta = \begin{bmatrix} B| & AB| & \ldots| & A^{n-1}B \end{bmatrix}$*, a sufficient condition for system (4.1) to be asymptotically stabilizable is*

$$n\,\|A^n\|\,2^{-(\frac{R_1}{n}+n)} + \|\zeta\|\,\left\|\zeta^{-1}A\right\|\,2^{-(R_2+1)} < 1.$$

Proof: Since there is now a rate constraint from the controller to the plant actuators, we can no longer apply the calculated control signal $u(k)$ directly to the plant. Instead, only the bits encoding $u(k)$ according to the available rate R_2 may be used. This encoded control signal $\widetilde{u}(k)$ is the one that is received by the plant. We then have

$$x(k+1) = Ax(k) + B\widetilde{u}(k).$$

Let us assume that we have exactly the same encoding and decoding schemes used in Theorem 4.4.1. The evolution of the system in the first n time steps is given by $x(n) = A^n x(0) + \zeta\widetilde{\mathbb{U}}$, where $\widetilde{\mathbb{U}} = \begin{bmatrix}\widetilde{u}(n-1) & \dots & \widetilde{u}(0)\end{bmatrix}'$. If we choose the control signal $\mathbb{U} = -\zeta^{-1}A^n\bar{x}(0)$, then $\|\mathbb{U}\| \leqslant \|\zeta^{-1}A^n L_0\| \leq \|\zeta^{-1}A^n\| L_0 = L2_0$. For other time k, the normalization value that is kept in the memory of the encoder/decoder of network II, i.e. $L2_k$, is given by $L2_k = \|\zeta^{-1}A^n\| L_k$. Since $\widetilde{u}(k)$ represents the R_2 most significant bits of $u(k)$ we know that

$$\left\|\mathbb{U} - \widetilde{\mathbb{U}}\right\| \leqslant \left\|\zeta^{-1}A^n\right\| L_0 2^{-R_2}. \tag{4.12}$$

From equation (4.12) and recalling that $x(0) = \bar{x}(0) + \epsilon(0)$ and $\|\epsilon(0)\| < \sqrt{n}L_0 2^{-\frac{R_1}{n}}$, we have

$$\begin{aligned}
\|x(n)\| &= \left\|A^n\bar{x}(0) + A^n\epsilon(0) + \zeta\widetilde{\mathbb{U}}\right\|; \\
&= \left\|\zeta\left(\zeta^{-1}A^n\bar{x}(0) + \widetilde{\mathbb{U}}\right) + A^n\epsilon(0)\right\|; \\
&\leqslant \|\zeta\| \left\|\mathbb{U} - \widetilde{\mathbb{U}}\right\| + \|A^n\epsilon(0)\|; \\
&\leqslant \|\zeta\| \left\|\zeta^{-1}A\right\| L_0 2^{-R_2} + \sqrt{n}\,\|A^n\|\, L_0 2^{-\frac{R_1}{n}}; \\
&< \frac{L_0}{\delta}.
\end{aligned}$$

To guarantee the decrease of $x(n)$, we force

$$\|\zeta\| \left\|\zeta^{-1}A\right\| L_0 2^{-R2} + \sqrt{n}L_0 \|A^n\|\, 2^{-\frac{R_1}{n}} < L_0,$$

i.e., $\sqrt{n}\,\|A^n\|\,2^{-\frac{R_1}{n}} + \|\zeta\|\,\|\zeta^{-1}A\|\,2^{-R_2} < 1$. As in previous proofs, we now select $x(n)$ as the new initial condition and using the same control law and rates, R_1 and R_2, the state $x(2n)$ will be a shrunken version of $x(n)$. Continuing in the same fashion, $x(tn)$ will tend to zero as $t \in \mathbb{N}$ grows and, therefore $x(k)$ will tend to zero and asymptotic stability is achieved. To take into account the sign of the state we add n bits to R_1, one per state component. We will we will need a minimum of $R_{p_1} = \left\lceil \frac{R_1}{D_{Max}} \right\rceil$ packets/time-step for the sensor-controller network and a minimum of $R_{p_2} = \left\lceil \frac{R_2}{D_{Max}} \right\rceil$ packets/time-step in the controller-actuator network.

■

4.5 Removing the Rate Dependency on $\|A\|$

The result of Theorem 4.4.1 (as well as Corollary 4.4.1 and Theorem 4.4.2) established a sufficient rate in terms of the norm of A. For different matrices A with the same eigenvalues however, this may lead to very different rates, some of which may also be very large compared to the minimum rates specified by the Data Rate Theorem. For example, the following two matrices A have the same eigenvalues (therefore, the same minimum stabilization rate according to the Data Rate Theorem) but different norms (therefore, different sufficient rates according to Theorem 4.4.1):

$$A_1 = \begin{bmatrix} 2 & 100000 \\ 0 & 2 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Then $\|A_1\| = 1 \times 10^5$. and $\|A_2\| = 2$ but A_1 and A_2 have the same eigenvalues $\lambda = \{2, 2\}$. One way to remove this disadvantage is to modify the control law used in the proof of Theorem 4.4.1. Instead of trying to asymptotically stabilize the state x, we attempt to stabilize the state $z = \Phi^{-1}x$, where Φ is a linear transformation such that $\Phi^{-1}A\Phi$ is the diagonal matrix equivalent to A (or more generally the Jordan-block matrix). The error $\epsilon_z(0)$ in the z space is given by $\Phi^{-1}(x_j(0) - \bar{x}_j(0))$. For stabilization analysis purposes, designing a control law to stabilize the state z is equivalent to stabilizing x since $z \to 0$ implies $x \to 0$. There will however be a difference in the transient response as we will see later. The change of variable implies that the control law in equation (4.7) no longer depends on the controllability matrix of the pair (A, B), i.e. ζ_μ. But will instead depend on the controllability matrix of the pair $(\Phi^{-1}A\Phi, \Phi B)$, denoted by ζ_{Φ_μ}. Therefore, the new control law is given by

$$\mathbb{U}_n = -\zeta_{\Phi_\mu}^{-1}(\Phi^{-1}A\Phi)^\mu \Phi^{-1}\bar{x}(0); \tag{4.13}$$

and in the z space, after μ time-steps, we will have

$$z(\mu) = (\Phi^{-1}A\Phi)^\mu \epsilon_z(0). \tag{4.14}$$

Then, from equations (4.6) and (4.14), and using the properties of matrix norms, we obtain

$$\begin{aligned}
\|z(\mu)\| &= \|(\Phi^{-1}A\Phi)^\mu \epsilon_z(0)\|; \\
&\leqslant \|(\Phi^{-1}A\Phi)^\mu\| \|\epsilon_z(0)\|; \\
&\leqslant \sqrt{n}2^{-r_n} \|(\Phi^{-1}A\Phi)^\mu\| \|\Phi^{-1}\| L_0.
\end{aligned}$$

Similarly, in order to force the state z to decrease in the norm (after μ steps), we shrink the upper bound of the state $z(\mu)$ by forcing it to be less than the lower bound of the state $z(0)$, i.e., $2^{-R_n}\sqrt{n}\|(\Phi^{-1}A\Phi)^\mu\|\|\Phi^{-1}\|L_0 < \|\Phi^{-1}\|L_0$. However, if $\Phi^{-1}A\Phi$ is a diagonal matrix then $\|(\Phi^{-1}A\Phi)^\mu\| = |\rho(A)|^\mu$ where $\rho(A)$ is the spectral radius of A. We can then replace in Theorem 4.4.1 the expression $R > n\left\lceil \log_2(\|A^\mu\|) + \frac{1}{2}\log_2(n)\right\rceil + n$ with

$$R > n\left\lceil \log_2(\rho(A)^\mu) + \frac{1}{2}\log_2(n)\right\rceil + n. \tag{4.15}$$

If matrix $\Phi^{-1}A\Phi$ is a Jordan-block matrix (for the case of repeated eigenvalues of A), we also know that $\|(\Phi^{-1}A\Phi)^{\mu}\| \approx |\rho(A)|^{\mu}$. This quantity, in general, is less than $\|A^{\mu}\|$. We can consider as an approximation that the rate is no longer a function of the norm of A but rather a function of $\rho(A)$. Therefore, this leads to a lower sufficient rate for stabilizability, but with the possible deterioration in the transient response.

4.6 New Encoder/Decoder Design: A Zoom-In Type Dynamic Quantizer

In Section 4.4 we obtained sufficient stabilization rates with an easily implementable encoder/decoder scheme for Network Type I. Although such rates are larger than the ones given by the Data Rate Theorem, the implementation of the truncation-based scheme requires less computational power than other published schemes. Specifically, the evolution of the quantizer in our scheme uses one scalar equation (equation (4.3)). On the other hand, encoder-decoder schemes such as the ones proposed in [16] or [53] achieve the minimum rate established by the Data Rate Theorem at the expense of a higher computational cost since they require state-space predictors, the use similarity transformation (to undo the rotations caused by the A matrix), and the calculation of the centroid of the region that traps the state space variables. In some scenarios, both the computational power and the rate may be constrained. Our purpose in this section is to design an encoder/decoder scheme that achieves a rate close to that provided by the Data Rate Theorem, while using less computational power. The following builds upon ideas described in [29], [16], [53].

4.6.1 Encoder-Decoder Design

Let the initial state be bounded by some value L_0, i.e. $\|x(0)\| \leq L_0$. This hypercube will have 2^n vertices. The set of 2^n vertices is denoted by V_0, and each vertex is denoted by, v_0. We allocate r_i bits for the state space component x_i, $\forall i \in \{1, 2, \ldots, n\}$. We introduce a matrix Q_R:

$$Q_R = \begin{bmatrix} 1/2^{r_1} & 0 & \ldots & 0 \\ 0 & 1/2^{r_2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1/2^{r_n} \end{bmatrix}. \tag{4.16}$$

Moreover, we will assume that $r_1, r_2, \ldots, r_n$ are such that the matrix $A_Q = AQ_R$ is a stable matrix (we will show later how to accomplish this). In the following steps, we focus on the analysis problem and assume that the plant is deterministic and undriven, described by $x(k+1) = Ax(k)$. The controller design problem will be discussed in Subsection 4.6.2. The first step is to generate a n dimensional cube centered at the origin with sides of length $2L_0$. The center of this first quantizer will be labeled $C_Q(0)$. The uncertainty region is divided in 2^{r_1} subregions in the x_1 direction, 2^{r_2} subregions in the x_2 direction, and so on until we obtain 2^{r_n} subregions for the x_n direction. After one time step, the state will land in one of these smaller n dimensional cubes and the total of small cubes will be $2^{r_1+r_2+\ldots+r_n}$. Therefore, the number of bits needed to represent all the cube centroids is $R = r_1 + r_2 + \ldots + r_n$ which is the actual rate in bits/time-step. After determining in which cube the state has landed, we calculate the centroid of this smaller cube. This centroid will be chosen by the encoder as the estimate of the state, $\bar{x}(0)$. The binary symbol, s, that represents $\bar{x}(0)$ is transmitted to the receiver. Note that the error between the state and the state estimate, $\epsilon(0)$, lies in the region $\{[-L_0/2^{r_1}, L_0/2^{r_1}], [-L_0/2^{r_2}, L_0/2^{r_2}], \ldots, [-L_0/2^{r_n}, L_0/2^{r_n}]\}$. This is the key property of this quantizer. Figure (4.6) shows an example of a two dimensional quantizer with $r_1 = 2$ and $r_2 = 1$. The encoder and decoder will evolve

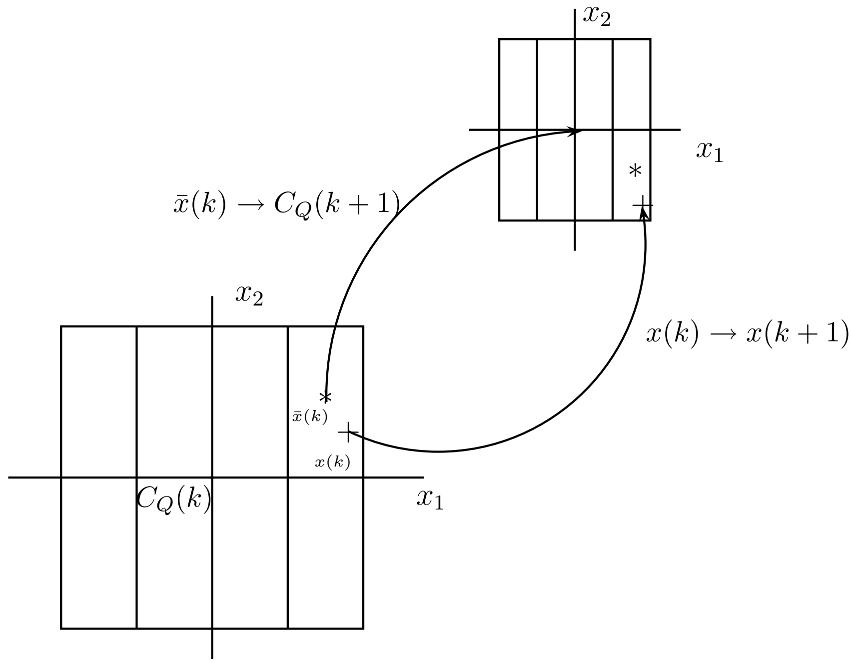

Figure 4.6: Quantizer Evolution Sample: Centroid, State and State Estimator.

the center of the quantizer, C_Q at time $k+1$:

$$C_Q(k+1) = A\bar{x}(k). \tag{4.17}$$

This new center is used to generate an uncertainty region that may be divided into another $2^{r_1+r_2+\ldots+r_n}$ subregions with the same 2^{r_i} subregions in the x_i direction as explained before. At time $k+1$, the length of each of the sides parallel to x_i is determined by the quantity Δ_{x_i}. These Δ_{x_i} quantities are determined using the matrix A_Q and the vertices v_0 of the original uncertain n-dimensional cube and given by:

$$\Delta_{x_i} = \max_{v_0} |(A_{Q,i})^{k+1} v_0|, \forall\ v_0 \in V_0. \tag{4.18}$$

where $A_{Q,i}$ is the "i-th" row of matrix A_Q. Equation (4.18) evaluates the maximum over absolute values, therefore, we can guarantee that the state $x(k+1)$ at time $k+1$ will land in an n-dimensional box (not necessarily a cube) that is centered on

$C_Q(k+1)$ and with sides of length $2\Delta_{x_i}$ in the x_i direction. In other words, the hyper-planes that are perpendicular to the x_i component direction will be located at $-\Delta_{x_i}$ and Δ_{x_i} units from $C_Q(k+1)$ in the x_i direction. The new uncertainty box, will again be divided into $2^{r_1+r_2+\ldots+r_n}$ boxes with 2^{r_i} in the x_i direction. We label these small boxes with binary symbols (a total of $2^{r_1+r_2+\ldots+r_n}$ binary symbols). We then determine in which of these boxes the actual state, $x(k+1)$, lies and use the centroid of this specific box as the state estimate $\bar{x}(k+1)$ at time $k+1$. We again transmit the binary symbol, s, that corresponds to the box where the state lies. Because of the way we have constructed this quantizer and since A_Q was assumed to be stable, the uncertainty box keeps on shrinking as k tends to infinity, which guarantees that our state estimate reaches the actual state and that $\|\epsilon\|$ tends to zero. Note that both encoder and decoder must know the original size L_0 of the uncertainty as well as the exact dynamics of the plant. Also, both encoder and decoder must be able to compute equations (4.17) and (4.18). This guarantees equimemory. The only remaining issue is to guarantee that A_Q is stable. This may be done by the following procedure:

1. Set $r_i = \lceil \log_2(|\lambda_i|) \rceil \; \forall i \in \{1, 2, \ldots, n\}$, where λ_i is any of the n eigenvalues of A such that for all $i \neq j$ the eigenvalue chosen is different. For the particular case where A is diagonal or a Jordan block matrix, then r_i is chosen to be $r_i \geq \lceil \log_2(|\lambda_i|) \rceil$, where λ_i is the eigenvalue associated with the state space component x_i.

2. Using rates r_i, form the matrix Q_R and obtain the eigenvalues of $A_Q = AQ_R$.

3. Check that all such eigenvalues are inside the unit circle, i.e., $|\lambda_{A_Q}| < 1$.

4. If $|\lambda_{A_Q}| < 1$, stop and use the rates r_i for transmission. If for any eigenvalue of A_Q we have $|\lambda_{A_Q}| \geq 1$, then look for the largest r_i in Q_R such that $r_i < \lceil \log_2(\rho(A)) \rceil$, and replace it by $r_i + 1$ and return to step 2.

We note that when A is not in Jordan form, there is a degree of freedom in the way we allocate the bits for every x_i; i.e., what eigenvalue is picked for every r_i. Therefore, the rate given by this algorithm is no unique and optimizing this allocation is part of a future work.

We test this algorithm in the following example. Given the following matrix, find r_1 and r_2 such that A_Q is stable.

$$A = \begin{bmatrix} 2 & 0.5 \\ 3 & 4 \end{bmatrix}.$$

Since the eigenvalues of A are $\lambda_A = \{1.418, 4.581\}$ we let $r_1 = 1$ and $r_2 = 3$. Then, Q_R is given by

$$Q_R = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.125 \end{bmatrix}. \tag{4.19}$$

We then obtain the eigenvalues of A_Q: $\lambda_{A_Q} = \{1.14, 0.35\}$. Since one of them is outside the unit circle, we add 1 to r_1 since r_2 already equals $\lceil \log_2(\rho(A)) \rceil$ and we replace $r_1 = 2$. The new eigenvalues of A_Q with the updated Q_R are $\{0.761, 0.283\}$. Now that A_Q is stable, the values $r_1 = 2$ and $r_2 = 3$ may be used as the rates for transmission.

4.6.2 Adding a Controller for Stabilization

We consider the system described by equation (4.1). Let us include this system in the encoder/decoder computations and modify equations (4.17) and (4.18) accordingly. The new equations are

$$C_Q(k+1) = A\bar{x}(k) + Bu(k) \tag{4.20}$$

and

$$\Delta_{x_i} = \max_{v_0} |(A_{Q,i})^{k+1} v_0|, \; \forall \; v_0 \in V_0. \tag{4.21}$$

where $A_{Q,i}$ is the "i-th" row of matrix A_Q. We assume that the encoder/decoder have access to the control signal or that it may be computed locally. The derivations of the previous subsection remain valid since the addition of the control law, only represent a *translation* of the centroid of the quantizer. At this point the simplest controller is the estimated state linear feedback controller, $u(k) = -K_c \bar{x}(k)$, which is motivated by the following lemma found in [53].

Lemma 4.6.1 *[53] Let A_s be a stable matrix. Let Bs_m a set of matrices such that $\|Bs_m\| \leq M$, $M \in \mathbb{R}$, $\forall m$, and $\lim_{m\to\infty} Bs_m \to 0$. Let $S_k = \sum_{m=0}^{k-1} A_s^{k-1-m} Bs_m$ then $\lim_{k\to\infty} S_k \to 0$.*

We will use this Lemma 4.6.1 as follows. For K_c such that $A - BK_c$ is stable, then we can solve iteratively $x(k+1) = Ax(k) + B(-K_c\bar{x}(k)) = Ax(k) + B(-K_c(x(k) - \epsilon(k)))$ with initial condition $\bar{x}(0)$:

$$x(k) = (A - BK_c)^k \bar{x}(0) + \sum_{m=0}^{k-1} (A - BK_c)^{k-1-m} BK_c \epsilon(m). \tag{4.22}$$

Our encoder/decoder scheme guarantees that $\|\epsilon(m)\| \leq \sup_k \|(A_{Q,i})^{k+1} v_0\|$. Moreover, it guarantees that $\|\epsilon(m)\|$ tends to zero when m grows. Since A_Q is stable, we know that $\sup_k \|(A_{Q,i})^{k+1} v_0\| \leq \infty$. If we let $A_s = A - BK_c$ and $Bs_m = BK_c\epsilon(m)$, we then may apply Lemma 4.6.1. We see that any stabilizing K_c asymptotically stabilize the system using the rates obtained earlier since the first additive term in equation (4.22) tends to zero (since $A - BK_c$ is stable), and the second additive term tends to zero by Lemma 4.6.1.

4.7 Comparison Between Encoding Schemes

The truncation-based scheme requires a larger data rate than the dynamic quantizer. To prove this fact, we note that the worst data rate that is required in the dynamic quantizer is when $r_i = \lceil \log_2(\rho(A)) \rceil$, $\forall i$. Let $r_\rho = \lceil \log_2(\rho(A)) \rceil$, then $Q_R = (1/2^{r_\rho})\mathbb{I}_{n\times n}$. This is the worst case since A_Q is guaranteed to be stable for this particular Q_R. This is easily proven since $A_Q = AQ_R = (1/2^{r_\rho})A$. From Linear Algebra, we know that the eigenvalues of A_Q are the eigenvalues of A multiplied by $1/2^{r_\rho}$. From the definition of r_ρ, the eigenvalues of A_Q are inside the unit circle. We note that for the worst case, the rate given by the dynamic quantizer is

$$R = nr_\rho = n\lceil \log_2(\rho(A)) \rceil.$$

The best case for the truncation-based encoding scheme is

$$R = n\left\lceil \log_2\left(\rho(A)^\mu\right) + \frac{1}{2}\log_2(n) \right\rceil + n$$

according to equation (4.15). It is then obvious that the dynamic quantizer achieves lower rate than the truncation-based one. In terms of the computational cost of both schemes, we note that the truncation-based only needs to compute the scalar equation (4.3) in order to decode correctly the transmitted signal. The dynamic quantizer however has to compute two equations, (4.20) and (4.21). Moreover, once the quantizer evolves from k to $k+1$ we need to compute in which of the $2^{r_1+r_2+\ldots+r_n}$ boxes the state is located, and this requires several comparison operations.

Previous schemes in literature, such as [29] and [53], achieve the minimum rate possible according to the Data Rate Theorem. Those schemes however require an extra step in their algorithms: the encoding scheme has to compute a matrix transformation to undo the rotations and state coupling caused by the matrix A, which in general is not a Jordan-block matrix. In this sense, our algorithm is simpler to implement. The scheme in [16] also achieves the minimum rate but under the strong assumption that the A matrix is a Jordan-block matrix. In terms of the data rate, we note that in the worst case, our scheme requires a data rate equal to $R = n\lceil \log_2(\rho(A)) \rceil$, while the minimum given by the Data Rate Theorem is $R = \sum_i \log_2 |\lambda_i(A)|$. If A happens to be a Jordan-block matrix, then our scheme will achieve the minimum data rate possible and will be comparable to the one in [16].

4.8 Simulations

To verify some of the results derived in the previous sections, we present several numerical examples using Matlab®. We consider a DTLI plant, so that, $x(k)$ exists only at the time instants $k = \{0, 1, 2, \ldots\}$. We do not consider the discretization of a continuous time system so the sampling time is not specified in the simulations. However, in all the plots, $x(k)$ was interpolated between sampling times for ease of visualization. We intentionally omit the packet maximum length D_{max} so we can compare the rates in *bit/time-step* and not in *packet/time-step*, which is equivalent to assuming that $D_{max} = 1$ *bit/packet*. The value L_0 that is known *a priori* by the encoder/decoder scheme was selected in the simulations to be $L_0 = 2\|x(0)\|$.

4.8.1 Example 1

We tested the results of Theorem 4.4.1 for the system

$$x(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u(k).$$

Let $L_0 = 71.68$, $x(0) = \begin{bmatrix} -16.333 & 30.768 & 8.44 \end{bmatrix}^T$, such that, $\|x(0)\| \leq L_0$. Since for this example $n = 3$ and $\mu = 2$, the rate obtained according to Theorem 4.4.1 is $R = 18$ *bit/time-step* and the simulation is shown in Figure 4.7. Note that asymptotic stability is indeed achieved. We note that for this system, the Data Rate Theorem gives 3.58 *bit/time-step* (a more accurately 4) while the dynamic quantizer requires a rate larger than 4 *bit/time-step*.

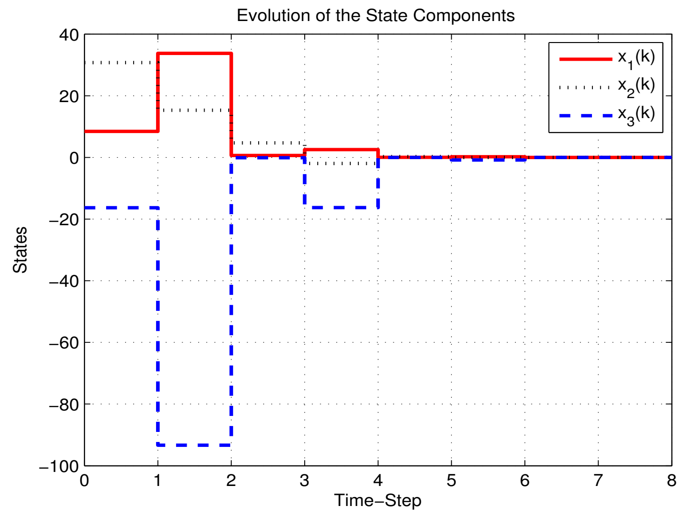

Figure 4.7: Truncation-based scheme: NCS (Type I) using $R = 18$ bit/time-step.

4.8.2 Example 2

To test the conservativeness of Corollary 4.4.1 , we considered a single-input system given by

$$x(k+1) = \begin{bmatrix} 20 & 0 & 10 \\ 0 & 10 & 0 \\ 0 & 10 & 30 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k).$$

Let $L_0 = 166.45$ and we assume initial condition $x(0) = \begin{bmatrix} 16.333 & 13.768 & -80.44 \end{bmatrix}^T$. Since for this example $n = \mu = 3$, the rate obtained using Corollary 4.4.1, is $R = 51$ *bit/time-step*. We then verify in Figure 4.8 the asymptotic stability claim of the corollary. Since our results provide sufficient conditions only, we tried for smaller values of R and found out that for this particular example, $R = 42$ *bit/time-step* leads to instability, see Figure 4.9. We note that for this system, the Data Rate Theorem gives 12.55 *bit/time-step* while the dynamic quantizer requires a rate of 15 *bit/time-step.*

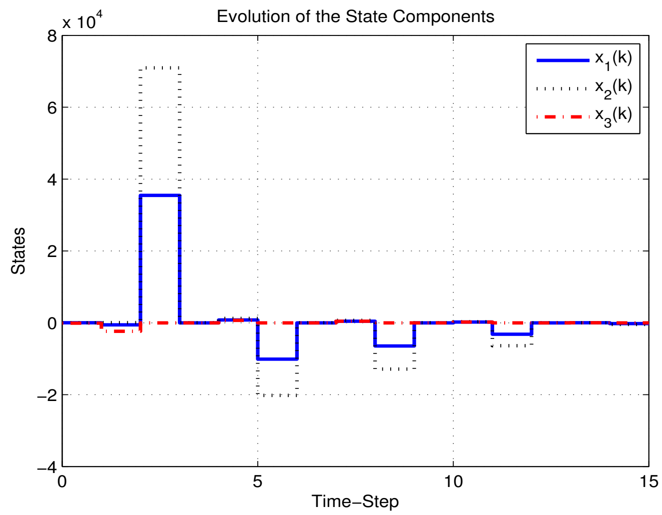

Figure 4.8: Truncation-based scheme: NCS (Type I) using $R = 51$ bit/time-step.

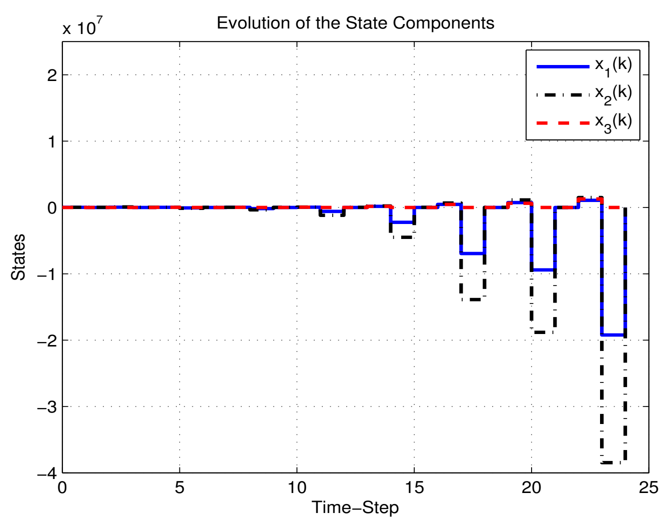

Figure 4.9: Truncation-based scheme: NCS (Type I) using $R = 42$ bit/time-step.

4.8.3 Example 3

Consider a second order system ($n = 2$) with time-delay $p = 2$ evolving according to the following dynamics

$$x(k+1) = \begin{bmatrix} 2 & 0 \\ 0 & 1.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k-2);$$

with $x(0) = \begin{bmatrix} -16.333 & 30.768 \end{bmatrix}^T$. Assuming $L_0 = 69.66$, the rate obtained using Theorem 4.4.2 is $R = 28$ *bit/time-step*. The corresponding simulation is shown in Figure 4.10. For this particular example we do not compare with the Data Rate Theorem or our dynamic quantizer since neither of those consider a delayed system.

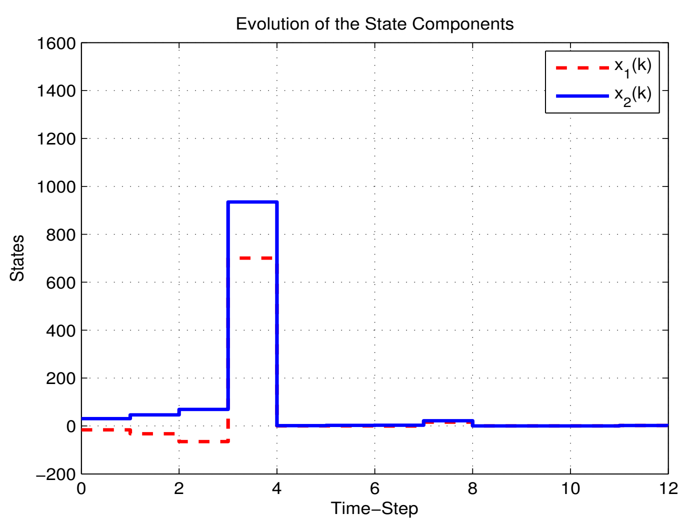

Figure 4.10: Closed-Loop NCS with Time-Delay

4.8.4 Example 4

Consider a third order system ($n = 3$) evolving according to the following dynamics

$$x(k+1) = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k);$$

with the initial condition state vector $x(0) = \begin{bmatrix} 1.33 & 3.768 & 8.44 \end{bmatrix}^T$. We assume that this plant is part of a Network Type II and we also assume $L_0 = 18.67$. The network rates obtained using Theorem 4.4.3 are $R_1 = 30$ *bit/time-step* and $R_2 = 10$ *bit/time-step* and the simulation is shown in Figure 4.11. For this particular example we do not compare with the Data Rate Theorem since this last one considers a Network Type I and not a Type II as in this example.

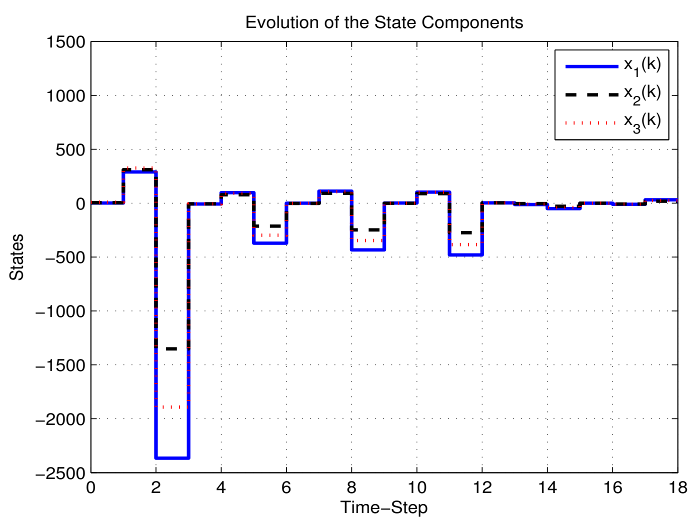

Figure 4.11: Truncation-based scheme: NCS (Type II)

4.8.5 Example 5

The following simulation shows the evolution of x when using the control law given in equation (4.13) with the rate given by $R = n \left\lceil \log_2 \left(|\lambda_{max}|^{\mu}\right) + \frac{1}{2}\log_2(n) \right\rceil + n$. Let us consider the following system:

$$x(k+1) = \begin{bmatrix} 2 & 100 & 100 \\ 0 & 4 & 100 \\ 0 & 1 & 4 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k).$$

Let the initial condition be $x(0) = \begin{bmatrix} 16.333 & 13.768 & -80.44 \end{bmatrix}^T$ and $L_0 = 166.45$. Using equation (4.15), we find that $R = 42$ *bit/time-step* is now sufficient for stabilization. The simulation using this control law is shown in Figure 4.12. We also show in Figure 4.13 the simulation using the results of Theorem 4.4.1 and the rate was $R = 57$ *bit/time-step*. The tradeoff is evident when comparing the two simulations: although a lower rate is needed in the simulation in Figure 4.12, the transient response (overshoot, settling time) in Figure 4.13 is actually better.

4.8.6 Example 6

We present next an example considering the following system:

$$x(k+1) = Ax(k) + Bu(k) = \begin{bmatrix} 2 & 0.5 \\ 3 & 4 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k);$$

$$u(k) = -K_c x(k) = -\begin{bmatrix} 2.533 & 2.566 \end{bmatrix} x(k).$$

With this K_c, the poles of $(A - BK_c)$ are located at 0.5 and 0.4. We assume that the initial condition state vector $x(0) = \begin{bmatrix} 2.1 & 2.8 \end{bmatrix}^T$ and $L_0 = 7$. We calculate the rates to stabilize A_Q are $r_1 = 2$ and $r_2 = 3$. This gives a total rate of $R = 5$ *bits/time-step*. Using the dynamic quantizer scheme we obtain the plots in Figure 4.14

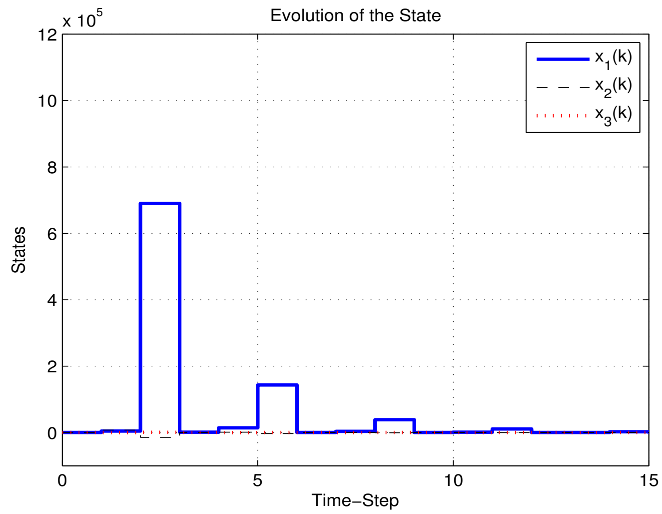

Figure 4.12: Closed-Loop NCS using $R = 42$ bit/time-step.

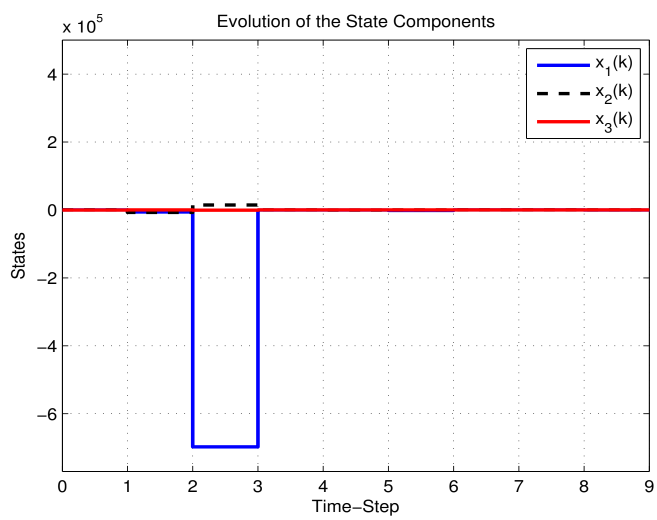

Figure 4.13: Closed-Loop NCS using $R = 57$ bit/time-step.

4.9 Summary

This chapter has extended previous results for determining the sufficient rate for stabilization of a packet-based NCS. While the rates obtained for Network Type I

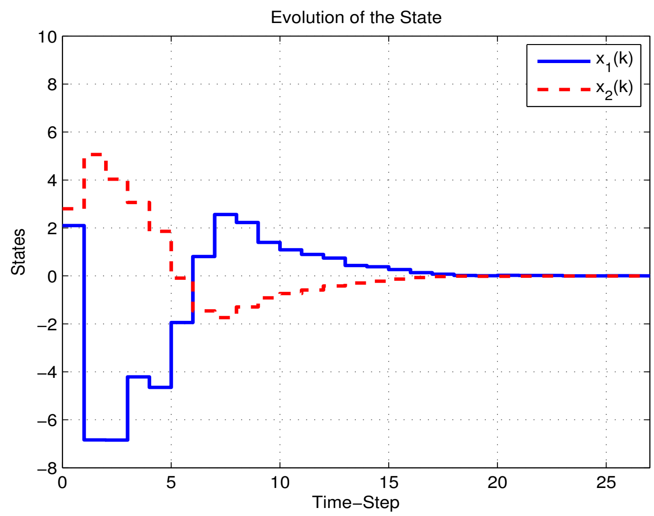

Figure 4.14: State evolution in NCS Type I using $R = 5$ bits/time-step

are higher that the limits set by the Data Rate Theorem, the computational cost of our encoding/decoding scheme implementation is lower than schemes proposed earlier. In this setup we were able to treat the case of a constant time delay in the network. We also obtained sufficient rates for stabilizing a system using a Network Type II.

In order to lower the required transmission rate and to include a general full-state feedback controller, we proposed a more complex encoder/decoder scheme that achieves rates close to those specified by the Data Rate Theorem. This scheme combines a dynamic quantizer that achieves asymptotic stability in the closed-loop without performing the linear transformation that previous works used.

In the next chapter we consider the tracking problem with a general finite-capacity noisy channel in the feedback and we obtain some limitations that such channels impose in tracking systems. It is a theoretical approach that will rely in information theory principles and it is universal since it is independent of the encoding schemes or the control law. The flavor of the results is similar to the ones in Chapter 2 in the sense that they are fundamental limitations for the general NCS.

Chapter 5

Limitations in Tracking Systems

5.1 Introduction

The goal of this chapter is to find fundamental limitations on feedback tracking systems in terms of information theoretical quantities. This is important since the emerging control applications involve the presence of a constraint communication channel in the feedback loop. Typically, control systems have been understood as signal processing blocks or systems interchanging energy. However, these approaches are not appropriate for the new scenarios. That is why we suggest that an interpretation in terms of information flow may be more suitable for the future design of control algorithms.

Previous related work in [23], [25], [26], [55], [60], [61] and [62] detailed some aspects of performance and limitations of control systems in terms of information theoretic quantities. Specifically, the work in [60] dealt with the tracking issues without a channel in the feedback link, while [23] dealt with disturbance rejection. A result in [60], shows that a necessary condition for efficient tracking is that the information flow from the reference signal to the output should be greater than the information flow between the disturbance and the output. We know that in the absence of noise, and without a communication channel in the feedback loop, the mutual information rate (or information rate) between reference signal and the output is infinite. From Chapter 2 we know, however, that if the feedback signal is transmitted by means of a finite capacity channel, the mutual information rate is upper bounded by $C_f - \sum_{i=1} \max \{0, \log_2 (|\lambda_i(A)|)\}$.

Following the same approach of [24], we expect that the parameters of the plant and feedback channel capacity C_f will be related, and that there will be a trade-off between these parameters. If by some reason this upper bound happens to be zero, then we reach a fundamental limitation where no information of the reference signal is available for feedback. This means that the two signals are independent, therefore, uncorrelated, and this is exactly the condition that implies that tracking is impossible. In other words, the feedback signal does not provide any useful information for the reference to be tracked.

We note that the condition for a non-zero mutual information between the reference and the feedback signal is a necessary condition for tracking, but not a sufficient one. A large mutual information between the reference signal and the feedback signal does not necessarily imply that tracking is possible (it only implies that the signals are highly correlated). This is expected because even in the case of a perfect infinite capacity channel, the tracking issue requires additional conditions to be satisfied.

We remark again what we mentioned before in Chapter 1. The results here do not have the intention to be applied in the design of a new control algorithm. These results are fundamental limitations in terms of information quantities that any control system designer must be aware of before trying to design a new control system.

5.2 Notation

We present next the notation used in the rest of this chapter.

- Let $\mathbf{x}^k = \{\mathbf{x}(1), \mathbf{x}(2), \ldots, \mathbf{x}(k)\}$ and $\mathbf{y}^k = \{\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(k)\}$ be sets of observations of stochastic processes $\mathbf{x}$ and $\mathbf{y}$. We follow the notation in [49] where bold letters represent stochastic processes.

- Let $\mathbf{x}(k)$ be a time sample of the stochastic process $\mathbf{x}$.

- Let $\mathbf{x_j}$ be the "j-th" state component. For example, if $\mathbf{x}$ has dimension $n = 3$, then $\mathbf{x_j}$ will denote any of the state components $\mathbf{x_1}$, $\mathbf{x_2}$ or $\mathbf{x_3}$.

- Let $\mathbf{x_J}$ denote the set of state components, $\mathbf{x_j}$, such that $\mathrm{j} \in \mathrm{J}$. For example, if $\mathrm{J} = \{1, 3\}$, then $\mathbf{x_J}$ is the set $\{\mathbf{x_1}, \mathbf{x_3}\}$.

- Let $|.|$ denote the absolute value and $\mathbf{det(.)}$ denotes the absolute value of the determinant of a matrix.

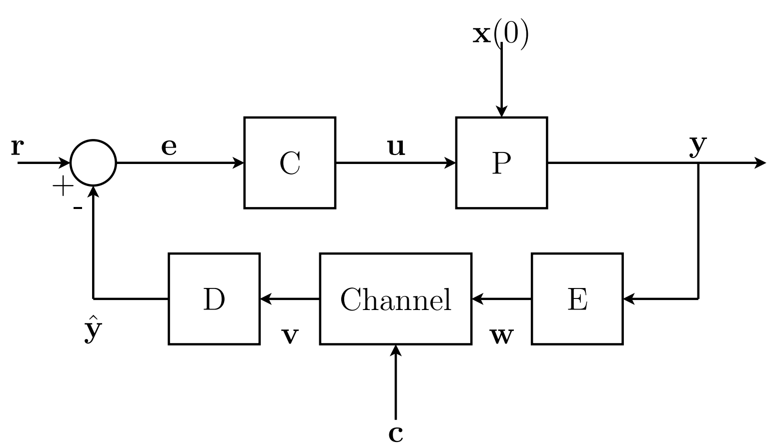

Figure 5.1: Closed-Loop System with Communication Channel in Feedback Link.

We also define the blocks in Figure 5.1:

- C is the controller, which does not have any constraints (it could be time-invariant, nonlinear, etc.).

- P is the plant to be controlled and is assumed to be discrete, linear, time-invariant, with state-space realization

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k); \tag{5.1}$$

$$\mathbf{y}(k) = \mathbb{C}\mathbf{x}(k). \tag{5.2}$$

- E is the encoder assumed to be a causal operator well defined in the input alphabet of the channel.

- D is the decoder assumed to be well defined and conserving equimemory with the encoder.

- The Channel block is any type of communication channel with finite capacity.

- c is the channel noise.

5.3 Information Theory Preliminaries

Before proceeding, we enumerate some well-known information-theoretical properties that will be very useful later on.

Properties 5.3.1 *Assume that* $\mathbf{z}$, $\mathbf{w}$, $\mathbf{u} \in \mathbb{R}$ *are random variables and* $f(\mathbf{z})$, $g(\mathbf{z})$ *are real functions. All of the following may be found in several references as [5], [24] and [38].*

(a) $h(\mathbf{z}|\mathbf{w}) \leq \mathbf{h}(\mathbf{z})$ *with equality if* $\mathbf{z}$ *and* $\mathbf{w}$ *are independent.*

(b) *Let* $\mathbf{z}$ *have mean* μ *and covariance* $Cov\{\mathbf{z}^n\}$*. Then*

$$h(\mathbf{z}^n) \leq \frac{1}{2}\log_2\Big((2\pi e)^n \mathbf{det}(Cov\{\mathbf{z}^n\})\Big)$$

with equality if $\mathbf{z}$ *has a multivariate normal distribution.*

(c) $h(a\mathbf{z}) = h(\mathbf{z}) + \log_2\Big(|a|\Big)$ *for nonzero constant* a*.*

(d) $h(A\mathbf{z}) = h(\mathbf{z}) + \log_2\Big(\mathbf{det}(A)\Big)$ *for nonsingular* A *matrix.*

(e) $h(\mathbf{z}|\mathbf{w}) = h(\mathbf{z} - g(\mathbf{w})|\mathbf{w})$*.*

(f) $I(\mathbf{z};\mathbf{w}) = I(\mathbf{w};\mathbf{z}) \geq 0$*.*

(g) $I(\mathbf{z};\mathbf{w}) \geq I(g(\mathbf{z}); f(\mathbf{w}))$*.*

(h) $I(\mathbf{z};\mathbf{w}|\mathbf{u}) = I((\mathbf{u},\mathbf{z});\mathbf{w}) - I(\mathbf{u};\mathbf{w}) = h(\mathbf{z}|\mathbf{u}) - h(\mathbf{z}|\mathbf{w},\mathbf{u}) = h(\mathbf{w}|\mathbf{u}) - h(\mathbf{w}|\mathbf{z},\mathbf{u})$*.*

(i) *For any random variable* $\mathbf{z}$ *and estimate* $\hat{\mathbf{z}}$*:* $E\{(\mathbf{z}-\hat{\mathbf{z}})^2\} \geq \frac{1}{2\pi e}2^{2h(\mathbf{z})}$*, with equality if and only if* $\mathbf{z}$ *is Gaussian and* $\hat{\mathbf{z}}$ *is the mean of* $\mathbf{z}$*.*

(j) *The variance of the error in the estimate* $\hat{\mathbf{z}}$ *of* $\mathbf{z}$ *given the infinite past is lower bounded as* $\sigma^2_\infty(\mathbf{z}) = \lim_{k\to\infty} E\{(\mathbf{z}-\hat{\mathbf{z}})^2(k)|(\mathbf{z}-\hat{\mathbf{z}})(k-1)\} \geq \frac{1}{2\pi e}2^{2h_\infty(\mathbf{z})}$ *with equality if* $\mathbf{z}$ *is Gaussian.*

(k) *If* z *is an asymptotically stationary process, then*

$$h_\infty(z) \leq \frac{1}{4\pi}\int_{\pi}^{\pi} \log_2\Big(2\pi e\widehat{\Phi}_z(\omega)\Big)d\omega$$

where $\widehat{\Phi}_z$ *is the asymptotic power spectral density of* z *and equality holds if, in addition,* z *is Gaussian auto-regressive.*

5.4 Signal Analysis

From Property 5.3.1.(a) and Property 5.3.1.(j) the functional dependencies among the signals involved in the closed-loop is shown in Figure 5.1 are the following:

$$\begin{aligned}
\mathbf{y}(k) &= f_1(\mathbf{r}^{k-1}, \mathbf{c}^{k-1}, \mathbf{x}(0)); \\
\mathbf{e}(k) &= f_2(\mathbf{r}^{k}, \hat{\mathbf{y}}^{k}) = \mathbf{r}(k) - \hat{\mathbf{y}}(k); \\
\mathbf{u}(k) &= f_3(\mathbf{e}^{k}); \\
\hat{\mathbf{y}}(k) &= f_4(\mathbf{y}^{k}, \mathbf{c}^{k}).
\end{aligned}$$

5.5 Assumptions

The matrix A in block P in Figure 5.1 is assumed to be diagonal with only unstable eigenvalues ($|\lambda_i(A)| > 1$) and therefore, A^k is invertible $\forall k$. We assume that A has unstable eigenvalues since it is the worst case. The general case in which A have stable eigenvalues is discussed in Remark 5.6.1. Since we are considering the tracking problem, the control law is a function of the error $\mathbf{e}^k = \mathbf{r}^k - \hat{\mathbf{y}}^k$, $\mathbf{u}(k) = f_3(\mathbf{e}^k)$. We note for now that the output is an n-dimensional vector, but this will be relaxed later on. In our setup f_3 is not limited to be a linear or time-invariant control law. We note that the solution of the difference equation (5.1) may be written as $\mathbf{x}(k) = A^k\mathbf{x}(0) + \sum_{i=0}^{k-1} A^{k-i-1}Bf_3(\mathbf{e}^i)$. If $\mathbb{C} = \mathbb{I}$, then from the tracking error, defined by $\boldsymbol{\epsilon}(k) = \mathbf{r}(k) - \mathbf{y}(k)$, we have

$$\mathbf{r}(k) - \boldsymbol{\epsilon}(k) = \mathbf{y}(k) = \mathbf{x}(k) = A^k\mathbf{x}(0) + \sum_{i=0}^{k-1} A^{k-i-1}Bf_3(\mathbf{e}^i). \tag{5.3}$$

We rearrange the terms as

$$\mathbf{x}(0) + A^{-k}\sum_{i=0}^{k-1} A^{k-i-1}Bf_3(\mathbf{e}^i) = -A^{-k}(\boldsymbol{\epsilon}(k) - \mathbf{r}(k)). \tag{5.4}$$

In a tracking problem, we do not necessarily assume that the state is bounded, since for unbounded reference signals, the state may grow unbounded. Instead, we assume that the closed-loop is such that the error is bounded, i.e.,

$$E\{\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}\} < \infty.$$

Since this implies that $\boldsymbol{\epsilon}$ is a second-order process, the mean $E\{\boldsymbol{\epsilon}\}$ and the covariance $\text{Cov}\{\boldsymbol{\epsilon}\} = E\{(\boldsymbol{\epsilon} + E\{\boldsymbol{\epsilon}\})(\boldsymbol{\epsilon} + E\{\boldsymbol{\epsilon}\})^T\}$ must be finite. For bounded reference signals, the condition $E\{\boldsymbol{\epsilon}^T\boldsymbol{\epsilon}\} < \infty$ guarantees stability since by the triangle inequality [36] we know that

$$\sqrt{E\{\mathbf{x}^2(k)\}} \leq \sqrt{E\{\mathbf{r}^2(k)\}} + \sqrt{E\{\boldsymbol{\epsilon}^2(k)\}}. \tag{5.5}$$

Since the two terms on the right side of equation (5.5) are finite, then we also get that $\sqrt{E\{\mathbf{x}^2(k)\}} < \infty$ and, therefore, the system remains stable.

5.6 Auxiliary Results

We first introduce some results that will later be used to obtain the limitations on tracking systems. Specifically, the following result will be used to prove Lemma 5.6.3. Let us consider the set P_j defined as $P_j = \{i \in \mathbb{N}, j \leq n : i \in \{1, 2, \ldots, n\} - \{j\}\}$. The following lemma holds for stabilization and is a slight modification of the result presented in [24].

Lemma 5.6.1 *Consider the closed-loop system in Figure 5.1, where the plant is a DTLI system described by equations (5.1) and (5.2), with $\mathbb{C} = \mathbb{I}$, and A diagonal in equation (5.2). If $E\{\mathbf{x}_{\mathrm{P}_j}(k)\mathbf{x}_{\mathrm{P}_j}^T(k)\} < \infty$, then*

$$\lim_{k\to\infty} \frac{I(\mathbf{x}_{\mathrm{P}_j}(0); \mathbf{e}^k | \mathbf{r}^k, \mathbf{x}_j(0))}{k} \geq \sum_{i \neq j} \log_2 \Big(|\lambda_i(A)|\Big).$$

Proof: By the chain rule expressed in Property 5.3.1.(h) we expand the expression given by $I(\mathbf{x}_{P_j}(0);\mathbf{e}^k|\mathbf{r}^k,\mathbf{x}_j)$ as

$$I(\mathbf{x}_{P_j}(0);\mathbf{e}^k|\mathbf{r}^k,\mathbf{x}_j(0)) = \sum_{i\neq j}^{n} I(\mathbf{x}_i(0);\mathbf{e}^k|\mathbf{r}^k,\mathbf{x}_j(0),\mathbf{x}_1(0),\ldots,\mathbf{x}_{i-1}(0)). \tag{5.6}$$

Each state component may be expressed as

$$\mathbf{x}_i(k) = \lambda_i^k \mathbf{x}_i(0) + g_i(\mathbf{e}^k); \tag{5.7}$$

for some function g_i. Therefore, each initial state component is given by

$$\mathbf{x}_i(0) = \lambda_i^{-k}(\mathbf{x}_i(k) - g_i(\mathbf{e}^k)).$$

From the definition of mutual information we expand the "i-th" additive term in equation (5.6).

$$\begin{aligned} I(\mathbf{x}_i(0);\mathbf{e}^k|\mathbf{r}^k,\mathbf{x}_j(0),\mathbf{x}_1(0),\ldots,\mathbf{x}_{i-1}(0)) &= h(\mathbf{x}_i(0)|\mathbf{r}^k,\mathbf{x}_j(0),\mathbf{x}_1(0),\ldots,\mathbf{x}_{i-1}(0)) \\ &- h(\mathbf{x}_i(0)|\mathbf{e}^k,\mathbf{r}^k,\mathbf{x}_j(0),\mathbf{x}_1(0),\ldots,\mathbf{x}_{i-1}(0)). \end{aligned}$$

From the independence between $\mathbf{x}(0)$ and $\mathbf{r}^k$, $\forall i \in P_j$, the term $\mathbf{r}^k$ may be eliminated in the first entropy term

$$\begin{aligned} &I(\mathbf{x}_i(0);\mathbf{e}^k|\mathbf{r}^k,\mathbf{x}_j(0),\mathbf{x}_1(0),\mathbf{x}_2(0),\ldots,\mathbf{x}_{i-1}(0)) \\ &= h(\mathbf{x}_i(0)|\mathbf{x}_j(0),\mathbf{x}_1(0),\mathbf{x}_2(0),\ldots,\mathbf{x}_{i-1}(0)) \\ &\quad - h(\mathbf{x}_i(0)|\mathbf{e}^k,\mathbf{r}^k,\mathbf{x}_j(0),\mathbf{x}_1(0),\mathbf{x}_2(0),\ldots,\mathbf{x}_{i-1}(0)). \end{aligned}$$

From equation (5.7), the term $h(\mathbf{x}_i(0)|\mathbf{e}^k,\mathbf{r}^k,\mathbf{x}_1(0),\mathbf{x}_2(0),\ldots,\mathbf{x}_{i-1}(0))$ may be rewritten as

$$\begin{aligned} &h(\mathbf{x}_i(0)|\mathbf{e}^k,\mathbf{r}^k,\mathbf{x}_1(0),\mathbf{x}_2(0),\ldots\,\mathbf{x}_{i-1}(0)) \\ &= h(\lambda_i^{-k}(\mathbf{x}_i(k) - g_i(\mathbf{e}^k))|\mathbf{e}^k,\mathbf{r}^k,\mathbf{x}_1(0),\mathbf{x}_2(0),\ldots\,\mathbf{x}_{i-1}(0)). \end{aligned}$$

By Properties 5.3.1.(c), 5.3.1.**??** and 5.3.1.(a) we have that

$$\begin{aligned}
&h(\lambda_i^{-k}\mathbf{x}_i(k)|\mathbf{e}^k, \mathbf{r}^k, \mathbf{x}_1(0), \mathbf{x}_2(0), \dots \mathbf{x}_{i-1}(0)) \\
&= h(\lambda_i^{-k}\mathbf{x}_i(k)|\mathbf{e}^k, \mathbf{r}^k, \mathbf{x}_1(0), \mathbf{x}_2(0), \dots \mathbf{x}_{i-1}(0)); \\
&= -k\log_2(|\lambda_i|) + h(\mathbf{x}_i(k)|\mathbf{e}^k, \mathbf{r}^k, \mathbf{x}_1(0), \mathbf{x}_2(0), \dots \mathbf{x}_{i-1}(0)); \\
&\leq -k\log_2(|\lambda_i|) + h(\mathbf{x}_i(k)); \\
&\leq -k\log_2(|\lambda_i|) + \frac{1}{2}\log_2\Big((2\pi e)\mathbf{det}(\mathrm{Cov}\{\mathbf{x}_i\})\Big).
\end{aligned}$$

Then

$$\begin{aligned}
&I(\mathbf{x}_i(0); \mathbf{e}^k|\mathbf{r}^k, \mathbf{x}_j(0), \mathbf{x}_1(0), \mathbf{x}_2(0), \dots, \mathbf{x}_{i-1}) \\
&\geq h(\mathbf{x}_i(0)|\mathbf{x}_j(0), \mathbf{x}_1(0), \mathbf{x}_2(0), \dots, \mathbf{x}_{i-1}(0)) \; + k\log_2(|\lambda_i|) \\
&\quad -\frac{1}{2}\log_2\Big((2\pi e)\mathbf{det}(\mathrm{Cov}\{\mathbf{x}_i\})\Big).
\end{aligned}$$

Dividing by k and taking the limit to infinity we obtain

$$\lim_{k\to\infty}\frac{I(\mathbf{x}_i(0); \mathbf{e}^k|\mathbf{r}^k, \mathbf{x}_j(0), \mathbf{x}_1(0), \dots, \mathbf{x}_{i-1}(0))}{k} \geq \log_2(|\lambda_i|). \tag{5.8}$$

From equations (5.6) and (5.8) we have

$$\lim_{k\to\infty}\frac{I(\mathbf{x}_{P_j}(0); \mathbf{e}^k|\mathbf{r}^k, \mathbf{x}_j(0))}{k} \geq \sum_{i\neq j}\log_2\Big(|\lambda_i(A)|\Big).$$

■

We next focus on the tracking problem which is different from the stabilization one treated in previous works. We first consider the following two lemmas.

Lemma 5.6.2 *Consider the closed-loop system in Figure 5.1, where the plant is a DLTI system described by equations (5.1) and (5.2),* $\mathbb{C} = \mathbb{I}$. *If* $E\{\boldsymbol{\epsilon}(k)\boldsymbol{\epsilon}^T(k)\} < \infty$, *then*

$$\lim_{k\to\infty}\frac{I(\mathbf{x}(0); \mathbf{e}^k|\mathbf{r}^k)}{k} \geq \sum_{i}\log_2\Big(|\lambda_i(A)|\Big).$$

Proof: The mutual information $I(\mathbf{x}(0);\mathbf{e}^k|\mathbf{r}^k)$ may be expanded as:

$$\begin{aligned} I(\mathbf{x}(0);\mathbf{e}^k|\mathbf{r}^k) &= h(\mathbf{x}(0)|\mathbf{r}^k) - h(\mathbf{x}(0)|\mathbf{e}^k,\mathbf{r}^k); \\ &= h(\mathbf{x}(0)) - h(\mathbf{x}(0)|\mathbf{e}^k,\mathbf{r}^k); \end{aligned}$$

where we have used the fact that $\mathbf{x}(0)$ and $\mathbf{r}^k$ are independent. If we focus on the quantity $h(\mathbf{x}(0)|\mathbf{e}^k,\mathbf{r}^k)$ and using the properties of entropy we obtain:

$$\begin{aligned} & h(\mathbf{x}(0)|\mathbf{e}^k,\mathbf{r}^k) \\ &= h(\mathbf{x}(0) + A^{-k}\sum_{i=0}^{k} A^{k-i-1}Bf_3(\mathbf{e}^i)|\mathbf{e}^k,\mathbf{r}^k); \\ &= h(-A^{-k}(\boldsymbol{\epsilon}(k) - \mathbf{r}(k))|\mathbf{e}^k,\mathbf{r}^k); && (5.9) \\ &= h(-A^{-k}\boldsymbol{\epsilon}(k)|\mathbf{e}^k,\mathbf{r}^k); && (5.10) \\ &\leq h(-A^{-k}\boldsymbol{\epsilon}(k)); && (5.11) \\ &\leq \frac{1}{2}\log_2\Big((2\pi e)^n \mathbf{det}(\mathrm{Cov}\{-A^{-k}\boldsymbol{\epsilon}\})\Big); && (5.12) \\ &= \frac{n}{2}\log_2\Big(2\pi e\Big) + \frac{1}{2}\log_2\Big(\mathbf{det}(-A^{-k}\mathrm{Cov}\{\boldsymbol{\epsilon}\}(-A^{-k})^T)\Big); \\ &= \frac{n}{2}\log_2\Big(2\pi e\Big) + \frac{1}{2}\log_2\Big(\mathbf{det}(A^{-k}(A^{-k})^T)\Big) + \frac{1}{2}\log_2\Big(\mathbf{det}(\mathrm{Cov}\{\boldsymbol{\epsilon}\})\Big); \\ &= \frac{n}{2}\log_2\Big(2\pi e\Big) - k\sum_i \log_2\Big(|\lambda_i(A)|\Big) + \frac{1}{2}\log_2\Big(\mathbf{det}(\mathrm{Cov}\{\boldsymbol{\epsilon}\})\Big). \end{aligned}$$

Where equation (5.9) is due to equation (5.4), equation (5.10) is due to Property 5.3.1.(e), equation (5.11) is due to Property 5.3.1.(a), and equation (5.12) is due to Property 5.3.1.**??**. From these simplifications we obtain

$$\begin{aligned} & I(\mathbf{x}(0);\mathbf{e}^k|\mathbf{r}^k) \\ &\geq h(\mathbf{x}(0)) - \frac{n}{2}\log_2(2\pi e) + k\sum_i \log_2\Big(|\lambda_i(A)|\Big) - \frac{1}{2}\log_2\Big(\mathbf{det}(\mathrm{Cov}\{\ \boldsymbol{\epsilon}\ \})\Big). \end{aligned}$$

Finally, if we divide by k and take the limit as $k \to \infty$, we obtain:

$$\lim_{k\to\infty} \frac{I(\mathbf{x}(0); \mathbf{e}^k|\mathbf{r}^k)}{k} \geq \sum_i \log_2 \Big(|\lambda_i(A)| \Big).$$

since $\boldsymbol{\epsilon}$ is a second order process.

■

We note that for Lemma 5.6.2, we have assumed that $\mathbf{y}(k) = \mathbf{x}(k)$, i.e. the entire state is available for measurement. However, the lemma still holds when the output is only one component of the state vector (single output), e.g. $\mathbf{y}(k) = \mathbf{x}_1(k)$. In that case, we need to guarantee that the components of the state that do not appear in the output remain bounded. The only component that can grow unbounded is the one that appears in the output (in the case of an unbounded reference signal). For example, if the plant is a third order system ($n = 3$) and $\mathbb{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, we have to guarantee that the difference between the reference signal and the output $\mathbf{y} = \mathbf{x}_1$ must remain bounded; and that the state components that do not appear in the output $\{\mathbf{x}_2, \mathbf{x}_3\}$ remain bounded, i.e., $E\{\mathbf{x}_j(k)\mathbf{x}_j(k)^T\} < \infty, \forall j \in \{2, 4\}$. Before generalizing Theorem 5.6.2 we introduce the following notation:

- $\mathbf{y} = \mathbf{x_j}$
- Let $\mathbf{x}_{\bar{Y}}$ be the vector of state components that do not appear in output $\mathbf{y}$.

For example, if $\mathbb{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, then $\mathbf{x}_j = \{\mathbf{x}_1\}$ whereas $\mathbf{x}_{\bar{Y}} = \{\mathbf{x}_2, \mathbf{x}_3\}$. We then prove the following.

Lemma 5.6.3 *Consider closed-loop system given in Figure 5.1, where the plant is a DLTI system described by equation (5.1) and* $\mathbf{y} = q\mathbf{x}_j$ *for some* $j \in \{1, \ldots, n\}$, q *a non-zero constant. If* $E\{\boldsymbol{\epsilon}(k)\boldsymbol{\epsilon}^T(k)\} < \infty$ *and* $E\{\mathbf{x}_{\bar{Y}}(k)\mathbf{x}_{\bar{Y}}^T(k)\} < \infty$, *then*

$$\lim_{k\to\infty} \frac{I(\mathbf{x}(0); \mathbf{e}^k|\mathbf{r}^k)}{k} \geq \sum_i \log_2 \Big(|\lambda_i(A)| \Big).$$

Proof: $I(\mathbf{x}(0);\mathbf{e}^k|\mathbf{r}^k)$ may be expanded as

$$I(\mathbf{x}(0);\mathbf{e}^k|\mathbf{r}^k) = I(\mathbf{x}_j(0);\mathbf{e}^k|\mathbf{r}^k) + I(\mathbf{x}_{\bar{Y}}(0);\mathbf{e}^k|\mathbf{r}^k,\mathbf{x}_j(0)); \tag{5.13}$$

where $\mathbf{x}_j = \mathbf{y}$, and $\mathbf{x}_{\bar{Y}}$ are the states that do not appear in $\mathbf{y}$. We also know that $\mathbf{y}(k) = \mathbf{r}(k) - \boldsymbol{\epsilon}(k) = q\mathbf{x}_j(k)$, then $\mathbf{y}(k)$ may be expressed as

$$\mathbf{y}(k) = q\lambda_{x_j}^k \mathbf{x}_j(0) + G(\mathbf{e}^k).$$

Where $G(\mathbf{e}^k)$ is a function of the error $\mathbf{e}^k$ and $\lambda_{x_j}^k$ is the eigenvalue corresponding to x_j. We may also expand $I(\mathbf{x}_j(0);\mathbf{e}^k|\mathbf{r}^k)$ using the definition of mutual information:

$$\begin{aligned} I(\mathbf{x}_j(0);\mathbf{e}^k|\mathbf{r}^k) &= h(\mathbf{x}_j(0)|\mathbf{r}^k) - h(\mathbf{x}_j(0)|\mathbf{e}^k,\mathbf{r}^k); \\ &= h(\mathbf{x}_j(0)) - h(\mathbf{x}_j(0)|\mathbf{e}^k,\mathbf{r}^k); \\ &= h(\mathbf{x}_j(0)) - h\Big(\lambda_{x_j}^{-k}(\boldsymbol{\epsilon}(k) - \mathbf{r}(k) - G(\mathbf{e}^k))|\mathbf{e}^k,\mathbf{r}^k\Big). \end{aligned}$$

Consider the term $h\Big(q^{-1}\lambda_{x_j}^{-k}(\mathbf{r}(k) - \boldsymbol{\epsilon}(k) - G(\mathbf{e}^k))|\mathbf{e}^{\mathbf{k}},\mathbf{r}^{\mathbf{k}}\Big)$ which may be simplified to

$$\begin{aligned} & h\Big(q^{-1}\lambda_{x_j}^{-k}(\mathbf{r}(k) - \boldsymbol{\epsilon}(k) - G(\mathbf{e}^k))|\mathbf{e}^k,\mathbf{r}^k\Big) \\ &= -k\log_2\Big(|\lambda_{x_j}|\Big) - \log_2\Big(|q|\Big) \\ &\quad + h\Big((\mathbf{r}(k) - \boldsymbol{\epsilon}(k) - G(\mathbf{e}^k))|\mathbf{e}^k,\mathbf{r}^k\Big); \\ &= -k\log_2\Big(|\lambda_{x_j}|\Big) - \log_2\Big(|q|\Big) + h\Big(\boldsymbol{\epsilon}(k)|\mathbf{e}^k,\mathbf{r}^k\Big); && (5.14) \\ &\leq -k\log_2\Big(|\lambda_{x_j}|\Big) - \log_2\Big(|q|\Big) + h\Big(\boldsymbol{\epsilon}(k)\Big); && (5.15) \\ &\leq -k\log_2\Big(|\lambda_{x_j}|\Big) - \log_2\Big(|q|\Big) + \frac{1}{2}\log_2\Big((2\pi e)\mathbf{det}(\mathrm{Cov}\{\boldsymbol{\epsilon}\})\Big); && (5.16) \end{aligned}$$

where equation (5.14) is due to Property 5.3.1.(e), equation (5.15) is due to Property 5.3.1.(a) and equation (5.16) is due to Property 5.3.1.**??**. We then have

$$\begin{aligned} I(\mathbf{x}_j(0);\mathbf{e}^k|\mathbf{r}^k) \geq\;& h(\mathbf{x}_j(0)) + k\log_2\Big(|\lambda_{x_j}|\Big) - \frac{1}{2}\log_2\Big(|q^{-2}2\pi e|\Big) \\ & -\frac{1}{2}\log_2\Big(\mathbf{det}(\mathrm{Cov}\{\boldsymbol{\epsilon}\})\Big). \end{aligned}$$

Taking the limit when k tends to infinity we obtain

$$\begin{aligned} &\lim_{k\to\infty}\frac{I(\mathbf{x}_j(0);\mathbf{e}^k|\mathbf{r}^k)}{k} \\ &\geq \lim_{k\to\infty}\frac{1}{k}\Bigg(h\Big(\mathbf{x}_j(0)\Big) + k\log_2\Big(|\lambda_{x_j}|\Big) - \log_2\Big(|q^{-2}2\pi e|\Big) - \log_2\Big(\mathbf{det}(\mathrm{Cov}\{\boldsymbol{\epsilon}\})\Big)\Bigg). \end{aligned}$$

Finally,

$$\lim_{k\to\infty}\frac{I(\mathbf{x}_j(0);\mathbf{e}^k|\mathbf{r}^k)}{k} \geq \log_2\Big(|\lambda_{x_Y}|\Big). \tag{5.17}$$

We note that the right side of equation (5.13) contains the term $I(\mathbf{x}_{\bar{Y}}(0);\mathbf{e}^k|\mathbf{r}^k,\mathbf{x}_j(0))$ but this term is bounded from below (Lemma 5.6.1) since $\mathrm{E}\{\mathbf{x}_{\bar{Y}}(k)\mathbf{x}_{\bar{Y}}^T(k)\} < \infty$ is required. Therefore

$$\lim_{k\to\infty}\frac{I(\mathbf{x}_{\bar{Y}}(0);\mathbf{e}^k|\mathbf{r}^k,\mathbf{x}_j(0))}{k} \geq \sum_{i\neq j}\log_2\Big(|\lambda_i(A)|\Big). \tag{5.18}$$

From inequalities (5.17) and (5.18) we have

$$\lim_{k\to\infty}\frac{I(\mathbf{x}(0);\mathbf{e}^k|\mathbf{r}^k)}{k} \geq \sum_{i}\log_2\Big(|\lambda_i(A)|\Big).$$

■

Remark 5.6.1 *The general case includes a matrix A with stable and unstable eigenvalues as follows*

$$A = \begin{bmatrix} A_s & \mathbf{0} \\ \mathbf{0} & A_u \end{bmatrix};$$

where A_s corresponds to the matrix with stable eigenvalues and A_u corresponds to the matrix with unstable eigenvalues. As in [24], if $A = A_s$ then we use the property of mutual information to conclude that

$$\lim_{k\to\infty} \frac{I(\mathbf{x}(0); \mathbf{e}^k|\mathbf{r}^k)}{k} \geq 0.$$

If A includes both A_s and A_u then for the proof of Lemmas 5.6.1, 5.6.2, and 5.6.3 we only need to consider the unstable state components (the ones associate with the eigenvalues of A_u) and conclude that

$$\lim_{k\to\infty} \frac{I(\mathbf{x}(0); \mathbf{e}^k|\mathbf{r}^k)}{k} \geq \sum_i \max\{0, \log_2\Big(|\lambda_i(A)|\Big)\}.$$

5.7 Results

Using the results in Section 5.6, we find limitations on tracking systems that are imposed by the presence of a finite capacity channel. We consider the expression $I(\mathbf{r}^k; \hat{\mathbf{y}}^k)$ instead of $I(\mathbf{r}^k; \mathbf{y}^k)$. Although $I(\mathbf{r}^k; \mathbf{y}^k)$ provides the actual information between the output and the reference signals, the former is easier to calculate than the later. The mutual information $I(\mathbf{r}^k; \hat{\mathbf{y}}^k)$ represents the information between the transmitted feedback, i.e., $\hat{\mathbf{y}}^k$, and the reference signal. If this mutual information happens to be zero, all information contained in the feedback signal about the reference signal was lost and the error $\mathbf{e}$ used to generate the control signal is useless. In fact, $I(\mathbf{r}; \hat{\mathbf{y}})$ measures the usefulness of feedback. By the properties of mutual information, we have

$$I((\mathbf{r}^k, \mathbf{x}(0)); \hat{\mathbf{y}}^k) = I(\mathbf{r}^k; \hat{\mathbf{y}}^k) + I(\mathbf{x}(0); \hat{\mathbf{y}}^k|\mathbf{r}^k). \tag{5.19}$$

From the definition of mutual information, Property 5.3.1.(e), and from the fact that $\mathbf{e}^k = \mathbf{r}^k - \hat{\mathbf{y}}^k$, we have

$$\begin{aligned} I(\mathbf{x}(0);\hat{\mathbf{y}}^k|\mathbf{r}^k) &= h(\hat{\mathbf{y}}^k|\mathbf{r}^k) - h(\hat{\mathbf{y}}^k|\mathbf{x}(0),\mathbf{r}^k); \\ &= h(\mathbf{e}^k|\mathbf{r}^k) - h(\mathbf{e}^k|\mathbf{x}(0),\mathbf{r}^k); \\ &= I(\mathbf{x}(0);\mathbf{e}^k|\mathbf{r}^k). \end{aligned} \tag{5.20}$$

From equation (5.19) and (5.20) we have

$$I((\mathbf{r}^k,\mathbf{x}(0));\hat{\mathbf{y}}^k) = I(\mathbf{r}^k;\hat{\mathbf{y}}^k) + I(\mathbf{x}(0);\mathbf{e}^k|\mathbf{r}^k). \tag{5.21}$$

From equation (5.21) and knowing that $kC_f \geq I((\mathbf{r}^k,\mathbf{x}(0));\hat{\mathbf{y}}^k)$ we obtain

$$I(\mathbf{r}^k;\hat{\mathbf{y}}^k) \leq kC_f - I(\mathbf{x}(0);\mathbf{e}^k|\mathbf{r}^k). \tag{5.22}$$

By Lemma 5.6.2, and dividing equation (5.22) by k and taking the limit as $k \to \infty$, we finally have

$$I_\infty(\mathbf{r};\hat{\mathbf{y}}) \leq C_f - \sum_i \log_2\Big(|\lambda_i(A)|\Big).$$

We summarize this result in the following lemma:

Lemma 5.7.1 *Consider the closed-loop system given in Figure 5.1, where the plant is a DLTI system described by equations (5.1) and (5.2), a feedback capacity C_f in the channel. If $E\{\boldsymbol{\epsilon}(k)\boldsymbol{\epsilon}(k)^T\} < \infty$, then*

$$I_\infty(\mathbf{r};\hat{\mathbf{y}}) \leq C_f - \sum_i \log_2\Big(|\lambda_i(A)|\Big).$$

We note from Lemma 5.7.1 that if the channel does not have a minimum capacity of $\sum_i \log_2(|\lambda_i(A)|)$, the feedback signal does not provide any information of the reference signal. Lemma 5.6.2 is one of the main contributions of this chapter. We note that Lemma 5.6.3 is needed when the output is only one of the state components and not the whole state.

5.7.1 Limitations on Reference Signals

The results of the previous sections deal with the idea of bounding the error signal, $\boldsymbol{\epsilon}(k) = \mathbf{r}(k) - \mathbf{y}(k)$. However, it is well known that given a plant and a particular controller, there will be limitations on the type of signals that may be tracked. We show next that a tracking system may be thought of as a channel where the reference signal is the input message, the closed-loop is a feedback channel (with the encoder-decoder embedded) and the system output is the received message. Under this scenario good message estimation is synonymous with good tracking. We consider $\boldsymbol{\epsilon} = \mathbf{r} - \mathbf{y}$ as the error estimate of the message. Note from Property 5.3.1.(i), that

$$E\{(\mathbf{r} - \mathbf{y})^2\} \geq \frac{1}{2\pi e} 2^{2h(\mathbf{r})}.$$

This inequality captures the idea that the greater is the entropy of the reference signal, the larger is the error signal, $\boldsymbol{\epsilon}$. Moreover, since $E\{(\mathbf{r}-\mathbf{y})^2\}$ is a nonnegative number, we note that the error between the output and the reference cannot reach zero unless the reference signal is deterministic ($h(\mathbf{r}) = -\infty$). In other words, perfect tracking is not possible and tracking gets worse for high entropy reference signals regardless of the type or quality of the channel and the controller. Moreover, the following result holds regardless of the plant. Let us consider that the expected value of $(\boldsymbol{\epsilon}^k)^2$ given the entire past $\boldsymbol{\epsilon}_0^{k-1}$ as k tends to infinity given by

$$\sigma_\infty^2(\mathbf{r}) = \lim_{k\to\infty} E\{\boldsymbol{\epsilon}^2(k)|\boldsymbol{\epsilon}(k-1)\}.$$

From information theory, the entropy rate lower-bounds the variance $\sigma_\infty^2(\mathbf{r})$:

$$\sigma_\infty^2(\mathbf{r}) \geq \frac{1}{2\pi e} 2^{2h_\infty(\mathbf{r})}.$$

We then obtain the following lemma.

Lemma 5.7.2 *Consider the closed-loop system given in Figure 5.1, where the plant is a DLTI system described by equations (5.1) and (5.2). Then the best estimator* $\mathbf{y}$

for $\mathbf{r}$ *is bounded as*

$$E\{(\mathbf{r}-\mathbf{y})^2\} \geq \frac{1}{2\pi e} 2^{2h(\mathbf{r})}. \tag{5.23}$$

Moreover, the variance of the best reference estimator, $\sigma^2_\infty(\mathbf{r})$*, is bounded from below as follows*

$$\sigma^2_\infty(\mathbf{r}) \geq \frac{1}{2\pi e} 2^{2h_\infty(\mathbf{r})}. \tag{5.24}$$

5.8 Examples

The results derived in so far are necessary conditions but not sufficient. Since the quantity $I_\infty(\mathbf{r};\hat{\mathbf{y}})$ implies correlation of signals and not necessarily that $\mathbf{y}$ is tracking $\mathbf{r}$. The following examples capture how conservative the results of this chapter are.

5.8.1 Example 1: Erasure Channel

We consider the tracking problem shown in Figure 5.1 for the reference signal, $\mathbf{r}(k)$. The reference signal is assumed to be a white Gaussian sequence, with zero-mean and with $\sigma_r^2 = 1$. We consider a memoryless erasure channel as shown in Figure 5.2 in the feedback link with limited rate and a probability of receiving the state measurement of $p_\gamma = 0.70479$. The probability of dropping a packet is therefore $1-p_\gamma$. We consider a two-part encoder-decoder scheme: First, the encoder converts the real state-vector measured, $\mathbf{x}(k)$, to its binary form, truncates the binary representation to its R most significant bits, then encapsulates the bits in a packet and send the packet through the channel. If the packet is not dropped, the decoder on the receiver site receives the packet, extracts the bits and converts them to its real number representation. If

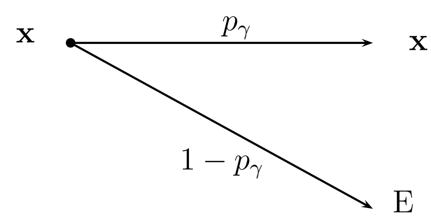

Figure 5.2: Erasure Channel Scheme.

the receiver does not receive a packet, the decoder will assume that a zero was sent and the controller does not apply any control signal. In [48] it is shown that for the scalar case, this scheme guarantees that the error between the actual measurement signal and the decoded signal, $\boldsymbol{\epsilon}(k) = \mathbf{x}(k) - \bar{\mathbf{x}}(k)$, is bounded and that the feedback channel capacity $C_f = \log_2(a)/p_\gamma$ is achieved. The scheme also assumes that the decoder knows exactly the operation of the encoder and that both have access to the control signal. Consider the following plant:

$$\begin{aligned} \mathbf{x}(k+1) &= 4.33\mathbf{x}(k) + \mathbf{u}(k); \\ \mathbf{y}(k) &= \mathbf{x}(k); \\ \mathbf{u}(k) &= 4.33(\mathbf{r}(k) - \bar{\mathbf{y}}(k)). \end{aligned}$$

One limitation of our result is that it is given in terms of the mutual information rate, which is difficult to compute for this type of problems. However, we know that it imposes a limit to guarantee that $\mathrm{E}\{\boldsymbol{\epsilon}(k)\boldsymbol{\epsilon}^T(k)\} < \infty$. In order to explore what happens to $\mathrm{E}\{\boldsymbol{\epsilon}(k)\boldsymbol{\epsilon}^T(k)\}$, we plot the power spectrum of $\boldsymbol{\epsilon}$, $S_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}}(\omega)$ whose enclosed area from $[-\pi, \pi]$ is equivalent to the squared output average of $\boldsymbol{\epsilon}$, i.e.,

$$E\{\boldsymbol{\epsilon}^2\} = \int_{-\pi}^{\pi} S_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}}(\omega) d\omega. \tag{5.25}$$

According to Theorem 5.7.1, the minimum feedback channel capacity for stabilization needed is 3 bits/time-step. The power spectrum density is shown in Figure 5.3, where we notice that the power spectrum is bounded and, therefore, $E\{\boldsymbol{\epsilon}^2(k)\}$ is finite. If, instead of using 3 bits/time-step, we use 2 bits/time-step, we obtain the new power spectrum of the error in Figure 5.4. Note that the power spectrum is becoming

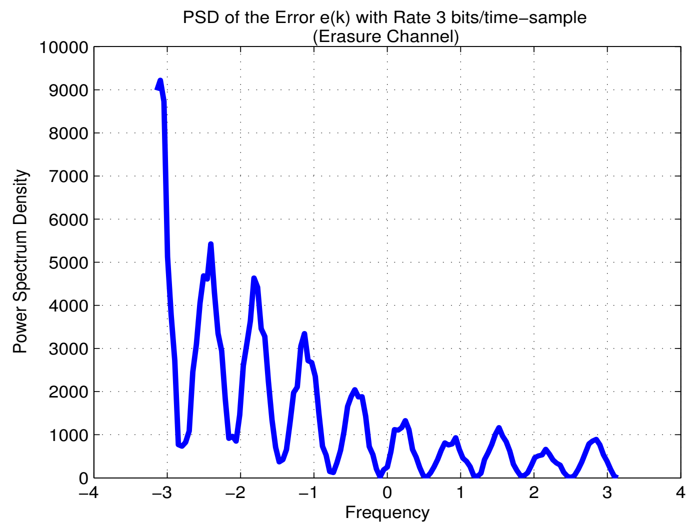

Figure 5.3: Example with Erasure Channel and Bit Rate of 3 bits/time-step.

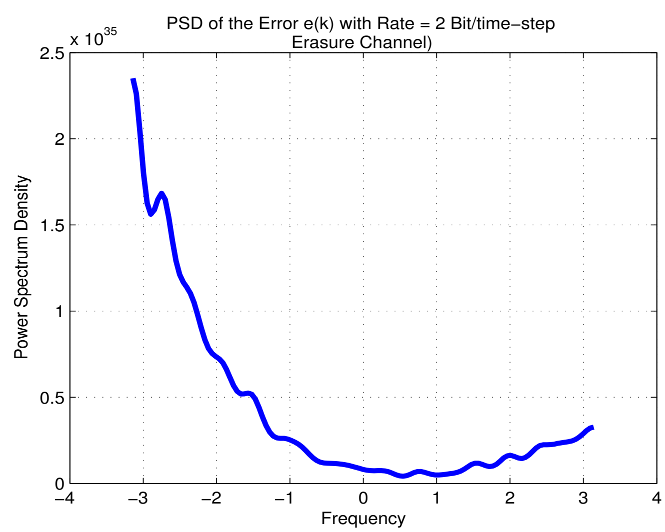

Figure 5.4: Example with Erasure Channel and Bit Rate of 2 bits/time-step.

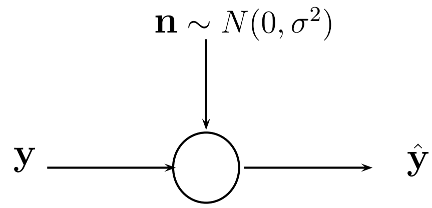

Figure 5.5: AGWN Channel Scheme.

unbounded and so the area below the curve, i.e., $E\{\boldsymbol{\epsilon}^2(k)\}$ is no longer finite.

5.8.2 Example 2: AWGN Channel

We consider the problem of tracking (see Figure 5.1) a reference signal, $\mathbf{r}(k)$, which is assumed to be a white Gaussian sequence with zero-mean and $\sigma_r^2 = 5000$. We consider a memoryless AWGN channel (Figure 5.5) in the feedback link with feedback channel capacity, $C_f = (1/2)\log_2(1 + P/\Phi)$, where Φ is the noise variance and P is the power constraint such that $E\{\hat{\mathbf{y}}^2\} \leq P$. The variance Φ is varied in the range $[1000; 200000]$, i.e, the SNR from the reference signal to the noise signal changes between 0.025 and 5. Let the plant be:

$$
\begin{aligned}
\mathbf{x}(k+1) &= 2\mathbf{x}(k) + \mathbf{u}(k); \\
\mathbf{y}(k) &= \mathbf{x}(k); \\
\mathbf{u}(k) &= 2(\mathbf{r}(k) - \hat{\mathbf{y}}(k)).
\end{aligned}
$$

In this example, we can actually measure the mutual information rate between the reference and the feedback signal for different SNR values, and monitor the upperbound $C_f - \log_2(a)$ given in Lemma 5.7.1. We use previous results from [38] to measure the mutual information rate, $I_\infty(\mathbf{r}; \hat{\mathbf{y}})$, and results from [2] to design a controller. Since the system is linear and all inputs are white Gaussian processes, the output $\hat{\mathbf{y}}$ is also a Gaussian process. From [38], we know that if $\mathbf{r}$ and $\hat{\mathbf{y}}$ are two jointly-Gaussian stationary processes, with spectral densities $\Phi_r(\omega)$ and $\Phi_{\hat{\mathbf{y}}}(\omega)$, and if we define $w = \begin{bmatrix} \mathbf{r} \\ \hat{\mathbf{y}} \end{bmatrix}$, with spectral density $\Phi_w(\omega)$, the mutual information rate of $\mathbf{r}$ and $\hat{\mathbf{y}}$ is given by

$$I_\infty(\mathbf{r};\hat{\mathbf{y}}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\det(\Phi_r(\omega))\det(\Phi_{\hat{y}}(\omega))}{\det(\Phi_w(\omega))} d\omega. \tag{5.26}$$

Figure 5.6 illustrates that we obtain the expected result. The mutual information rate tends to zero for low SNR and, for this particular case reaches its upper bound, i.e. $C_f - \log_2(a)$, for high SNR. We see that this upper bound never reaches a value of zero (actually, for a SNR of 0, its value is 0.61 bits/time). We conclude, however, that the bound for good tracking, as measured by $I_\infty(\mathbf{r};\hat{\mathbf{y}})$, is higher that the one for stabilization.

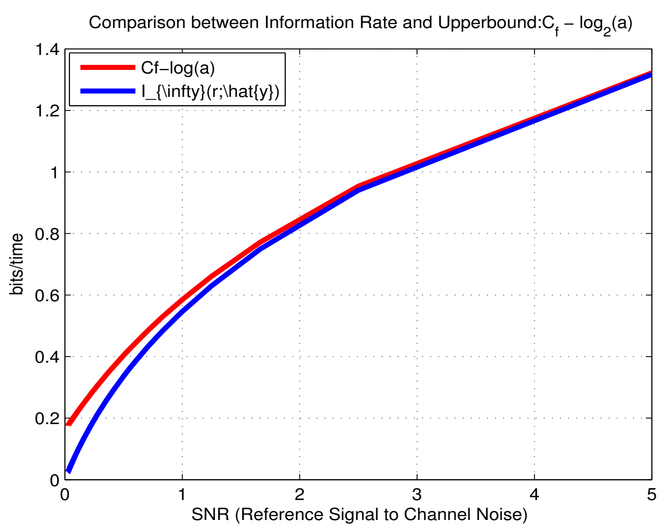

Figure 5.6: Example with AWGN Channel for different SNR levels.

5.8.3 Example 3: Limitations Due to the Entropy of the Reference

This example is presented to illustrate the results in Subsection 5.7.1. Let us consider the following system/controller:

$$\begin{aligned}\mathbf{x}(k+1) &= 1.2\mathbf{x}(k)+\mathbf{u}(k);\\ \mathbf{y}(k) &= \mathbf{x}(k);\\ \mathbf{u}(k) &= \mathbf{r}(k)-0.3\bar{\mathbf{y}}(k).\end{aligned}$$

Assume that the reference signal is given by

$$\mathbf{r}(k)=2+\mathbf{n}_r(k);$$

where $\mathbf{n}_r$ has a Gaussian distribution with zero mean and variance σ^2 with $\sigma=2$. Moreover, we assume a perfect feedback of the output. Since the reference is a Gaussian signal, by substitution in Property 5.3.1.**??**, the differential entropy of the reference signal is given by $h(\mathbf{r})=(1/2)\log_2\left(2\pi e\sigma^2\right)$. According to Lemma 5.7.2, the lower bound in the right side of equation (5.23) is σ^2. In order to plot $E\{(\mathbf{r}-\mathbf{y})^2\}=E\{(\boldsymbol{\epsilon})^2\}$, we ran 1000 simulations and averaged then. The average result of these simulations is shown in Figure 5.7, which clearly illustrates the result of Lemma 5.7.2.

5.9 A Misleading Case: Non-minimum Phase Zeros

The mutual information rate $I_\infty(\mathbf{r},\hat{\mathbf{y}})$ between the reference signal $\mathbf{r}(k)$ and the feedback signal $\hat{\mathbf{y}}(k)$ has been our performance measure in the previous sections of

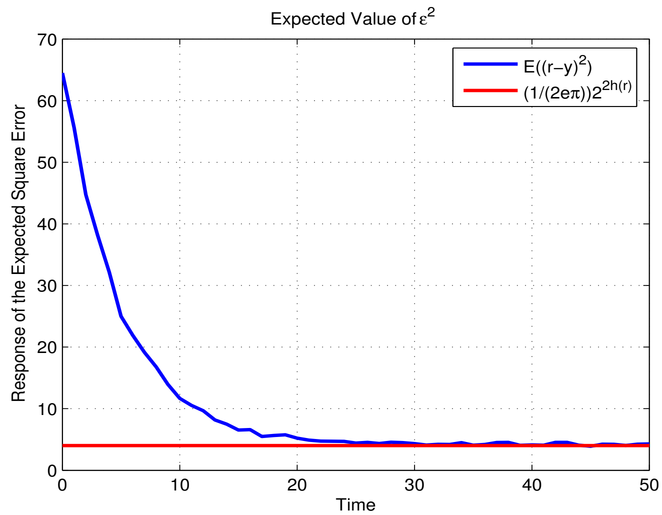

Figure 5.7: Example with Gaussian Reference Signal.

this chapter. Although it may be adequate to determine the relationship between channel capacity, unstable poles, and the possibility of achieving tracking, $I_\infty(\mathbf{r}, \hat{\mathbf{y}})$ is limited in predicting other important properties.

In order to illustrate the limitations of $I_\infty(\mathbf{r}; \hat{\mathbf{y}})$, we choose an AWGW channel. Let us consider the same LTI plant $P(z)$ as before and let us restrict the controller to be a linear time-invariant controller $C(z)$. We assume that the open loop transfer function is given by

$$C(z)P(z) = \gamma \frac{\prod_{i=1}^{n_z}(z - z_i)}{\prod_{i=1}^{n_p}(z - p_i)}.$$

Since we consider an AWGN channel and if we assume $\mathbf{r}(k)$ to be a Gaussian signal, $\mathbf{r}(k)$ and $\hat{\mathbf{y}}(k)$ are jointly Gaussian and we can then evaluate the mutual information rate exactly using equation (5.26). We start with the following relation

$$\hat{\mathbf{y}} = T(e^{i\omega})\mathbf{r} + S(e^{i\omega})\mathbf{n};$$

where $T(e^{i\omega})$ is the Complementary Sensitivity function and $S(e^{i\omega})$ is the Sensitivity function. Let $w = \begin{bmatrix} \mathbf{r} \\ \hat{\mathbf{y}} \end{bmatrix}$, then $\Phi_w = \Phi_r \Phi_{\hat{y}} - \Phi_{r\hat{y}} \Phi_{\hat{y}r}$ so that

$$\begin{aligned} \Phi_{\hat{y}} &= |T|^2 \Phi_r + |S|^2 \Phi_n; \\ \Phi_w &= \Phi_r \Phi_n |S|^2. \end{aligned}$$

Substituting these relations in equation (5.26) we obtain

$$I_\infty(\mathbf{r}; \hat{\mathbf{y}}) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \log_2 \Big(\frac{|T(e^{iw}|^2 \Phi_r + |S(e^{iw})|^2 \Phi_n}{\Phi_n |S(e^{iw})|^2} \Big) d\omega; \quad (5.27)$$

$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \log_2 \Big(\frac{\Phi_r}{\Phi_n} |C(e^{iw}) P(e^{iw})|^2 + 1 \Big) d\omega; \quad (5.28)$$

where

$$|C(z))P(z)|^2 = \Big| \gamma \frac{\prod_{i=1}^{n_z} (z - z_{pi})}{\prod_{i=1}^{n_p} (z - p_{pi})} \Big|^2.$$

Now, from equation (5.28) and using the properties of logarithms, we have:

$$\begin{aligned} I_\infty(\mathbf{r}; \hat{\mathbf{y}}) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \log_2 \Big(\frac{\Phi_r}{\Phi_n} |C(e^{iw}) P(e^{iw})|^2 + 1 \Big) d\omega; \\ &\geq \frac{1}{4\pi} \int_{-\pi}^{\pi} \log_2 \Big(\frac{\Phi_r}{\Phi_n} |C(e^{iw}) P(e^{iw})|^2 \Big) d\omega; \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \log_2 \Big(\frac{\Phi_r}{\Phi_n} \Big) d\omega + \frac{1}{4\pi} \int_{-\pi}^{\pi} \log_2 \Big(|C(e^{iw}) P(e^{iw})|^2 \Big) d\omega; \\ &= \log_2(|\gamma|) + \frac{1}{4\pi} \int_{-\pi}^{\pi} \log_2 \Big(\frac{\Phi_r}{\Phi_n} \Big) d\omega + \sum_{i=1}^{n_z} \frac{1}{4\pi} \int_{-\pi}^{\pi} \log_2 |z - z_i|^2 d\omega \\ &\quad - \sum_{i=1}^{n_p} \frac{1}{4\pi} \int_{-\pi}^{\pi} \log_2 |z - p_i|^2 d\omega. \end{aligned}$$

From complex variable Calculus we have the following result:

$$\int_{-\pi}^{\pi} \log_2 |z-p|^2 d\omega = \begin{cases} 0 & \text{if } |p| \leq 1; \\ 2\pi \log_2(p^2) & \text{if } |p| > 1. \end{cases}$$

Finally, we obtain the following lower bound for the mutual information rate:

$$I_\infty(\mathbf{r};\hat{\mathbf{y}}) > \log_2(|\gamma|) + \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(\frac{\Phi_r}{\Phi_n}\right) d\omega + \sum_{i=1}^{n_z} \log_2\Big(|z_i|\Big) - \sum_{i=1}^{n_p} \log_2\Big(|p_i|\Big).$$

We note that the right hand side contains a signal-to-noise ratio term, a gain term, a term that corresponds to the unstable open loop poles, and one that corresponds to the open-loop unstable zeros. We note first as expected, that the greater the signal-to-noise ratio is, the greater the mutual information rate between the reference and the output signal. Second, we note that the unstable open-loop poles decrease the mutual information rate. Finally, we note that the non-minimum phase zeros term increases the mutual information rate. This is unexpected since we know from control theory that the presence of non-minimum phase zeros decreases the performance of a tracking systems, therefore, it seems that we reach a contradiction.

We have another interpretation to this issue. Since the unstable poles decreases the information flow, the presence of the unstable zeros can help to cancel this effect (with perfect zero-pole cancelation). From control theory, we now that this is not an option if we want to preserved internal stability. But this issue was not consider in the analysis, i.e., the only analysis of the mutual information rate is not enough when designing a tracking feedback system and we see that it could be misleading.

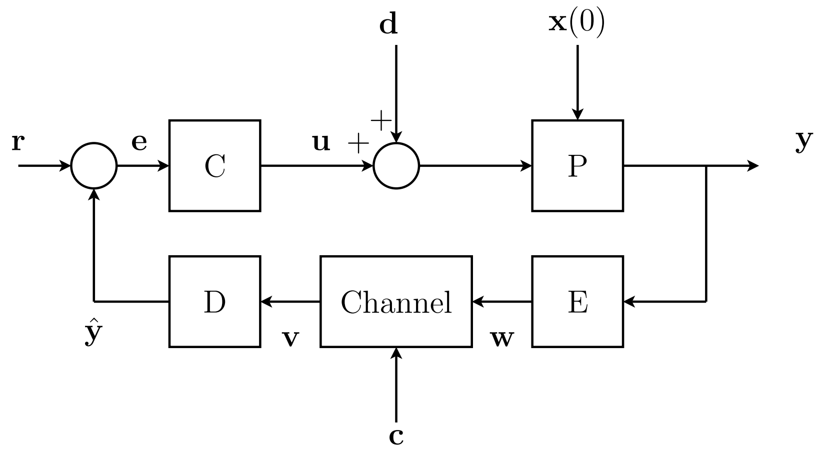

Figure 5.8: Closed-Loop System with Additive Disturbance.

5.10 Tracking under the Presence of Disturbances

5.10.1 Upper Bound of the Information Flow in the Presence of Disturbance

Let us suppose that a disturbance is present at the sensor and that the disturbance $\mathbf{d}^k$ is independent of $\mathbf{x}(0)$ and of $\mathbf{r}^k$. The new diagram is shown in Figure 5.8. We try next to find conditions for tracking. We first redefine the feedback capacity in this new setup. Recall that the feedback capacity is the quantity C_f that satisfies

$$\sup_{k \in \mathbb{N}_+} \frac{I((\mathbf{r}^k, \mathbf{d}^k, \mathbf{x}(0)); \hat{\mathbf{y}}^k)}{k} \leq C_f.$$

If we expand the quantity $I((\mathbf{r}^k, \mathbf{d}^k, \mathbf{x}(0)); \hat{\mathbf{y}}^k)$, by Property 5.3.1.(h) we obtain

$$I((\mathbf{r}^k, \mathbf{d}^k, \mathbf{x}(0)); \hat{\mathbf{y}}^k) = I(\mathbf{r}^k; \hat{\mathbf{y}}^k) + I(\mathbf{x}(0); \hat{\mathbf{y}}^k | \mathbf{r}^k) + I(\mathbf{d}^k; \hat{\mathbf{y}}^k | \mathbf{x}(0), \mathbf{r}^k). \quad (5.29)$$

Let us focus on $I(\mathbf{d}^k;\hat{\mathbf{y}}^k|\mathbf{x}(0),\mathbf{r}^k)$ to obtain

$$\begin{aligned} I(\mathbf{d}^k;\hat{\mathbf{y}}^k|\mathbf{x}(0),\mathbf{r}^k) &= h(\mathbf{d}^k|\mathbf{x}(0),\mathbf{r}^k) - h(\mathbf{d}^k|\mathbf{x}(0),\mathbf{r}^k,\hat{\mathbf{y}}^k); \\ &= h(\mathbf{d}^k) - h(\mathbf{d}^k|\mathbf{x}(0),\mathbf{r}^k,\hat{\mathbf{y}}^k); \quad (5.30) \\ &\geq h(\mathbf{d}^k) - h(\mathbf{d}^k|\hat{\mathbf{y}}^k); \quad (5.31) \\ &= I(\mathbf{d}^k;\hat{\mathbf{y}}^k); \quad (5.32) \end{aligned}$$

where equations (5.30) and (5.31) are due to Property 5.3.1.(a) and equation (5.32) results form the mutual information definition. We showed in equation (5.20) that $I(\mathbf{x}(0);\hat{\mathbf{y}}^k|\mathbf{r}^k) = I(\mathbf{x}(0);\mathbf{e}^k|\mathbf{r}^k)$. If we revisit Lemma 5.6.2's proof, we see that the lemma holds even with disturbances. Therefore, $I(\mathbf{x}(0);\mathbf{e}^k|\mathbf{r}^k) \geq k\sum_i \log_2(|\lambda_i(A)|)$. Moreover, from the definition of feedback capacity we know that

$$kC_f \geq I((\mathbf{r}^k,\mathbf{d}^k,\mathbf{x}(0));\hat{\mathbf{y}}^k)$$

then, from equation (5.29) we obtain

$$kC_f - k\sum_i \log_2\Big(|\lambda_i(A)|\Big) \geq I(\mathbf{r}^k;\hat{\mathbf{y}}^k) + I(\mathbf{d}^k;\hat{\mathbf{y}}^k).$$

If we divide by k and take the limit as $k \to \infty$ we finally have:

$$I_\infty(\mathbf{r};\hat{\mathbf{y}}) + I_\infty(\mathbf{d};\hat{\mathbf{y}}) \leq C_f - \sum_i \log_2\Big(|\lambda_i(A)|\Big).$$

This result may be summarized in the following theorem:

Theorem 5.10.1 *Consider the closed-loop system given by Figure 5.1, where the plant is a DLTI system described by equations (5.1) and (5.2), a feedback capacity C_f in the channel. If $E\{\boldsymbol{\epsilon}(k)\boldsymbol{\epsilon}(k)^T\} < \infty$, then*

$$I_\infty(\mathbf{r};\hat{\mathbf{y}}) + I_\infty(\mathbf{d};\hat{\mathbf{y}}) \leq C_f - \sum_i \log_2\Big(|\lambda_i(A)|\Big).$$

From this result we can see that if $I_\infty(\mathbf{d};\hat{\mathbf{y}})$ is large enough, compared with

$$C_f - \sum_i \log_2(|\lambda_i(A)|);$$

no useful information about the reference appears in the feedback, since the inequality may also be interpreted as

$$I_\infty(\mathbf{r};\hat{\mathbf{y}}) \leq C_f - \sum_i \log_2\Big(|\lambda_i(A)|\Big) - I_\infty(\mathbf{d};\hat{\mathbf{y}}).$$

Similarly,

$$I_\infty(\mathbf{d};\hat{\mathbf{y}}) \leq C_f - \sum_i \log_2\Big(|\lambda_i(A)|\Big) - I_\infty(\mathbf{r};\hat{\mathbf{y}}).$$

If $I_\infty(\mathbf{r};\hat{\mathbf{y}})$ is large enough, compared with $C_f - \sum_i \log_2(|\lambda_i(A)|)$, no useful information about the disturbance appears in the feedback.

5.10.2 Disturbance Rejection and Tracking Tradeoff

The previous subsection is concluded with Theorem 5.10.1. The goal in this section is to interpret Theorem 5.10.1 in the frequency domain. For this purpose, we assume that the following conditions hold:

- The signals $\mathbf{r}$ and $\mathbf{d}$ are Gaussian.
- The signals $\mathbf{r}$ and $\mathbf{e}$ are jointly asymptotically stationary.
- The signals $\mathbf{d}$ and $\mathbf{e}$ are jointly asymptotically stationary.

These conditions are needed to replace the stochastic processes by their corresponding asymptotic power spectra. Next, we start with the definition of the mutual

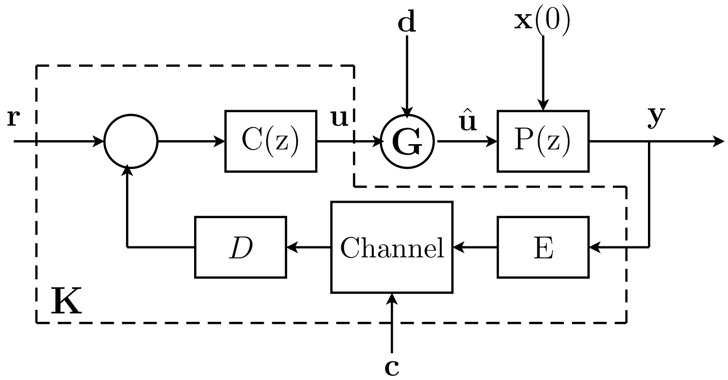

Figure 5.9: Equivalence of Tracking Closed-Loop with Block Diagram 2.3.

information between $\mathbf{r}$ and $\hat{\mathbf{y}}$:

$$\begin{aligned} I(\mathbf{r}^k;\hat{\mathbf{y}}^k) &= h(\mathbf{r}^k) - h(\mathbf{r}^k|\hat{\mathbf{y}}^k); \\ &= h(\mathbf{r}^k) - h(\mathbf{e}^k|\hat{\mathbf{y}}^k); \qquad (5.33) \\ &\geq h(\mathbf{r}^k) - h(\mathbf{e}^k); \qquad (5.34) \end{aligned}$$

where equation (5.33) is due to the fact that $\mathbf{e} = \mathbf{r} - \hat{\mathbf{y}}$ and Property 5.3.1.(e), equation (5.34) is due to Property 5.3.1.(a). If we divide by k and let $k \to \infty$ we obtain:

$$\begin{aligned} I_\infty(\mathbf{r};\hat{\mathbf{y}}) &\geq h_\infty(\mathbf{r}) - h_\infty(\mathbf{e}); \\ &\geq \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(2\pi e\widehat{\Phi}_r\right)d\omega - \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(2\pi e\widehat{\Phi}_e\right)d\omega; \qquad (5.35) \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(\frac{\widehat{\Phi}_r}{\widehat{\Phi}_e}\right)d\omega; \qquad (5.36) \end{aligned}$$

where equation (5.35) is due to Property 5.3.1.(k). Changing the sign in inequality (5.36), we get

$$-I_\infty(\mathbf{r};\hat{\mathbf{y}}) \leq \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(\frac{\widehat{\Phi}_e}{\widehat{\Phi}_r}\right)d\omega.$$

Then, using the inequality of Theorem 5.10.1 we obtain

$$I_\infty(\mathbf{d};\hat{\mathbf{y}}) \leq C_f - \sum \log(\lambda) + \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(\frac{\widehat{\Phi}_e}{\widehat{\Phi}_r}\right) d\omega. \tag{5.37}$$

In Figure 5.9, we group together the blocks enclosed within the dashed line. By doing so, we obtain the same block diagrams shown in Figure 2.3. We notice that some of the internal signals are labeled differently: in Figure 5.9 the signals $\mathbf{u}$ and $\hat{\mathbf{u}}$ correspond to signals $\mathbf{z}$ and $\mathbf{e}$ in Figure 2.3, respectively. According to Theorem 2.3.3 in Chapter 2, we know that $I_\infty(\mathbf{d};\hat{\mathbf{u}})$ is related to a disturbance rejection measure as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \min\{0, \log_2(S_{\hat{u},d}(\omega))\} d\omega \geq -I_\infty(\mathbf{d};\mathbf{u});$$

where, $S_{\hat{u},d}(\omega) = \sqrt{\frac{\widehat{\Phi}_{\hat{u}}}{\widehat{\Phi}_d}}$. We note that the smaller the term

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \min\{0, \log_2(S_{\hat{u},d}(\omega))\} d\omega$$

is, the better is the disturbance rejection. From Property 5.3.1.(g) we know that $I_\infty(\mathbf{d};\mathbf{u}) \leq I_\infty(\mathbf{d};\hat{\mathbf{y}})$. Substituting this expression in equation (5.37) we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \min\{0, \log_2(S_{\hat{u},d}(\omega))\} d\omega \geq \sum_{\lambda(A)} \log_2(\lambda) - C_f - \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(\frac{\widehat{\Phi}_e}{\widehat{\Phi}_r}\right) d\omega.$$

We summarize this result in the following theorem:

Theorem 5.10.2 *Consider the closed-loop system given by Figure 5.1, where the plant is a linear system described by equations (5.1) and (5.2), a feedback capacity* C_f *in the channel. If* $E\{\boldsymbol{\epsilon}(k)\boldsymbol{\epsilon}(k)^T\} < \infty$, $\mathbf{r}$ *and* $\mathbf{d}$ *are Gaussian signals,* $\mathbf{r}$ *and* $\mathbf{e}$ *are jointly asymptotically stationary,* $\mathbf{d}$ *and* $\mathbf{e}$ *are jointly asymptotically stationary, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \min\{0, \log_2(S_{\hat{u},d}(\omega))\} d\omega \geq \sum_{\lambda(A)} \log_2(\lambda) - C_f - \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(\frac{\widehat{\Phi}_e}{\widehat{\Phi}_r}\right) d\omega. \tag{5.38}$$

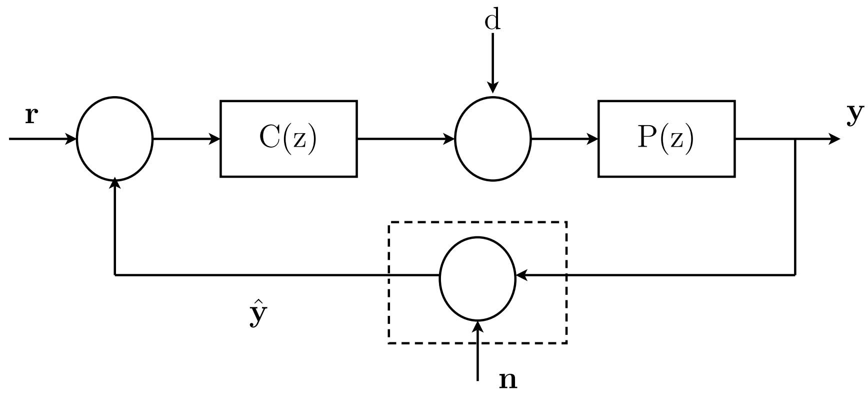

Figure 5.10: Closed-Loop System with AWGN Channel and Disturbance Presence.

where $S_{\hat{u},d}(\omega) = \sqrt{\frac{\widehat{\Phi}_{\hat{u}}}{\widehat{\Phi}_d}}$ *is a sensitivity-like function,* $\widehat{\Phi}_{\hat{u}}$*,* $\widehat{\Phi}_d$*,* $\widehat{\Phi}_e$ *and* $\widehat{\Phi}_r$ *are the asymptotical power spectrum densities of the signals* $\hat{\mathbf{u}}$*,* $\mathbf{d}$*,* $\mathbf{e}$ *and* $\mathbf{r}$*, respectively.*

We therefore observe that for good tracking, formally defined as being $\widehat{\Phi}_e$ near zero, implies $\log_2\left(\frac{\widehat{\Phi}_e}{\widehat{\Phi}_r}\right)$ to be negative and the whole integral term in inequality (5.38) positive. Therefore, the lower bound will be larger than the one where no tracking is required. In other words, if we improve tracking performance, we loose the information between the disturbance $\mathbf{d}$ and the feedback signal $\hat{\mathbf{y}}$, and the disturbance can no longer be rejected.

5.10.3 Design Ideas for Disturbance Rejection

Let us consider the case of achieving tracking in the presence of disturbance. From Theorem 5.10.1 we know that a finite capacity channel limits the information of the reference signal and the disturbance signal available at the feedback. Let us analyze the case of a feedback configuration with a AWGN channel shown in Figure 5.1. To accomplish good tracking, we need to force the information rate between $\mathbf{r}$ and $\mathbf{y}$ to be greater than the information rate $\mathbf{d}$ and $\mathbf{y}$, i.e.:

$$I_\infty(\mathbf{r};\mathbf{y}) > I_\infty(\mathbf{d};\mathbf{y}). \tag{5.39}$$

If we assume that $\mathbf{r}$ is zero white noise and that the disturbance is also white noise, we may calculate exactly the two mutual information rates in inequality (5.39). Before we proceed, let us introduce the following transfer functions:

$$\begin{aligned}\frac{\mathbf{y}}{\mathbf{r}} &= \frac{C(z)P(z)}{1+C(z)P(z)};\\ \frac{\mathbf{y}}{\mathbf{d}} &= \frac{1}{1+C(z)P(z)};\\ \frac{\mathbf{y}}{\mathbf{c}} &= \frac{-C(z)P(z)}{1+C(z)P(z)}.\end{aligned}$$

To simplify notation, we define

$$\begin{aligned}T(z) &= \frac{C(z)P(z)}{1+C(z)P(z)};\\ V(z) &= \frac{P(z)}{1+C(z)P(z)}.\end{aligned}$$

We use these transfer functions to calculate the spectral density of $\mathbf{y}$

$$\Phi_y = (\Phi_r + \Phi_c)|T(z)|^2 + \Phi_d|V(z)|^2.$$

Let $w_{ry} = \begin{bmatrix}\mathbf{r}\\ \mathbf{y}\end{bmatrix}$, then $\Phi_{w_{ry}} = \Phi_r\Phi_y - \Phi_{ry}\Phi_{yr}$, where $\Phi_{ry} = \Phi_r T$ and $\Phi_{yr} = \Phi_r T^*$, and $T(z)^*$ is the complex conjugate of $T(z)$. Similarly, we define $w_{dy} = \begin{bmatrix}\mathbf{d}\\ \mathbf{y}\end{bmatrix}$, then $\Phi_{w_{dy}} = \Phi_d\Phi_y - \Phi_{dy}\Phi_{yd}$, where $\Phi_{dy} = \Phi_d V$ and $\Phi_{yd} = \Phi_y V^*$, and V^* is the complex conjugate of $V(z)$. Next, from equation (5.26) we calculate the mutual information rate between $\mathbf{r}$ and $\mathbf{y}$:

$$I_\infty(\mathbf{r};\mathbf{y}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(1 + \frac{\Phi_r}{\Phi_d(\frac{1}{|C(e^{i\omega})P(e^{i\omega})|^2}) + \Phi_c}\right)d\omega.$$

Similarly, we have the mutual information rate between $\mathbf{d}$ and $\mathbf{y}$:

$$I_\infty(\mathbf{d};\mathbf{y}) = \frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(1 + \frac{\Phi_d}{(\Phi_r + \Phi_c)|C(e^{i\omega})P(e^{i\omega})|^2}\right)d\omega.$$

Since inequality (5.39) is a necessary condition to achieve a good tracking, we obtain the following inequality

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}\log_2\left(\frac{\frac{\Phi_r|C(e^{i\omega})|^2+\Phi_c|C(e^{i\omega})|^2+\Phi_d}{|C(e^{i\omega})|^2\Phi_c+\Phi_d}}{\frac{(\Phi_r+\Phi_c)|C(e^{i\omega})|^2+\Phi_d}{(\Phi_r+\Phi_c)|C(e^{i\omega})|^2}}\right)d\omega > 0.$$

This may be simplified to:

$$\begin{aligned}\frac{1}{4\pi}\int_{-\pi}^{\pi}\log_2\left(\frac{(\Phi_r+\Phi_c)|C(e^{i\omega})|^2}{\Phi_d+\Phi_c|C(e^{i\omega})|^2}\right)d\omega &> 0;\\ \log_2\left(\frac{\Phi_r}{\Phi_d}+\frac{1}{\gamma}\right)^{\frac{1}{2}}+\frac{1}{4\pi}\int_{-\pi}^{\pi}\log_2\left(\frac{|C(e^{i\omega})|^2}{1+\frac{1}{\gamma}|C(e^{i\omega})|^2}\right)d\omega &> 0; \qquad (5.40)\end{aligned}$$

where $\gamma = \frac{\Phi_d}{\Phi_c}$. Inequality (5.40) provides guidelines for controller design. We will consider next two cases: a) $\gamma > 1$, b) $\gamma < 1$. These two cases are shown in Figures 5.11 and 5.12 where we plot the integrand $\log_2\left(\frac{|C(e^{i\omega})|^2}{1+\frac{1}{\gamma}|C(e^{i\omega})|^2}\right)$.

Case $\gamma > 1$

For the case illustrated in Figure 5.11, $\gamma > 1$, we note that a large gain controller, for all $\omega \in [-\pi, \pi]$, is sufficient to satisfy inequality (5.39) since

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}\log_2\left(\frac{|C(e^{i\omega})|^2}{1+\frac{1}{\gamma}|C(e^{i\omega})|^2}\right)d\omega \approx \log_2(\gamma)^{\frac{1}{2}}. \qquad (5.41)$$

If we add to equation (5.41) the quantity $\log_2\left(\frac{\Phi_r}{\Phi_d}+\frac{1}{\gamma}\right)^{\frac{1}{2}}$, we obtain that:

$$\log_2(\gamma)^{\frac{1}{2}}+\log_2\left(\frac{\Phi_r}{\Phi_d}+\frac{1}{\gamma}\right)^{\frac{1}{2}} = \log_2\left(\frac{\Phi_r}{\Phi_n}+1\right)^{\frac{1}{2}} > 0.$$

To keep $|C(e^{i\omega})|$ large at all frequencies is not always possible nor recommended. The condition

$$\frac{1}{4\pi}\int_{-\pi}^{\pi}\log_2\left(\frac{|C(e^{i\omega})|^2}{1+\frac{1}{\gamma}|C(e^{i\omega})|^2}\right)d\omega > 0$$

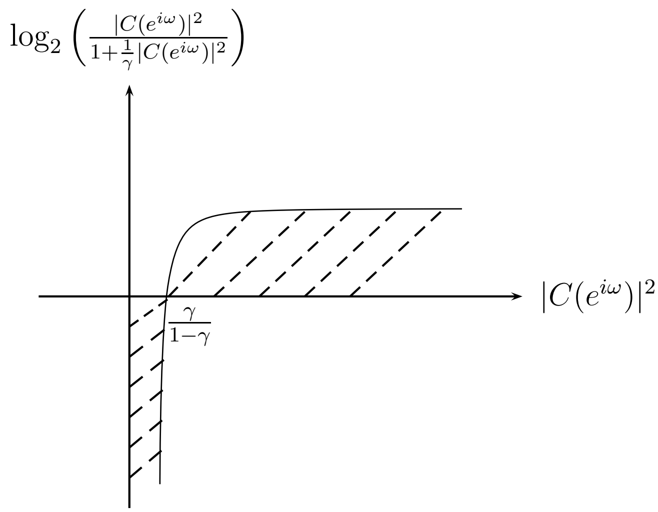

Figure 5.11: Plot of $\log_2\left(\frac{|C(e^{i\omega})|^2}{1+\frac{1}{\gamma}|C(e^{i\omega})|^2}\right)$ with $\gamma > 1$.

can however be used to shape the controller in the frequency-domain.

Case $\gamma < 1$

In this case, the integral term in inequality (5.40) is negative, i.e,

$$\frac{1}{4\pi}\int_{-\pi}^{\pi} \log_2\left(\frac{|C(e^{i\omega})|^2}{1+\frac{1}{\gamma}|C(e^{i\omega})|^2}\right)d\omega < 0.$$

Therefore, to keep inequality valid, we need to make $\log_2\left(\frac{\Phi_r}{\Phi_d}+\frac{1}{\gamma}\right)^{\frac{1}{2}}$ large and to keep the magnitude of the controller large for all frequencies. We note in Figure 5.12 that the smallest the magnitude of the controller, the more negative is the integral and, therefore, we need a much better signal-to-noise ratio $\frac{\Phi_r}{\Phi_d}$ or a smaller signal-to-noise ratio $\gamma = \frac{\Phi_d}{\Phi_n}$. This last observation about γ is a consequence of the fact that we can only reject a disturbance if we have good information about it, i.e., when the noise signal of the channel, n, does not distort the disturbance signal too much. In

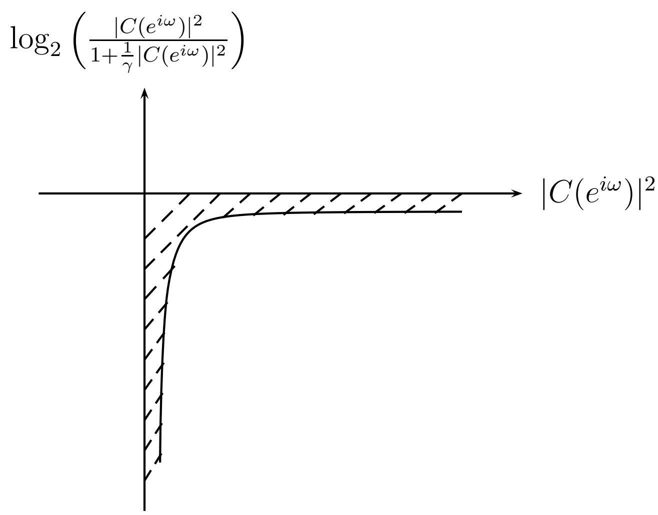

Figure 5.12: Plot of $\log_2\left(\frac{|C(e^{i\omega})|^2}{1+\frac{1}{\gamma}|C(e^{i\omega})|^2}\right)$ with $\gamma < 1$.

addition to the two analyzed cases, we note that, for our particular channel, the input to the AWGN channel is $\mathbf{y}$ and by definition is assumed to be constrained in power. The power limit is given by some constant $\mathbb{P}$:

$$||\mathbf{y}||_{POW} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_y(\omega)d\omega < \mathbb{P}. \tag{5.42}$$

But $||\mathbf{y}||_{POW}$ is given by

$$\begin{aligned} ||\mathbf{y}||_{POW} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_y(\omega)d\omega; \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |T|^2(\Phi_r(\omega) + \Phi_c(\omega)) + \Phi_d(\omega)|V|^2 d\omega. \end{aligned} \tag{5.43}$$

In other words, equations (5.42) and (5.43) provide another limitation that has to be considered while designing a controller.

5.11 Summary

This chapter has provided information theoretic conditions for tracking control systems. Our results are in terms of the mutual information rate between the feedback signal and the reference signal, the channel capacity, and the unstable eigenvalues of the DLTI system. We also obtained a lower bound for the maximum achievable accuracy for a tracking system, even in the absence of a channel. This bound is in terms of the entropy of the reference signal. These results were verified with several examples and simulations.

We also reported some limitations of the mutual information rate approach. In particular, we analyzed the case where non-minimum phase zeros, counterintuitively, increase the mutual information rate instead of decreasing it as expected from control theory.

Finally, we analyzed the case where both good tracking and good disturbance rejection are required at the same time. We noted that the finite-capacity channel imposes a tradeoff between the two objectives. This limitation was interpreted in the frequency-domain.

Chapter 6

General Conclusions and Future Work

6.1 General Conclusions

In this dissertation we addressed the stabilization of unstable DLTI systems under communication constraints. The problem was formulated considering a noiseless channel but with limitations in the data rate and we obtain sufficient conditions for the stabilization rate.

We considered several encoding schemes that were very simple to implement, although they do not achieve the minimum rate given by the Data Rate Theorem. The first scheme proposed was a 2-Bit Delta-Modulation-like scheme to encode the output of a scalar system. This scheme guaranteed the stabilization of scalar systems with eigenvalue $a < 3$. Moreover, the scheme guarantees stabilization even in the case of an specific number of lost bits. The drawback of the scheme is that it does not exploit all the potential of every bit in terms of stabilization. By this we mean that the degree of unstability of the systems that can be stabilized is far away from the one given by the Data Rate Theorem.

We then addressed the stabilization of multi-dimensional DLTI systems. We show a truncation-based system that is very simple to implement but that requires a larger stabilization rate compare to previous schemes in the literature. The main limitation of this approach is that it depends on an specific control law and not in a general linear state feedback.

Another approach that does work with a general state feedback control law is the dynamic quantizer that we proposed. This achieves lower rates than the truncation-based scheme and it is simpler to implement compared to others in the literature. However, one possible drawback of the scheme is that its zoom-in feature requires a very precise model of the plant to predict the evolution of the quantizer in the encoder and decoder.

We also obtain fundamental limitations for tracking systems under communication constraints. We obtain an upper bound for the information flow between the reference and the feedback signals in terms of the feedback channel capacity and the unstable eigenvalues of the plant. This bound is universal in the sense that it is independent of the encoding scheme or the control law. Although this result is not surprising, it provides an interesting interpretation of the tracking problem in terms of information theoretical quantities. We also notice that the solely information theoretical interpretation is not completed to analyze the tracking problem, in particular for the non-minimum phase zeros.

Finally, we provide an interpretation of the tracking-rejection problem in the frequency domain. It was shown that a finite capacity channel limits more the possibility of having good tracking and disturbance rejection simultaneously. Moreover, we see that the exact measurements of the information rate for the Gaussian Channel gives some guidelines of the shape of the controller in the frequency domain.

6.2 Future Work

The current research topic is far from being completed. We have identified some paths for future research. We plan to explore a Delta-Modulation encoding-decoding scheme for multidimensional systems and include time-delays in the channel. The idea of the 2-Bit Delta Modulator may also be extended for the M-Bit case in order to control systems with arbitrary magnitude eigenvalues a and/or to allow for a larger number of dropped packets. A time-varying scheduling policy to reach global stability may be explored in a multi-bit setup.

For the results in Chapter 4 we want to include time delay in a NCS Type II, and the extension of the general case of m inputs of this type of closed-loop system. Some other ideas include dealing with noise in the loop and the generalization to the case of packet drops and saturation in the control signal. For the dynamic quantizer there is the open question of how to select the values of r_i in terms of the eigenvalues. This is an optimization problem that may be interesting for future research. Similarly, these encoding schemes must be generalized at least to consider noisy channels and inaccuracies in the plant model.

Finally, we see that the the fundamental limitations that we obtain for tracking systems were limited to guarantee boundedness of the tracking error. We think that the accuracy of the tracking problem, measure in terms of some metric of the tracking error, is also lower bounded by the finite capacity of the channel and not only by the entropy of the reference signal. This would be another interesting research area for the future.

Appendices

Appendix A

Convergence of Random Sequences

According to [49] we have the following types of convergence for random sequences:

Definition A.0.1 *The random sequence x^k converges surely to the random variable x if the sequence of functions $x^k(\zeta)$ converges to the function $x(\zeta)$ as $k \to \infty$ for all $\zeta \in \Omega$.*

Definition A.0.2 *The random sequence x^k converges almost surely to the random variable x if the sequence of functions $x^k(\zeta)$ converges to the function $x(\zeta)$ as $k \to \infty$ for all $\zeta \in \Omega$ except possibly on a set of probability zero. In other words: $P\Big(\lim_{k\to\infty} x^k(\zeta) = x(\zeta)\Big) = 1$.*

Definition A.0.3 *Given the random sequence x^k and the limiting random variable x, we say that x^k converges in probability to x if for every $\epsilon > 0$, then*

$$\lim_{k\to\infty} P\Big(|x(k) - x| > \epsilon\Big) = 0$$

.

Definition A.0.4 *A random sequence* x^k *converges in the mean-square sense to the random variable* x *if* $E\left\{|x(k) - x|^2\right\} \to 0$ *as* $k \to \infty$.

Definition A.0.5 *A random sequence* x^k *with probability distribution function* $F_k(x)$ *converges in distribution to the random variable* x *with probability distribution function* $F(x)$ *if* $\lim_{k\to\infty} F_k(x) = F(x)$ *at all* x *for which* F *is continuous.*

Appendix B

Information Theory Review

Definition B.0.6 *[5] The differential entropy of a random variable z with pdf $f(z)$ is*

$$h(z) = -\int_S f(z) \log_2 f(z) dz;$$

where S is the support set of the random variable.

Definition B.0.7 *[5] The joint differential entropy of a set z^k of random variables with density $f(z_1, z_2, \ldots, z_k)$ is defined as*

$$h(z_1, z_2, \ldots, z_k)$$
$$= -\int f(z_1, \ldots, z_k) \log_2 f(z_1, \ldots, z_k) dz_1, \ldots, dz_k.$$

Definition B.0.8 *[5] The mutual information $I(z; w)$ between two random variables, z and w, with joint density $f(z, w)$ is defined as*

$$I(z; w) = \int f(z, w) \log_2 \frac{f(z, w)}{f(z) f(w)} dz dw.$$

Definition B.0.9 *[5] The entropy rate of z is given by*

$$h_\infty(z) = \lim_{k\to\infty} \frac{h(z^k)}{k};$$

where $h(z^k)$ *is the joint differential entropy of* z^k.

Definition B.0.10 *[5] The mutual information rate of two stochastic processes z and w is defined*

$$I_\infty(z; w) = \lim_{k\to\infty} \frac{I(z^k; w^k)}{k};$$

where $I(z^k; w^k)$ *is the mutual information of* z^k *and* w^k *as defined by (B.0.8).*

References

[1] P. Antsaklis and A. N. Michel. *Linear Systems*. Mc Graw Hill, New York, New York, 1997.

[2] J. Braslavsky, R. Middleton, and J. Freundenber. Feedback stabilization over signal-to-noise ratio constrained channels. *IEEE Transactions on Automatic Control*, 52(8):1391–1403, August 2007.

[3] R. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Transactions on Automatic Control*, 45(7):1279–1289, July 2000.

[4] C. Canudas de Wit, F. Rubio, J. Fornes, and F. Gomez-Estern. Differential coding in networked controlled linear systems. pages 4177–4182, June 2006.

[5] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. John Wiley & Sons, Inc., Hoboken, New Jersey, 2006.

[6] N. Elia. When Bode meets Shannon: Control-oriented feedback communication schemes. *IEEE Transactions on Automatic Control*, 49(9):1477–1488, September 2004.

[7] N. Elia and S. Mitter. Stabilization of linear systems with limited information. *IEEE Transactions on Automatic Control*, 46(9):1384–1400, September 2001.

[8] F. Fagnani and S. Zampieri. Stability analysis and synthesis for scalar linear systems with a quantized feedback. 48(9):1569–1584, September 2003.

[9] V. Gupta, A. Dana, R. Murray, and B. Hassibi. On the effect of quantization on performance at high rates. In *Proceedings of the American Control Conference*, pages 1364–1369, June 2006.

[10] V. Gupta, D. Spanos, B. Hassibi, and R. Murray. On LQG control accross a stochastic packet-dropping link. In *Proceedings of the American Control Conference*, pages 360–365, June 2005.

[11] J. Hespanha, A. Ortega, and L. Vasudevan. Towards the control of linear systems with minimum bit-rate. *Proceedings of the Int. Symp. on the Mathematical Theory of Networks and Systems*, pages 1–15, August 2002.

[12] P. Hokayem. Stability analysis of networked control systems. Master's thesis, University of New Mexico, 2003.

[13] G. W. Kim, K. Ji, and A. Ambike. Networked real-time control strategies dealing with stochastic time delays and packet losses. *Proceedings of the American Control Conference*, 1:621–626, June 2005.

[14] F. Lian, J. Moyne, and D. Tilbury. Time delay modeling and sample time selection for networked control systems. *Proceedings of the ASME-DSC*, 10:1–8, 2001.

[15] D. Liberzon and J. Hespanha. Stabilization of nonlinear systems with limited information feedback. *IEEE Transactions on Automatic Control*, 50(6):910–915, June 2005.

[16] Q. Ling and M. Lemmon. Stability of quantized control systems under dynamic bit assignment. *IEEE Transactions on Automatic Control*, 50(5):734–739, May 2005.

[17] J. Liu and N. Elia. Convergence of fundamental limitations in information, estimation, and control. In *Proceedings of the IEEE Conference on Decision and Control*, pages 5609–5614, December 2006.

[18] I. Lopez and C. Abdallah. Recent advances on control theory under communications constrainsts: A survey. *Proceedings of the Mediterranean Conference on Control and Automation*, pages 1–6, June 2006.

[19] I. Lopez and C. Abdallah. Data rates conditions for network control system stabilization. *Proceedings of the Mediterranean Conference on Control and Automation*, pages 1–6, June 2007.

[20] I. Lopez and C. Abdallah. Rate-limited stabilization for network control systems. *Proceedings of the American Control Conference*, pages 275–280, July 2007.

[21] I. Lopez, C. Abdallah, and C. C. de Wit. Gain-scheduling multi-bit delta-modulator for networked controlled system. *Proceedings of the European Control Conference*, July 2007.

[22] I. Lopez, J. Piovesan, C. Abdallah, D. Lee, O. Martinez, M. Spong, and R. Sandoval. Practical issues in networked control systems. *Proceedings of the American Control Conference*, pages 4201–4206, June 2006.

[23] N. Martins. *Information Theoretic Aspects of the Control and Mode Estimation of Stochastic Systems*. PhD thesis, Massachusetts Institute of Technology, 2004.

[24] N. Martins and M. Dahleh. Feedback control in the presence of noisy channels: "Bode-like" fundamental limitations of performance. *To appear in IEEE Transactions of Automatic Control*, 2007.

[25] N. Martins and M. Dahleh. Fundamental limitations of disturbance attenuation in the presence of side information. *IEEE Transactions on Automatic Control*, 52(1):56–66, January 2007.

[26] N. Martins, M. Dahleh, and N. Elia. Feedback stabilization of uncertain systems using a stochastic digital link. *Proceedings of the IEEE Conference on Decision and Control*, 2:1889–1895, December 2004.

[27] A. Matveev and A. Savkin. An analogue of Shannon information theory for networked control systems: State estimation via a noisy discrete channel. In *Proceedings of the IEEE Conference on Decision and Control*, pages 4485–4490, December 2004.

[28] S. Mitter. Control with limited information: the role of systems theory and information theory. *Proceedings IEEE Int. Sympt. Information Theory*, 7:1–23, July 2001.

[29] L. A. Montestruque and P. Antsaklis. Stability of model-based networked control, systems with time-varying transmission times. *IEEE Transactions on Automatic Control*, 49(9):2876–2880, September 2004.

[30] R. Murray. Control in an information rich world. *Report of the Panel on Future Directions in Control*, June 2002.

[31] G. Nair, J. Evans, I. Mareels, and W. Moran. Feedback data rates of nonlinear systems. In *Proceedings of the European Control Conference*, pages 731–736, September 2003.

[32] G. Nair and R. Evans. Communication limited stabilization of linear systems. In *Proceedings of the IEEE Conference on Decision and Control*, volume 1, pages 1005–1010, December 2000.

[33] G. Nair and R. Evans. Stabilizability of stochastic linear systems with finite feedback data rate. *SIAM Journal Control & Optimization*, 43(2):413–436, July 2004.

[34] J. Nilsson. *Real-Time Control Systems with Delays*. PhD thesis, Lund Institute of Technology, 1998.

[35] J. Nocedal and S. Wright. *Numerical Optimization*. Springer, New York, New York, 1999.

[36] A. Papoulis. *Probability Random Variables, and Stochastic Processes*. Mc Graw Hill, New York, New York, 1991.

[37] C. Persis. Results on stabilization of nonlinear systems under finite data-rate constraints. In *Proceedings of the IEEE Conference on Decision and Control*, pages 3986–3991, December 2004.

[38] M. Pinsker. *Information and Information Stability of Random Variables and Processes*. Holden Day, San Francisco, California, 1964.

[39] A. Sahai. Any-time capacity and a separation theorem for tracking unstable processes. *Proceedings IEEE Int. Sympt. Information Theory*, pages 500–506, June 2000.

[40] A. Sahai. Evaluating channels for control: Capacity reconsidered. *Proceedings of the American Control Conference*, 4:2358–2362, June 2000.

[41] R. Sandoval-Rodriguez, C. Abdallah, H. Jerez, and R. Byrne. Experimental results on the effects of 802.11b WLAN on networked control system. *Proceedings of the Mediterranean Conference on Control and Automation*, pages 1113–1118, June 2005.

[42] S. V. Sarma. *Finite-Rate Control: Stability and Performance*. PhD thesis, Massachusetts Institute of Technology, 2006.

[43] M. Seron, J. Braslavsky, and G. Goodwin. *Fundamental Limitations in Filtering and Control*. Springer, London, 1997.

[44] K. Shanmugan and A. Breipohl. *Random Signals: Detection, Estimation and Data Analysis*. John Wiley & Sons, Inc., New York, New York, 1988.

[45] L. Shi, M. Epstein, and R. Murray. Networked control systems with norm bounded uncertainties: A stability analysis. *Proceedings of the American Control Conference*, pages 2783–2788, June 2006.

[46] L. Shi, M. Epstein, and R. Murray. Towards robust control over a packet dropping network. In *International Symposium on Mathematical Theory of Networks and Systems*, 2006.

[47] L. Shi and R. Murray. Towards a packet-based control theory - Part I: Stabilization over a packet-based network. *Proceedings of the American Control Conference*, 2:1251–1256, June 2005.

[48] L. Shi and R. Murray. Towards a packet-based control theory - Part II: Rates issues. *Proceedings of the American Control Conference*, pages 3482–3487, June 2006.

[49] H. Stark and J. Woods. *Probability and Random Processes with Applications to Signal Processing*. Pearson Education, Upper Saddle River, New Jersey, 2002.

[50] S. Tarbouriech, C. Abdallah, and J. Chiasson. *Advances in Communication Control Networks*. Springer, 2005.

[51] S. Tatikonda. *Control Under Communications Constraints*. PhD thesis, Massachusetts Institute of Technology, 2000.

[52] S. Tatikonda and S. Mitter. Control over noisy channels. *IEEE Transactions on Automatic Control*, 49(7):1196–1201, July 2004.

[53] S. Tatikonda and S. Mitter. Control under communication constraints. *IEEE Transactions on Automatic Control*, 49(7):1056–1068, July 2004.

[54] S. Tatikonda, A. Sahai, and S. Mitter. Stochastic linear control over a communication channel. *IEEE Transactions on Automatic Control*, 49(9):1549–1561, September 2004.

[55] H. Touchette and S. Lloyd. Information-theoretic limits of control. *Physical Review Letters*, 6(84):1156–1159, February 2000.

[56] G. Walsh, H. Ye, and L. Bushnell. Stability analysis of networked control systems. *IEEE Transactions on Control Systems Technology*, 10(3):2876–2880, May 2002.

[57] W. Wong and R. Brockett. Systems with finite communication bandwidth - Part II: Stabilization with limited information feedback. *IEEE Transactions on Automatic Control*, 44(5):1049–1053, May 1999.

[58] L. Xiao, A. Hassibi, and J. How. Control with random communication delays via a discrete-time jump system approach. In *Proceedings of the American Control Conference*, volume 3, pages 2199–2204, June 2000.

[59] S. Yuksel and T. Basar. Quantization for LTI systems with noiseless channels. In *Proceedings of the IEEE Conference on Control Applications*, volume 1, pages 423–429, June 2003.

[60] H. Zhang and Y. Sun. Bode integrals and laws of variety in linear control systems. In *Proceedings of the American Control Conference*, volume 1, pages 66–70, June 2003.

[61] H. Zhang and Y. Sun. Information theoretic limit and bound of disturbance rejection in LTI systems: Shannon entropy and H_∞ entropy. In *Proceedings of the IEEE International Conference on Systems, Man and Cybernetics*, volume 2, pages 1378–1383, October 2003.

[62] H. Zhang and Y. Sun. Directed information and mutual information in linear feedback tracking systems. In *Proceedings of the IEEE World Congress on Intelligent Control and Automation*, volume 1, pages 723–727, June 2006.

[63] L. Zhang, Y. Shi, T. Chen, and B. Huang. A new method for stabilization of networked control systems with random delays. *IEEE Transactions on Automatic Control*, 50(8):1177–1181, August 2005.